Stability Results for Steady, Spatially–Periodic Planforms

Benoit Dionne
Dept. of Mathematics
University of Ottawa
585 King Edward
Ottawa, Ontario K1N 6N5
Canada

Mary Silber
Dept. of Engineering Sciences
and Applied Mathematics
Northwestern University
Evanston, IL 60208
U.S.A.

Anne C. Skeldon
Dept. of Mathematics
City University
Northampton Square
London, EC1V 0HB
United Kingdom

September 25, 1995, revised July 31, 1996

Abstract

Equivariant bifurcation theory has been used extensively to study pattern formation via symmetry–breaking steady state bifurcation in various physical systems modeled by E(2)–equivariant partial differential equations. Much attention has focussed on solutions that are doubly–periodic with respect to a square or hexagonal lattice, for which the bifurcation problem can be restricted to a finite–dimensional center manifold. Previous studies have used four– and six–dimensional representations for the square and hexagonal lattice symmetry groups respectively, which in turn allows the relative stability of squares and rolls or hexagons and roll to be determined. Here we consider the countably infinite set of eight– and twelve–dimensional irreducible representations for the square and hexagonal cases, respectively. This extends earlier relative stability results to include a greater variety of bifurcating planforms, and also allows the stability of rolls, squares and hexagons to be established to a countably infinite set of perturbations. In each case we derive the Taylor expansion of the equivariant bifurcation problem and compute the linear, orbital stability of those solution branches guaranteed to exist by the equivariant branching lemma. In both cases we find that many of the stability results are established at cubic order in the Taylor expansion, although to completely determine the stability of certain states, higher order terms are required. For the hexagonal lattice, all of the solution branches guaranteed by the equivariant branching lemma are, generically, unstable due to the presence of a quadratic term in the Taylor expansion. For this reason we consider two special cases: the degenerate bifurcation problem that is obtained by setting the coefficient of the quadratic term to zero, and the bifurcation problem when an extra reflection symmetry is present.

1 Introduction.

Equivariant bifurcation theory [12] is a powerful tool for investigating pattern-forming instabilities in physical systems. This approach distinguishes between those aspects of the bifurcation problem that are a consequence of symmetry and those aspects that depend on the specifics of the mathematical model. For example, the general form of the bifurcation equations is derived using symmetry considerations alone, with details of the mathematical model appearing only in the numerical values of the coefficients. Consequently, disparate physical systems, that nonetheless share the same symmetries, can exhibit strikingly similar behavior. In this paper we extend previous work on the evolution of symmetry-breaking, steady state bifurcations in parameterized families of E(2)-equivariant partial differential equations (PDEs), where E(2) is the Euclidean group of rotations, reflections and translations in a plane. The results, which apply to spatially-periodic patterns, are based solely on the symmetries of the PDEs and certain features of the linear instability. They apply to a wide variety of pattern forming systems, *e.g.*, Rayleigh-Bénard convection [3], models of steady cellular patterns in combustion [26] and solidification [5], and chemical reaction-diffusion systems in the Turing instability regime [28].

The non-compactness of the Euclidean group presents a fundamental difficulty in applying standard methods of equivariant bifurcation theory, as described in [12], to bifurcation problems with E(2) symmetry. (See recent work by Melbourne [20] for a classification and treatment of bifurcation problems with Euclidean symmetry.) The approach used both here and previously is to restrict the solutions of the PDEs to those that are spatially doubly-periodic. This restriction reduces the relevant symmetry group to one that is compact – the translation symmetry of the problem is reduced from $\mathbf{R}^2$ to a torus $\mathbf{T}^2$. In this vein, Buzano and Golubitsky [4], and Golubitsky, Swift and Knobloch [13], considered steady state bifurcation of a spatially uniform equilibrium state to steady planforms periodic on a hexagonal lattice. Buzano and Golubitsky used singularity theory to derive the normal form of a degenerate $\mathbf{D}_6\dot{+}\mathbf{T}^2$-equivariant bifurcation problem on $\mathbf{C}^3$; one in which the coefficient of the quadratic term is zero. (In the non-degenerate problem, *all* of the primary solution branches bifurcate unstably.) They analyzed a universal unfolding of this degenerate bifurcation problem to determine all steady state solution branches, and their stability properties, in a neighborhood of the bifurcation point. Their approach enabled a rigorous analysis of the relative stability of the primary branches, namely hexagons and rolls and, in particular, the study of the well-know hysteretic transition between these two states. Golubitsky, Swift and Knobloch contrasted the degenerate problem investigated by Buzano and Golubitsky with the case where there is an additional reflection symmetry that kills all even terms in the bifurcation equations. They also determined stability and branching of steady states by analyzing an equivariant bifurcation problem posed on $\mathbf{C}^3$. The additional symmetry introduces two more primary branches, called regular triangles and patchwork quilt solutions in [13]; the latter are shown to be unstable. Both the degenerate problem and the problem with additional reflection symmetry arise naturally in, and are motivated by, Rayleigh-Bénard convection; the degenerate bifurcation problem arises when the linearized operator is self-adjoint [23]; and the $\mathbf{Z}_2$ symmetry corresponds to a reflection in the midplane of the fluid layer that is present in the Boussinesq approximation when the boundary conditions at the top and bottom of the fluid layer are identical [13].

Bifurcation to steady planforms periodic on a square lattice are considered in [27], where the general form of the $\mathbf{D}_4\dot{+}\mathbf{T}^2$-equivariant bifurcation problem on $\mathbf{C}^2$ is derived. This bifurcation problem reduces to one with $\mathbf{D}_4$ symmetry on $\mathbf{R}^2$, which has been studied extensively in the context of O(2)-equivariant Hopf bifurcation [12]. The primary branches for this bifurcation problem are squares and rolls, and again the relative stability of these two states can be found in terms of the coefficients of the leading terms in the Taylor expansion of the bifurcation equations.

Subsequently, Dionne [7] used entirely group theoretic methods to classify, by symmetry and spatial periodicity, all spatially-periodic planforms that are guaranteed to exist by the equivariant branching lemma [12, 29]. (Also see Dionne and Golubitsky [8].) This classification applies to a broad class of E(2)-equivariant steady state bifurcation problems. In addition to the rolls, squares and hexagons discussed above, there is a continuum of rectangles, and a countably infinite set of "super squares", "anti-squares" and "super hexagons". All of these planforms bifurcate simultaneously from the fully symmetric equilibrium state. The branching of super hexagons in Rayleigh-Bénard convection was investigated by Kirchgässner [16].

In this paper, we re-visit the issue of relative stability of solutions which are doubly-periodic on hexagonal and square lattices by now considering the remaining irreducible representations of $\mathbf{D}_6\dot{+}\mathbf{T}^2$ and $\mathbf{D}_4\dot{+}\mathbf{T}^2$, which are on $\mathbf{C}^6$ and $\mathbf{C}^4$, respectively. This enables us to study the relative stability of rolls, squares and hexagons to some of the new states shown to exist in [7, 8]. In all cases, the bifurcation problems associated with the lower-dimensional representations, analyzed in [4], [13] and [27], are regained by restricting the bifurcation equations to an appropriate subspace. There are a countably infinite number of representations of $\mathbf{D}_6\dot{+}\mathbf{T}^2$ and $\mathbf{D}_4\dot{+}\mathbf{T}^2$ on $\mathbf{C}^6$ and $\mathbf{C}^4$, respectively. For each representation there are six planforms guaranteed to exist by the equivariant branching lemma. In the case of the square lattice these are: rolls (stripes), simple squares, two different types of rhombs (rectangles), super squares, and anti-squares. In the case of the hexagonal lattice, the planforms are rolls, simple hexagons, three different rhombs, and super hexagons. The precise form of the rhombs, super squares, anti-squares and super hexagons differs from representation to representation. (Some examples of the different planforms are shown in figures 3 and 4 below.)

Following a similar rationale to Golubitsky, Swift and Knobloch, we proceed by first restricting the space of solutions of the PDEs to those that are periodic with respect to a square or hexagonal lattice. Then, within this subspace of solutions, we invoke the center manifold theorem to reduce the bifurcation problem to a finite-dimensional one

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \lambda), \quad \mathbf{g} : \mathbf{C}^s \times \mathbf{R} \to \mathbf{C}^s, \tag{1.1}$$

where $s = 4$ for the square lattice problem, $s = 6$ for the hexagonal one, and λ is the bifurcation parameter.

We treat the square and hexagonal lattice bifurcation problems separately. In each case we determine the Taylor expansion of the equivariant bifurcation problem to sufficiently high order so that we can determine the linear (orbital) stability of the planforms to perturbations that lie on the same lattice. In the case of the hexagonal lattice the Taylor expansion of $\mathbf{g}$ contains quadratic terms that force the solution branches to bifurcate unstably [15]. Thus, as in [13], we consider the following two problems: (1) the degenerate bifurcation problem in which the coefficient of the quadratic term is zero, and (2) the bifurcation problem for PDEs that are E(2)$\oplus\mathbf{Z}_2$-equivariant, where the extra $\mathbf{Z}_2$ reflection symmetry kills the even terms in the Taylor expansion of $\mathbf{g}$. When the PDEs are E(2)$\oplus\mathbf{Z}_2$-equivariant, we use the equivariant branching lemma to show that there are five additional solution branches to those given above. These are: simple triangles (called "regular triangles" in [13]), rhombs (called the "patchwork quilt" in [13]), anti-hexagons, super triangles, and anti-triangles. (See figure 5 below for examples of these planforms.) The countable set of anti-hexagons, super triangles and anti-triangles solution branches is new; all are periodic on some hexagonal lattice. The presence of the extra $\mathbf{Z}_2$ symmetry does not change the bifurcation problems on square lattices. For each problem, we exploit the symmetry of the solution branch to determine the eigenvalues of the Jacobian matrix D$\mathbf{g}$ and their multiplicities.

This study enables us to achieve two ends. Firstly, for a particular irreducible representation, it allows us to determine the relative stability of the primary branches enumerated above in terms of the coefficients of the leading terms in the Taylor expansion of (1.1). For example, we determine the relative stability of the six primary branches known to exist for each representation of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ on $\mathbf{C}^4$. Our analysis also allows us to make a number of general statements about (bi)stability and branching of solutions. These results are especially pertinent to E(2) equivariant PDEs posed on square or hexagonal domains with periodic boundary conditions. Secondly, since rolls, simple squares, simple hexagons and simple triangles are primary branches for all of the countably infinite set of representations, our analysis presents a framework for determining the stability of these primary branches to a countable set of perturbations in E(2)-equivariant problems. We find that the perturbation calculations necessary for determining these stability results in specific applications are no more involved than those executed to determine the relative stability of squares and rolls. (See [17] for a similar stability computation of simple hexagons in the Bénard problem.)

Our paper is organized as follows. In section 2 we give a mathematical formulation of the bifurcation problem, by stating our assumptions about the linear instability and giving the action of the symmetry group on the space of spatially-periodic solutions on the square and hexagonal lattices. In section 3 we characterize the solutions guaranteed by the equivariant branching lemma in terms of their symmetries. The role of "hidden" Euclidean symmetries is described. We also present some examples of the planforms associated with these primary solution branches. Section 4 contains our analysis of the square lattice bifurcation problem. We compute the eigenvalues of the solution branches in terms of the coefficients of the leading terms in the Taylor expansion of the general bifurcation problem. From this information we draw a number of conclusions about the branching and (bi)stability of the solutions. In section 5 we consider two bifurcation problems associated with the hexagonal lattices. We compute stability of the solutions for the degenerate bifurcation problem in which the coefficient of the quadratic term is zero. We also briefly discuss the unfolding of this bifurcation problem and present an example bifurcation diagram that indicates the secondary bifurcation points on the primary solution branches. We then consider the bifurcation problem in the case that there is an extra $\mathbf{Z}_2$ symmetry. Section 6 contains our conclusions.

2 Problem Formulation.

2.1 Symmetries of the PDEs.

We consider parameterized families of partial differential equations which we write in evolutionary form,

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x}, y, t) = \mathbf{F}(\mathbf{u}(\mathbf{x}, y, t), \lambda) \ , \tag{2.1}$$

where $\mathbf{F} : \mathcal{X} \times \mathbf{R} \to \mathcal{Y}$ is a nonlinear operator between suitably chosen function spaces, $\mathcal{X}$ and $\mathcal{Y}$, and $\lambda \in \mathbf{R}$ is the bifurcation parameter. Here $\mathbf{u} : \mathbf{R}^2 \times \Omega \times \mathbf{R} \to \mathbf{R}^n$ is a function in $\mathcal{X}$ of a spatial variable $\mathbf{x} \in \mathbf{R}^2$, (possibly) a bounded spatial variable $y \in \Omega$, and time t.

We assume that (2.1) has Euclidean symmetry. The Euclidean group E(2) is the group of motions in $\mathbf{R}^2$ that preserve distances, *i.e.* rotations, reflections and translations. We denote elements of E(2) by $(h, \mathbf{d})$ where $h \in \mathrm{O}(2)$ is an orthogonal transformation (a reflection or rotation) and $\mathbf{d} \in \mathbf{R}^2$ is a translation. The action of $(h, \mathbf{d}) \in \mathrm{E}(2)$ on $\mathbf{x} \in \mathbf{R}^2$ is defined by

$$(h, \mathbf{d})\mathbf{x} = h\mathbf{x} + \mathbf{d} \ . \tag{2.2}$$

This action forces the product of $(h_1, \mathbf{d_1})$ and $(h_2, \mathbf{d_2})$ to be defined by

$$(h_1, \mathbf{d_1})(h_2, \mathbf{d_2}) = (h_1 h_2, \mathbf{d_1} + h_1 \mathbf{d_2}) \ . \tag{2.3}$$

Hence E(2) is the semi-direct product (denoted by $\dot{+}$) of the groups of orthogonal transformations and translations; specifically, E(2)=O(2)$\dot{+}\mathbf{R}^2$, where $\mathbf{R}^2$ is a normal subgroup of E(2).

We assume that the Euclidean group acts on the vector-valued function $\mathbf{u} : \mathbf{R}^2 \times \Omega \times \mathbf{R} \to \mathbf{R}^n$ as follows

$$\gamma.\mathbf{u}(\mathbf{x}, y, t) = \mathbf{A}_h \mathbf{u}(\gamma^{-1}\mathbf{x}, y, t) \tag{2.4}$$

for all $\gamma = (h, \mathbf{d}) \in$ E(2). Here $\mathbf{A}_h$ is an $n \times n$ orthogonal matrix; the collection of all $\mathbf{A}_h$ is a representation of O(2) on $\mathbf{R}^n$. Our assumption that (2.1) has Euclidean symmetry means that $\mathbf{F}$ is E(2)–equivariant, *i.e.*,

$$\gamma.\mathbf{F}(\mathbf{u}(\mathbf{x}, y, t), \lambda) = \mathbf{F}(\gamma.\mathbf{u}(\mathbf{x}, y, t), \lambda) \ . \tag{2.5}$$

In the remainder of the paper we suppress any possible dependence of $\mathbf{u}$ on the bounded spatial variable y.

The symmetry of the problem is enlarged from E(2) to E(2)$\oplus\mathbf{Z}_2$ for some of the motivating applications. For example, in certain Rayleigh-Bénard convection problems $\mathbf{Z}_2$ is a reflection in the mid-plane of the fluid layer [13].

2.2 Linear analysis and the symmetry-breaking bifurcation.

We assume that there is a Euclidean-invariant time-independent solution of (2.1) for all values of λ. This corresponds to a spatially uniform equilibrium, which, without loss of generality, we take to be $\mathbf{u} = \mathbf{0}$. We assume that this trivial solution is linearly stable for $\lambda < 0$, unstable for $\lambda > 0$, and that $\lambda = 0$ corresponds to a symmetry–breaking steady state bifurcation point. At this bifurcation point, the zero solution is neutrally stable to perturbations in the form of spatial Fourier modes $e^{2\pi i \mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} \in \mathbf{R}^2$, $|\mathbf{k}| = k_c$, where we assume that k_c is nonzero. We refer to the equilibrium solutions $\{\mathbf{u_k} e^{2\pi i \mathbf{k}\cdot\mathbf{x}},\ |\mathbf{k}| = k_c\}$, of the linearized problem at $\lambda = 0$, as the *critical* or *neutral* modes, and the circle $|\mathbf{k}| = k_c$ in the two–dimensional $\mathbf{k}$–space as the *critical circle*. Here $\mathbf{u_k}$ is a constant n-dimensional vector, which we take to be unique, up to scalar multiplication.

Melbourne [20] has recently shown that, generically, there are two distinct classes of symmetry–breaking, steady state bifurcation problems for systems of E(2)–equivariant PDEs, each of which can be reduced (locally) to a single PDE. Following [1], Melbourne refers to the two types as *scalar* and *pseudoscalar*, where the scalar action of E(2) on $v : \mathbf{R}^2 \to \mathbf{R}$ is $v(\mathbf{x}) \mapsto v(\gamma^{-1}\mathbf{x})$, and the pseudoscalar action is $v(\mathbf{x}) \mapsto \det(h) v(\gamma^{-1}\mathbf{x})$, for all $\gamma = (h, \mathbf{d}) \in \mathbf{R}^2$. In this paper, we consider the scalar case only, *i.e.*, we assume that the kernel of the linearized PDEs transforms under the scalar action of E(2). This is the case for all of the applications mentioned in the introduction. (See [1] for examples of "pseudo-scalar" PDEs, and a classification of the spatially–periodic planforms guaranteed by the equivariant branching lemma in this case.)

2.3 Spatially doubly-periodic solutions.

We restrict our bifurcation analysis to solutions $\mathbf{u}(\mathbf{x}, t)$ of (2.1) that are doubly-periodic with respect to some square or hexagonal lattice $\mathcal{L}$. Specifically, the planar lattice $\mathcal{L}$ is generated by two linearly independent vectors $\ell_1, \ell_2 \in \mathbf{R}^2$, *i.e.*,

$$\mathcal{L} = \{n_1\ell_1 + n_2\ell_2 \in \mathbf{R}^2 : n_1, n_2 \in \mathbf{Z}\}. \tag{2.6}$$

We say that a function $\mathbf{u}(\mathbf{x},t)$ is $\mathcal{L}-periodic$ if

$$\mathbf{u}(\mathbf{x}+\ell,t)=\mathbf{u}(\mathbf{x},t) \quad \text{for all} \quad \ell\in\mathcal{L}. \tag{2.7}$$

We assume that $\mathcal{L}$-periodic solutions of (2.1) can be expressed in a Fourier series

$$u_j(\mathbf{x},t)=\sum_{\mathbf{k}\in\mathcal{L}^*}\left(\hat{u}_{j,\mathbf{k}}(t)\,e^{2\pi i\mathbf{k}\cdot\mathbf{x}}+c.c.\right)\ , \ \ j=1,\ldots,n\ , \tag{2.8}$$

where $\hat{u}_{j,\mathbf{k}}\in\mathbf{C}$ is the time-dependent amplitude of the $\mathbf{k}^{th}$ Fourier mode. The wave vectors $\mathbf{k}$ lie in the dual lattice to $\mathcal{L}$, denoted $\mathcal{L}^*$. Specifically, $\mathcal{L}^*$ is generated by two linearly independent vectors $\mathbf{k}_1,\mathbf{k}_2\in\mathbf{R}^2$, where $\mathbf{k}_i\cdot\ell_j=\delta_{i,j}$ (the Kronecker delta):

$$\mathcal{L}^*=\{n_1\mathbf{k}_1+n_2\mathbf{k}_2\in\mathbf{R}^2:n_1,n_2\in\mathbf{Z}\}. \tag{2.9}$$

In this paper we consider two cases that satisfy $|\ell_1|=|\ell_2|$: (1) the square lattice, where the spatial variable $\mathbf{x}$ is scaled so that

$$\ell_1=(1,0), \qquad \ell_2=(0,1), \tag{2.10}$$

and (2) the hexagonal lattice, with $\mathbf{x}$ scaled so that

$$\ell_1=\Big(\frac{1}{\sqrt{3}},1\Big), \qquad \ell_2=\Big(\frac{2}{\sqrt{3}},0\Big). \tag{2.11}$$

An important consequence of restricting the solution space of (2.1) to $\mathcal{L}$-periodic functions is that the spectrum of the linear operator $\mathbf{L}_\lambda$ is rendered discrete. Hence, we expect the center manifold theorem [19] to apply at the bifurcation point. Specifically, for the problems of interest, this restriction ensures that there are only a finite number of zero eigenvalues at the bifurcation point, with all other eigenvalues bounded away from the imaginary axis. The dimension of the bifurcation problem depends on the number of points $\mathbf{k}\in\mathcal{L}^*$ that lie on the critical circle of radius k_c. For the square and hexagonal lattices we consider the cases where the critical circle intersects 8 and 12 points points in $\mathcal{L}^*$ respectively (see Figure 1).

In what follows we identify the kernel of the linear operator $\mathbf{L}_0$,

$$\ker(\mathbf{L}_0)=\{\mathbf{u}=\sum_{j=1}^{s}z_je^{2\pi i\mathbf{K}_j\cdot\mathbf{x}}\mathbf{u}_j+c.c.:\ z_j\in\mathbf{C},\ |\mathbf{K}_j|=k_c\}\ , \tag{2.12}$$

with the vector space

$$V=\{v=\sum_{j=1}^{s}z_je^{2\pi i\mathbf{K}_j\cdot\mathbf{x}}+c.c.:\ z_j\in\mathbf{C},\ |\mathbf{K}_j|=k_c\}\cong\mathbf{C}^s\ , \tag{2.13}$$

where the isomorphism between V and $\mathbf{C}^s$ is defined by

$$v\mapsto\mathbf{z}=(z_1,z_1,\ldots,z_s)\ . \tag{2.14}$$

As a vector space over the reals, $\dim(V)=2s$. As mentioned above, this paper focuses on the case $s=4$ for the square lattice and $s=6$ for the hexagonal lattice.

The PDEs, restricted to the center manifold, lead to a system of ordinary differential equations

$$\dot{\mathbf{z}}=\mathbf{g}(\mathbf{z},\lambda), \quad \mathbf{g}:\mathbf{C}^s\times\mathbf{R}\to\mathbf{C}^s. \tag{2.15}$$

Here $g(\mathbf{0},\lambda)=\mathbf{0}$ and the Jacobian matrix at the bifurcation point, $\mathrm{D}\mathbf{g}(\mathbf{0},0)$, is the zero matrix. In the next section we describe the symmetries inherited by the bifurcation problem from the PDEs. In particular, if Γ is the symmetry group of the bifurcation problem (2.15), then $\mathbf{g}(\mathbf{z},\lambda)$ satisfies the usual equivariance condition

$$\gamma\mathbf{g}(\mathbf{z},\lambda)=\mathbf{g}(\gamma\mathbf{z},\lambda), \quad \text{for all } \gamma\in\Gamma\ . \tag{2.16}$$

2.4 Symmetry of the restricted bifurcation problem.

The symmetry of the PDEs (2.1), reformulated in the space $\mathcal{X}_{\mathcal{L}}$ of $\mathcal{L}$-periodic functions, is a compact group Γ. Specifically, Γ is the largest group, constructed from E(2), that preserves $\mathcal{X}_{\mathcal{L}}$, *i.e.*, $\gamma.\mathcal{X}_{\mathcal{L}} \subset \mathcal{X}_{\mathcal{L}}$ for all $\gamma \in \Gamma$. As with E(2), Γ has a semi-direct product structure, namely $\Gamma = \mathbf{H}\dot{+}\mathbf{T}^2$, where $\mathbf{H} \subset$O(2) is the finite group of rotations and reflections that preserve the lattice and $\mathbf{T}^2 \simeq \mathbf{R}^2/\mathcal{L}$ is the torus of translations. The discrete group $\mathbf{H}$ is called the *holohedry* of the lattice; in the case of the square lattice, $\mathbf{H} = \mathbf{D}_4$, while $\mathbf{H} = \mathbf{D}_6$ for the hexagonal lattice. (Recall that $\mathbf{D}_n$, the dihedral group of order $2n$, is the group of symmetries of a regular n-gon.) In this paper we also consider the case where Γ is enlarged to $\Gamma \oplus \mathbf{Z}_2$. In the remainder of the paper, let $\Gamma_s \equiv \mathbf{D}_4\dot{+}\mathbf{T}^2$ and $\Gamma_h \equiv \mathbf{D}_6\dot{+}\mathbf{T}^2$, while Γ, without a subscript, refers to $\Gamma_s(\oplus\mathbf{Z}_2)$ and/or $\Gamma_h(\oplus\mathbf{Z}_2)$.

Square lattice case.

For doubly-periodic solutions on a square lattice we take the generators of the dual lattice $\mathcal{L}^*$ to be

$$\mathbf{k}_1 = (1,0) \quad \text{and} \quad \mathbf{k}_2 = (0,1) \ . \tag{2.17}$$

Thus the wave vectors $\mathbf{k} \in \mathcal{L}^*$ in (2.8) have the form (n_1, n_2), where n_1 and n_2 are integers. Moreover, we assume that lengths in the original PDEs have been scaled so that $k_c = \sqrt{\alpha^2+\beta^2}$ for some integers α and β. Alternatively, we could have held k_c fixed and scaled the lattice $\mathcal{L}$.

The relevant representation of the symmetry group $\Gamma_s = \mathbf{D}_4\dot{+}\mathbf{T}^2$ is determined by considering its action on the complex amplitudes z_j of the critical Fourier modes in (2.13). The irreducible representations of Γ_s are either 4-dimensional or 8-dimensional, in which case there are two or four complex Fourier amplitudes, respectively. Examples of these two different cases are depicted in Figure 1a for $k_c = 1$ and $k_c = \sqrt{3}$, *i.e.*, for $(\alpha,\beta) = (1,0)$ and $(\alpha,\beta) = (2,1)$. Note that it is also possible for the critical circle to intersect more than eight points in the dual lattice, *e.g.*, if $k_c = 5$ then there are four (real) Fourier modes associated with $(\alpha,\beta) = (5,0)$ and eight associated with $(\alpha,\beta) = (4,3)$. We do not consider these special cases here. (See Crawford [6] for an application of these higher-dimensional reducible representations.)

Following Dionne and Golubitsky [8] we require the representation of Γ_s to be not only irreducible, but also *translation free*. A representation is translation free if there are no (non-trivial) translations in Γ_s that act trivially on (2.13). This requirement ensures that we have found the finest lattice $\mathcal{L}$ that supports the neutral modes (2.13) [8]. Table 1 gives the values of the critical wave vectors for the translation free (absolutely) irreducible representations, henceforth simply called representations. Note that there is just one four-dimensional representation. It is the one that applies when the periodicity of functions in $\mathcal{X}_{\mathcal{L}}$ coincides with the wavelength of the instability, *i.e.* $k_c = |\mathbf{k}_1| = |\mathbf{k}_2|$. We focus on the eight-dimensional representations associated with the integer pairs (α,β) where $\alpha > \beta > 0$ (see Figure 2a). The additional requirements in Table 1, namely that α and β are relatively prime and not both odd, ensure that the representation is translation free, and hence that the set of all critical modes (2.13) cannot be supported by a finer lattice $\mathcal{L}$ [8].

$\mathbf{D}_4 \subset \Gamma_s$ is generated by a counterclockwise rotation $R_{\pi/2}$ by $\pi/2$ about the origin and a reflection τ_{x_1} through the x_1-axis. The elements of $\mathbf{T}^2 \subset \Gamma_s$ are denoted by $\Theta = (\theta_1, \theta_2)$, where $\theta_1, \theta_2 \in [0,1)$. The action of Γ_s on V for $s = 4$ in Table 1 induces an action of Γ_s on $\mathbf{C}^4$ generated by (*cf.* Figure 2a)

$$R_{\pi/2}(\mathbf{z}) = (\overline{z}_2, z_1, \overline{z}_4, z_3) \ , \tag{2.18}$$

$$\tau_{x_1}(\mathbf{z}) = (\overline{z}_4, \overline{z}_3, \overline{z}_2, \overline{z}_1) \ , \tag{2.19}$$

Table 1: Translation Free (absolutely) Irreducible Representations for the Square Lattice.

$\dim(V)$	$\mathbf{K}'s$
4 (s=2)	$\mathbf{K}_1=\mathbf{k}_1=(1,0)$ $\mathbf{K}_2=\mathbf{k}_2=(0,1)$
8 (s=4)	$\mathbf{K}_1=\alpha\mathbf{k}_1+\beta\mathbf{k}_2=(\alpha,\beta)$ $\mathbf{K}_2=-\beta\mathbf{k}_1+\alpha\mathbf{k}_2=(-\beta,\alpha)$ $\mathbf{K}_3=\beta\mathbf{k}_1+\alpha\mathbf{k}_2=(\beta,\alpha)$ $\mathbf{K}_4=-\alpha\mathbf{k}_1+\beta\mathbf{k}_2=(-\alpha,\beta)$ $\alpha,\beta\in\mathbf{Z}$, $\alpha>\beta>0$, α and β are relatively prime and not both odd.

and

$$\begin{aligned}\Theta(\mathbf{z}) &= (e^{-2\pi i\mathbf{K}_1\cdot\Theta}z_1, e^{-2\pi i\mathbf{K}_2\cdot\Theta}z_2, e^{-2\pi i\mathbf{K}_3\cdot\Theta}z_3, e^{-2\pi i\mathbf{K}_4\cdot\Theta}z_4) \\ &= (e^{-2\pi i(\alpha\theta_1+\beta\theta_2)}z_1, e^{-2\pi i(-\beta\theta_1+\alpha\theta_2)}z_2, e^{-2\pi i(\beta\theta_1+\alpha\theta_2)}z_3, e^{-2\pi i(-\alpha\theta_1+\beta\theta_2)}z_4)\ .\end{aligned} \tag{2.20}$$

Hexagonal lattice case.

For doubly-periodic solutions on a hexagonal lattice the generators of the dual lattice $\mathcal{L}^*$ are

$$\mathbf{k}_1=(0,1) \quad \text{and} \quad \mathbf{k}_2=(\sqrt{3}/2,-1/2)\ . \tag{2.21}$$

We assume that lengths in the original PDEs have been scaled so that $k_c=\sqrt{\alpha^2+\beta^2-\alpha\beta}$ for some integers α and β.

The relevant representation of the symmetry group $\Gamma_h=\mathbf{D}_6\dot{+}\mathbf{T}^2$ is determined by considering its action on the complex amplitudes of the critical Fourier modes at the bifurcation point. The irreducible representations of Γ_h are either 6-dimensional or 12-dimensional. Examples of these two different cases are depicted in Figure 1b for $k_c=1$ and $k_c=\sqrt{7}$, *i.e.*, for $(\alpha,\beta)=(1,0)$ and $(\alpha,\beta)=(3,2)$. The values of the critical wave vectors for the translation free (absolutely) irreducible representations are summarized in Table 2. Note that there is just one six-dimensional representation which is associated with the case where the periodicity of functions in $\mathcal{X}_{\mathcal{L}}$ coincides with the wavelength of the instability, *i.e.* $k_c=|\mathbf{k}_1|=|\mathbf{k}_2|$. The bifurcation problem associated with this representation of Γ_h has been studied extensively [4, 13]. In this paper we focus on the twelve-dimensional representations associated with the integer pairs (α,β), $\alpha>\beta>\alpha/2>0$ (see Figure 2b). The restrictions that α and β be relatively prime and that $\alpha+\beta$ not be divisible by 3 ensure that the representations are translation free [8].

$\mathbf{D}_6\subset\Gamma_h$ is generated by a counterclockwise rotation $R_{\pi/3}$ by $\pi/3$ and a reflection τ_{x_1} through the x_1-axis. The elements of $\mathbf{T}^2\subset\Gamma_h$ are denoted by $\Theta=\theta_1\ell_1+\theta_2\ell_2$, where $\ell_1=(1/\sqrt{3},1)$, $\ell_2=(2/\sqrt{3},0)$, and $\theta_1,\theta_2\in[0,1)$. The action of Γ_h on V for $s=6$ in Table 2 induces an action of Γ_h on $\mathbf{C}^6$ generated by (*cf.* Figure 2b)

$$R_{\pi/3}(\mathbf{z})=(\overline{z}_2,\overline{z}_3,\overline{z}_1,\overline{z}_5,\overline{z}_6,\overline{z}_4)\ , \tag{2.22}$$

$$\tau_{x_1}(\mathbf{z})=(z_6,z_5,z_4,z_3,z_2,z_1) \tag{2.23}$$

and

$$\begin{aligned}\Theta(\mathbf{z}) = (&e^{-2\pi i(\alpha\theta_1+\beta\theta_2)}z_1, e^{-2\pi i((-\alpha+\beta)\theta_1-\alpha\theta_2)}z_2, e^{-2\pi i(-\beta\theta_1+(\alpha-\beta)\theta_2)}z_3, \\ &e^{-2\pi i(\alpha\theta_1+(\alpha-\beta)\theta_2)}z_4, e^{-2\pi i(-\beta\theta_1-\alpha\theta_2)}z_5, e^{-2\pi i((-\alpha+\beta)\theta_1+\beta\theta_2)}z_6)\ .\end{aligned} \tag{2.24}$$

Table 2: Translation Free (absolutely) Irreducible Representations for the Hexagonal Lattice.

dim(V)	$\mathbf{K}'s$
6 s=3	$\mathbf{K}_1 = \mathbf{k}_1 = (0,1)$ $\mathbf{K}_2 = \mathbf{k}_2 = (\sqrt{3}/2, -1/2)$ $\mathbf{K}_3 = -\mathbf{k}_1 - \mathbf{k}_2 = (-\sqrt{3}/2, -1/2)$
12 s=6	$\mathbf{K}_1 = \alpha\mathbf{k}_1 + \beta\mathbf{k}_2$ $\mathbf{K}_2 = (-\alpha+\beta)\mathbf{k}_1 - \alpha\mathbf{k}_2$ $\mathbf{K}_3 = -\beta\mathbf{k}_1 + (\alpha-\beta)\mathbf{k}_2$ $\mathbf{K}_4 = \alpha\mathbf{k}_1 + (\alpha-\beta)\mathbf{k}_2$ $\mathbf{K}_5 = -\beta\mathbf{k}_1 - \alpha\mathbf{k}_2$ $\mathbf{K}_6 = (-\alpha+\beta)\mathbf{k}_1 + \beta\mathbf{k}_2$ $\alpha, \beta \in \mathbf{Z}$, $\alpha > \beta > \alpha/2 > 0$, α and β are relatvely prime and $\alpha + \beta$ is not a multiple of 3.

Additional $\mathbf{Z}_2$ symmetry.

In this paper we consider the possibility that there is an additional $\mathbf{Z}_2$ symmetry so that the bifurcation problems are equivariant with respect to $(\mathbf{H}\dot{+}\mathbf{T}^2) \oplus \mathbf{Z}_2$, where $\mathbf{H} = \mathbf{D}_4$ or $\mathbf{H} = \mathbf{D}_6$. We assume that $\kappa \in \mathbf{Z}_2$ takes v to $-v$, where $v \in V$ is given by (2.13). This induces the following action on $\mathbf{z} \in \mathbf{C}^s$:

$$\kappa(\mathbf{z}) = -\mathbf{z} \ . \tag{2.25}$$

The additional reflection symmetry has no effect on the bifurcation problems associated with the square lattice. This observation, for the four-dimensional representation of $(\mathbf{D}_4\dot{+}\mathbf{T}^2) \oplus \mathbf{Z}_2$, is made in [25]. The case of the eight-dimensional representations in Table 1 is the same. Specifically, we note that the translation $(\frac{1}{2}, \frac{1}{2}) \in \mathbf{T}^2$ in (2.20) acts on $\mathbf{z}$ in the same way as the reflection κ in (2.25). Hence, we need only consider the effect of the additional reflection for the hexagonal lattice bifurcation problems.

3 Group Theoretic Results.

3.1 Axial subgroups.

In this paper, we consider solution branches that are guaranteed to exist by the equivariant branching lemma [12, 29]. This lemma provides an algebraic criterion for existence of solution branches associated with particular subgroups of Γ. Specifically, we specify the symmetry of an equilibrium solution $\mathbf{z} \in \mathbf{C}^s$ by the isotropy subgroup $\Sigma_{\mathbf{z}} \subset \Gamma$, where

$$\Sigma_{\mathbf{z}} = \{\sigma \in \Gamma : \sigma\mathbf{z} = \mathbf{z}\} \ . \tag{3.1}$$

A subgroup $\Sigma \subset \Gamma$ is an *isotropy* subgroup if there exists a $\mathbf{z} \in \mathbf{C}^s$ for which $\Sigma_{\mathbf{z}} = \Sigma$. Associated with each isotropy subgroup $\Sigma \subset \Gamma$ is a vector subspace of $\mathbf{C}^s$, called the fixed point subspace and denoted Fix(Σ), where

$$\text{Fix}(\Sigma) = \{\mathbf{z} \in \mathbf{C}^s : \sigma\mathbf{z} = \mathbf{z}, \ \text{for all } \sigma \in \Sigma\} \ . \tag{3.2}$$

The equivariant branching lemma states that provided certain (generic) conditions are satisfied by the bifurcation, there exists a branch of equilibrium solutions, bifurcating from the origin at $\lambda = 0$, with symmetry Σ for each isotropy subgroup $\Sigma \subset \Gamma$ that satisfies dim(Fix(Σ))=1.

Table 3: Axial subgroups Σ (up to conjugacy) of Γ_s. The generators of Γ_s are given in (2.22)-(2.24).

Nomenclature	Σ	Generators of Σ	Fix(Σ)
Super Squares (SuS$_{\alpha,\beta}$)	$\mathbf{D}_4$	$R_{\pi/2}$, τ_{x_1}	$z_1 = z_2 = z_3 = z_4 \in \mathbf{R}$
Anti-squares (AS$_{\alpha,\beta}$)	$\widetilde{\mathbf{D}}_4$	$R_{\pi/2}$, $(\tau_{x_1}, (\frac{1}{2}, \frac{1}{2}))$	$z_1 = z_2 = -z_3 = -z_4 \in \mathbf{R}$
Rolls (R)	$\mathbf{Z}_2^c \dot{+} \mathbf{S}^1$	R_π, $\mathbf{S}^1$ $^{(a)}$	$z_1 \in \mathbf{R}$, $z_2 = z_3 = z_4 = 0$
Simple Squares (SiS)	$\mathbf{Z}_4 \dot{+} \mathbf{S}_{1,2}$	$R_{\pi/2}$, $\mathbf{S}_{1,2}$ $^{(b)}$	$z_1 = z_2 \in \mathbf{R}$, $z_3 = z_4 = 0$
Rhombs (Rh$_{s1,\alpha,\beta}$)	$\mathbf{D}_2^d \dot{+} \mathbf{S}_{1,3}$	R_π, $\tau_{x_1} R^3_{\pi/2}$, $\mathbf{S}_{1,3}$ $^{(c)}$	$z_1 = z_3 \in \mathbf{R}$, $z_2 = z_4 = 0$
Rhombs (Rh$_{s2,\alpha,\beta}$)	$\mathbf{D}_2^x \dot{+} \mathbf{S}_{1,4}$	R_π, τ_{x_1}, $\mathbf{S}_{1,4}$ $^{(d)}$	$z_1 = z_4 \in \mathbf{R}$, $z_2 = z_3 = 0$

$^{(a)}$ $\mathbf{S}^1 = \{(\beta s, -\alpha s) \in \mathbf{T}^2 : s \in \mathbf{R}\}$.

$^{(b)}$ $\mathbf{S}_{1,2}$ is generated by $(\frac{\alpha}{\alpha^2+\beta^2}, \frac{\beta}{\alpha^2+\beta^2}), (\frac{-\beta}{\alpha^2+\beta^2}, \frac{\alpha}{\alpha^2+\beta^2}) \in \mathbf{T}^2$.

$^{(c)}$ $\mathbf{S}_{1,3}$ is generated by $(\frac{\alpha}{\alpha^2-\beta^2}, \frac{-\beta}{\alpha^2-\beta^2}), (\frac{-\beta}{\alpha^2-\beta^2}, \frac{\alpha}{\alpha^2-\beta^2}) \in \mathbf{T}^2$.

$^{(d)}$ $\mathbf{S}_{1,4}$ is generated by $(\frac{1}{2\alpha}, \frac{1}{2\beta}), (\frac{-1}{2\alpha}, \frac{1}{2\beta}) \in \mathbf{T}^2$.

Following [11], we refer to isotropy subgroups with 1-dimensional fixed point spaces as *axial* and the associated spatially doubly-periodic solutions as *axial planforms*.

This section gives all axial subgroups for the 8-dimensional representations of Γ_s, and for the 12-dimensional representations of Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$. The axial subgroups of Γ_s and Γ_h are obtained from the classification of axial planforms given by Dionne and Golubitsky [8], while the results for $\Gamma_h \oplus \mathbf{Z}_2$ are new. We follow the convention of identifying all solution branches that are on the group orbit $\Gamma \mathbf{z}_\lambda$ of a particular branch $\mathbf{z}_\lambda$. Thus we classify isotropy subgroups of Γ by conjugacy class since the isotropy of a point $\mathbf{z}_\lambda \in \mathbf{C}^s$ is conjugate to the isotropy of a point on its group orbit; specifically, $\Sigma_{\gamma \mathbf{z}_\lambda} = \gamma \Sigma_{\mathbf{z}_\lambda} \gamma^{-1}$.

Dionne and Golubitsky [8] considered rhombic, square and hexagonal lattices, and determined the symmetry of the axial planforms on the finest lattice that supports the solution. This was accomplished by insisting that the isotropy subgroups be translation-free. Here we must extend their results to the case where the lattice is fixed and hence the representation of Γ given. In this case, the same solutions are obtained, but the isotropy subgroups are not necessarily translation-free. The pure translation symmetries of a solution branch play a role in determining the eigenvalue structure of the Jacobian matrix evaluated on that solution branch (see, for example, Section 4.2).

Square lattice case.

We list in Table 3 the six axial subgroups of Γ_s acting on $\mathbf{C}^4$ together with their generators, and their one-dimensional fixed point subspaces. Note that the pure translation subgroups, denoted $\mathbf{S}^1$, $\mathbf{S}_{1,2}$, $\mathbf{S}_{1,3}$ and $\mathbf{S}_{1,4}$, depend on the values α and β and hence are not the same for all 8-dimensional representations. Associated with these fixed point subspaces are planforms that are periodic with respect to a finer lattice than $\mathcal{L}$. For instance, the rolls are periodic on a finer one-dimensional lattice, while each type of rhombs solution is periodic on a finer rhombic lattice, and simple squares are periodic on a finer square lattice.

Hexagonal lattice cases.

Table 4 lists the axial subgroups, up to conjugacy, of Γ_h acting on $\mathbf{C}^6$ together with their generators, and their fixed point subspaces. The axial subgroups associated with $\Gamma_h \oplus \mathbf{Z}_2$ are listed

Table 4: Axial subgroups Σ (up to conjugacy) of Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$. The generators of Γ_h are given in (2.18)-(2.20), and the action of $\kappa \in \mathbf{Z}_2$ is given by (2.25).

Nomenclature	Σ	Generators of Σ $^{(a)}$	Fix(Σ)
Super Hexagons ($\mathrm{SuH}^{\pm}_{\alpha,\beta}$)	$\mathbf{D}_6$	$R_{\pi/3}$, τ_{x_1}	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 \in \mathbf{R}^{\pm}$
Rolls (R)	$\mathbf{Z}_2^c \dot{+} \mathbf{S}^1$	R_π, $\mathbf{S}^1$	$z_1 \in \mathbf{R}$, $z_2 = z_3 = z_4 = z_5 = z_6 = 0$
Rhombs ($\mathrm{Rh}_{h1,\alpha,\beta}$)	$\mathbf{D}_2^n \dot{+} \mathbf{S}_{1,4}$	R_π, $\tau_n R_{\pi/3} \tau_{x_1}$, $\mathbf{S}_{1,4}$	$z_1 = z_4 \in \mathbf{R}$, $z_2 = z_3 = z_5 = z_6 = 0$
Rhombs ($\mathrm{Rh}_{h2,\alpha,\beta}$)	$\mathbf{D}_2^m \dot{+} \mathbf{S}_{1,5}$	R_π, $R^5_{\pi/3} \tau_{x_1}$, $\mathbf{S}_{1,5}$	$z_1 = z_5 \in \mathbf{R}$, $z_2 = z_3 = z_4 = z_6 = 0$
Rhombs ($\mathrm{Rh}_{h3,\alpha,\beta}$)	$\mathbf{D}_2^x \dot{+} \mathbf{S}_{1,6}$	R_π, τ_{x_1}, $\mathbf{S}_{1,6}$	$z_1 = z_6 \in \mathbf{R}$, $z_2 = z_3 = z_4 = z_5 = 0$
Simple Hexagons ($\mathrm{SiH}^{\pm}$)	$\mathbf{Z}_6 \dot{+} \mathbf{S}_{1,2,3}$	$R_{\pi/3}$, $\mathbf{S}_{1,2,3}$	$z_1 = z_2 = z_3 \in \mathbf{R}^{\pm}$, $z_4 = z_5 = z_6 = 0$

$^{(a)}$ The generators of $\mathbf{S}^1$, $\mathbf{S}_{1,4}$, $\mathbf{S}_{1,5}$, $\mathbf{S}_{1,6}$, and $\mathbf{S}_{1,2,3}$ are given in Table 6.

Table 5: Additional axial subgroups Σ (up to conjugacy) of $\Gamma_h \oplus \mathbf{Z}_2$. Also see Table 4.

Nomenclature	Σ	Generators of Σ $^{(a)}$	Fix(Σ)
Anti-triangles ($\mathrm{AT}_{\alpha,\beta}$)	$\mathbf{D}_6$	$((R_{\pi/3},0),\kappa),((\tau_{x_1},0),\kappa)$	$z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6 \in \mathbf{R}i$
Super Triangles ($\mathrm{SuT}_{\alpha,\beta}$)	$\mathbf{D}_6$	$((R_{\pi/3},0),\kappa),((\tau_{x_1},0),Id)$	$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 \in \mathbf{R}i$
Anti-hexagons ($\mathrm{AH}_{\alpha,\beta}$)	$\mathbf{D}_6$	$((R_{\pi/3},0),Id),((\tau_{x_1},0),\kappa)$	$z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6 \in \mathbf{R}$
Simple Triangles (SiT)	$\mathbf{Z}_6 \dot{+} \mathbf{S}_{1,2,3}$	$((R_{\pi/3},0),\kappa)$, $\mathbf{S}_{1,2,3}$	$z_1 = z_2 = z_3 \in \mathbf{R}i$, $z_4 = z_5 = z_6 = 0$
Rhombs (Rh_{h0})	$\mathbf{Z}_2^c \dot{+} \mathbf{S}_{1,2}$	$((R_\pi,0),Id)$, $\mathbf{S}_{1,2}$	$z_1, z_2 \in \mathbf{R}$ $^{(b)}$, $z_3 = z_4 = z_5 = z_6 = 0$

$^{(a)}$ The generators of $\mathbf{S}_{1,2,3}$ and $\mathbf{S}_{1,2}$ are given in Table 6.

$^{(b)}$ A hidden reflection further fixes $z_1 = z_2$.

in Tables 4 and 5. In the case of Γ_h, there are two branches each of simple and super hexagons, which differ by the sign of z_1, *e.g.* SuH^+ has $z_1 > 0$ while SuH^- has $z_1 < 0$. The equivariant branching lemma applies to the six axial subgroups in the case that $\Gamma = \Gamma_h$ and ten isotropy subgroups in the case that there is an extra $\mathbf{Z}_2$ symmetry. An eleventh solution branch, rhombs Rh_{h0}, is known to exist from an analysis of the six-dimensional representation of $\Gamma_h \oplus \mathbf{Z}_2$ presented in [13], where it is called the "pathwork quilt". This state is discussed further in the next subsection on hidden symmetries; a hidden reflection fixes $z_1 = z_2$.

Table 6 indicates how some of the isotropy subgroups in Table 4 are modified by the extra $\mathbf{Z}_2$ symmetry. In this table we denote elements of $\Gamma_h = \mathbf{D}_6 \dot{+} \mathbf{T}^2$ by (h, Θ) and elements of $\Gamma_h \oplus \mathbf{Z}_2$ by $((h,\Theta), Id)$ and $((h,\Theta),\kappa)$. Here $h \in \mathbf{D}_6$, $\Theta \in \mathbf{T}^2$ and $\mathbf{Z}_2 = \{Id, \kappa\}$, where Id specifies the identity element of a group.

3.2 Hidden symmetries.

The computations, in sections 4 and 5, of the equivariant bifurcation equations take certain *hidden symmetries* into account. The hidden symmetries are elements of O(2), which are not in the holohedry of the lattice, but which nonetheless leave invariant certain fixed point subspaces of the bifurcation problem restricted to the lattice. The hidden symmetries act only on these fixed point subspaces and may place some additional restrictions on the form of the bifurcation problem (2.15). (See Crawford [6] for a detailed treatment of hidden Euclidean symmetries in Γ_s mode interaction problems.)

Dionne and Golubitsky [8] classified, by symmetry, the spatially doubly-periodic axial planforms corresponding to translation-free isotropy subgroups. They showed that any planform corresponding to an isotropy subgroup containing nontrivial translations is in fact supported by a finer planar lattice. In the context of this finer lattice, the isotropy subgroup contains all the

Table 6: Generators of subgroups $\mathbf{S}^1$ and $\mathbf{S}_*$ of Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$.

$\mathbf{S}^1, \mathbf{S}_*$	Generators of $\mathbf{S}^1, \mathbf{S}_* \subset \Gamma_h$	Generators of $\mathbf{S}^1, \mathbf{S}_* \subset \Gamma_h \oplus \mathbf{Z}_2$
$\mathbf{S}^1$	$(Id,\ \beta s\ \ell_1 - \alpha s\ \ell_2)$, where $s \in \mathbf{R}$	$((Id,\ \beta s\ \ell_1 + (\frac{1}{2\beta} - \alpha s)\ \ell_2),\ \kappa)$, where $s \in \mathbf{R}$
$\mathbf{S}_{1,4}$	$(Id,\ \frac{\alpha-\beta}{\alpha^2-2\alpha\beta}\ \ell_1 - \frac{1}{\alpha-2\beta}\ \ell_2)$, $(Id,\ \frac{-\beta}{\alpha^2-2\alpha\beta}\ \ell_1 + \frac{1}{\alpha-2\beta}\ \ell_2)$	$((Id,\ \frac{\alpha-2\beta}{2(\alpha^2-2\alpha\beta)}\ \ell_1),\ \kappa)$, $((Id,\ \frac{1}{2(\alpha-2\beta)}\ (\ell_1 - 2\ell_2)),\ \kappa)$
$\mathbf{S}_{1,5}$	$(Id,\ \frac{\alpha}{\alpha^2-\beta^2}\ \ell_1 - \frac{\beta}{\alpha^2-\beta^2}\ \ell_2)$, $(Id, \frac{\beta}{\alpha^2-\beta^2}\ \ell_1 - \frac{\alpha}{\alpha^2-\beta^2}\ \ell_2)$	$((Id,\ \frac{1}{2(\alpha-\beta)}\ (\ell_1 - \ell_2)),\ \kappa)$ $((Id,\ \frac{1}{2(\alpha+\beta)}\ (\ell_1 + \ell_2)),\ \kappa)$
$\mathbf{S}_{1,6}$	$(Id,\ \frac{1}{2\alpha-\beta}\ \ell_1 + \frac{\alpha-\beta}{2\alpha\beta-\beta^2}\ \ell_2)$, $(Id,\ \frac{-1}{2\alpha-\beta}\ \ell_1 + \frac{\alpha}{2\alpha\beta-\beta^2}\ \ell_2)$	$((Id,\ \frac{1}{2\beta}\ \ell_2),\ \kappa)$, $((Id,\ \frac{1}{2(2\alpha-\beta)}\ (2\ell_1 - \ell_2)),\ \kappa)$.
$\mathbf{S}_{1,2,3}$	$(Id,\ \frac{\alpha}{\alpha^2-\alpha\beta+\beta^2}\ \ell_1 - \frac{\alpha-\beta}{\alpha^2-\alpha\beta+\beta^2}\ \ell_2)$, $(Id,\ \frac{\beta}{\alpha^2-\alpha\beta+\beta^2}\ \ell_1 - \frac{\alpha}{\alpha^2-\alpha\beta+\beta^2}\ \ell_2)$	$((Id,\ \frac{\alpha}{\alpha^2-\alpha\beta+\beta^2}\ \ell_1 - \frac{\alpha-\beta}{\alpha^2-\alpha\beta+\beta^2}\ \ell_2),\ Id)$, $((Id,\ \frac{\beta}{\alpha^2-\alpha\beta+\beta^2}\ \ell_1 - \frac{\alpha}{\alpha^2-\alpha\beta+\beta^2}\ \ell_2),\ Id)$
$\mathbf{S}_{1,2}$	not applicable	$((Id,\ \frac{\alpha+\beta}{2(\alpha^2-\alpha\beta+\beta^2)}\ \ell_1 - \frac{2\alpha-\beta}{2(\alpha^2-\alpha\beta+\beta^2)}\ \ell_2),\ \kappa)$, $((Id,\ \frac{\alpha-\beta}{2(\alpha^2-\alpha\beta+\beta^2)}\ \ell_1 + \frac{\beta}{2(\alpha^2-\alpha\beta+\beta^2)}\ \ell_2),\ \kappa)$

possible symmetries of the planform – there are no hidden symmetries. In the case of the hexagonal (square) lattice, the finer lattices are either one-dimensional, rhombic, or hexagonal (square). The finer rhombic, square, and hexagonal lattices are invariant under D_2, D_4, and D_6 subgroups of O(2), respectively. Solutions supported by these finer lattices lie in four– or six–dimensional fixed-point subspaces of the appropriate Γ_s- or Γ_h-equivariant bifurcation problems. For example, the four-dimensional subspace $\{\mathbf{z} = (z_1, z_2, 0, 0) : z_1, z_2 \in \mathbf{C}\}$ of the square lattice problem contains all solutions that are periodic on the finer square lattice. This subspace is invariant under $Z_4 \subset \Gamma_s$. A hidden reflection in E(2) enlarges Z_4 to D_4. Similarly, a hidden reflection enlarges $Z_6 \subset \Gamma_h$ to $D_6 \subset$O(2) for the six-dimensional subspace of solutions that are periodic on a finer hexagonal lattice.

We determine the hidden symmetries for the non-translation-free axial planforms in Tables 3, 4 and 5 by considering their symmetries on the finer lattice (see [8, 13]). For example, according to Table 4, simple hexagons have Z_6 symmetry, while on the finest lattice that supports this planform they have D_6 symmetry; it is the hidden reflection mentioned above that enlarges Z_6 to D_6. We summarize the hidden symmetries of the axial planforms as follows:

(i) Simple hexagons ($\mathrm{SiH}^{\pm}$) have $\mathbf{D}_6$ symmetry where $\mathbf{D}_6$ is generated by $R_{\pi/3} \in \Gamma_h$ and a (hidden) reflection through the line containing the vector $\beta\ell_1 - \alpha\ell_2$, denoted by $\widetilde{\tau}_{x_1} \in E(2)$. This reflection acts as follows

$$\widetilde{\tau}_{x_1}(z_1, z_2, z_3, 0, 0, 0) = (\overline{z}_1, \overline{z}_3, \overline{z}_2, 0, 0, 0). \tag{3.3}$$

(ii) Simple triangles (SiT) have $\mathbf{D}_6$ symmetry where $\mathbf{D}_6$ is generated by $((R_{\pi/3}, 0), \kappa) \in \Gamma_h \oplus \mathbf{Z}_2$ and the (hidden) reflection $((\widetilde{\tau}_{x_1}, 0), \kappa)$.

(iii) Rhombs (Rh_{h0}) have $\mathbf{D}_2$ symmetry where $\mathbf{D}_2$ is generated by $((R_\pi, 0), Id) \in \Gamma_h \oplus \mathbf{Z}_2$ and the (hidden) reflection $\widetilde{\tau}_{x_1} R_{\pi/3}$.

(iv) Simple squares (SiS) have $\mathbf{D}_4$ symmetry where $\mathbf{D}_4$ is generated by $R_{\pi/2} \in \Gamma_s$ and a (hidden) reflection $\widetilde{\tau}_{x_1}$ that acts as follows:

$$\widetilde{\tau}_{x_1}(z_1, z_2, 0, 0) = (z_1, \overline{z}_2, 0, 0). \tag{3.4}$$

Table 7: Characterization of the rhombs.

Lattice	Rhombs	aspect ratio	angle
Square	$\mathrm{Rh}_{s1,\alpha,\beta}$	$\frac{\alpha-\beta}{\alpha+\beta}$	$\cos^{-1}\left(\frac{2\alpha\beta}{\alpha^2+\beta^2}\right)$
Square	$\mathrm{Rh}_{s2,\alpha,\beta}$	$\frac{\beta}{\alpha}$	$\cos^{-1}\left(\frac{\beta^2-\alpha^2}{\alpha^2+\beta^2}\right)$
Hexagonal	$\mathrm{Rh}_{h1,\alpha,\beta}$	$\frac{2\beta-\alpha}{\sqrt{3}\ \alpha}$	$\cos^{-1}\left(\frac{\alpha^2+2\alpha\beta-2\beta^2}{2(\alpha^2-\alpha\beta+\beta^2)}\right)$
Hexagonal	$\mathrm{Rh}_{h2,\alpha,\beta}$	$\frac{\sqrt{3}\ (\alpha-\beta)}{\alpha+\beta}$	$\cos^{-1}\left(\frac{\alpha^2-4\alpha\beta+\beta^2}{2(\alpha^2-\alpha\beta+\beta^2)}\right)$
Hexagonal	$\mathrm{Rh}_{h3,\alpha,\beta}$	$\frac{2\alpha-\beta}{\sqrt{3}\ \beta}$	$\cos^{-1}\left(-\frac{2\alpha^2-2\alpha\beta-\beta^2}{2(\alpha^2-\alpha\beta+\beta^2)}\right)$
Hexagonal	Rh_{h0}	$\frac{1}{\sqrt{3}}$	$\frac{2\pi}{3}$

(v) Each roll (R) (square or hexagonal case) has $\mathbf{D}_2\dot{+}\mathbf{S}^1$ symmetry where $\mathbf{D}_2$ is generated by R_π and $\widetilde{\tau}_{x_1}$.

We find that the hidden reflection (3.3) places additional restrictions on the form of the Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$ bifurcation problems. The hidden reflection (3.4) does not change the general form of the Γ_s-equivariant bifurcation problem.

3.3 Axial planforms.

In this section we present examples of the planforms associated with the axial subgroups. Specifically, for each conjugacy class of subgroups $\Sigma \subset \Gamma$ that fixes a one-dimensional subspace, we present a grey scale plot of the function $v(\mathbf{x})$ in (2.13) for a representative point $\mathbf{z} \in \mathrm{Fix}(\Sigma)$.

Figure 3 presents examples of square lattice planforms associated with the axial subgroups listed in Table 3 in the case that $(\alpha,\beta) = (2,1)$. The rhombic, super square and anti-square states depend on (α,β). The rolls and simple squares are, up to scaling of the spatial variable $\mathbf{x}$, the same for each 8-dimensional representation of Γ_s.

Examples of planforms associated with the axial subgroups of Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$ listed in Tables 4 and 5 are depicted in Figures 4 and 5. In the case of $\Gamma_h \oplus \mathbf{Z}_2$, the SiH^- (SuH^-) branch of simple (super) hexagons is on the group orbit of the SiH^+ (SuH^+) branch since $\kappa(\mathbf{z}) = -\mathbf{z}$. Note that the only states that are the same (after rescaling $\mathbf{x}$) for every value of α and β are the rolls, the rhombs Rh_{h0}, the simple hexagons, and the simple triangles. Rolls are the only state that is common to both the square and the hexagonal lattices.

Only the super- and the anti-states of Tables 3, 4 and 5 are characterized by translation free isotropy subgroups. Hence these are the only axial planforms with (smallest) periodicity determined by ℓ_1 and ℓ_2. All of the other axial planforms are periodic on a finer square, hexagonal, or rhombic lattice. In particular, the wavelength of their periodicity is $1/k_c$, where $\mathbf{x}$ has been scaled so that $k_c = \sqrt{\alpha^2+\beta^2}$ for square lattice states and $k_c = \sqrt{\alpha^2+\beta^2-\alpha\beta}$ for hexagonal lattice states. Note that while the periodicity of super- and anti-states is given by $|\ell_1| = |\ell_2| \gg 1/k_c$, the lengthscale $1/k_c$ is also evident in the patterns. This lengthscale shows up as small scale structure in the patterns; compare, for example, simple hexagons with super hexagons in Figure 4.

Finally, we note that there is a countable set of rhombs that are periodic on square or hexagonal lattices. In Table 7 we characterize the rhombs on the square and hexagonal lattices in two different ways. We give the angle between the wave vectors associated with the critical modes, *e.g.* the angle between $\mathbf{K}_1$ and $\mathbf{K}_3$ for $\mathrm{Rh}_{s1,\alpha,\beta}$. We also give the aspect ratio of the rectangles evident in the rhomb patterns in Figures 3-5.

4 Stability Results: Square Lattice.

In this section we compute the linear stability, at bifurcation, of the six axial planforms listed in Table 3. We do this within the center manifold framework of a general Γ_s-equivariant bifurcation problem $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \lambda)$, $\mathbf{g} : \mathbf{C}^4 \times \mathbf{R} \to \mathbf{C}^4$. An equilibrium solution branch $\mathbf{z}_\lambda$ of the bifurcation problem is linearly, orbitally stable if all eigenvalues of the Jacobian matrix $\mathbf{Dg}(\mathbf{z}_\lambda, \lambda)$, not forced by symmetry to be zero, have negative real part for λ sufficiently close to zero. If any eigenvalue has positive real part then the planform is unstable. Implicit in this is that the stability is only determined with respect to perturbations which can be supported by the lattice.

4.1 $\mathbf{D}_4 \dot{+} \mathbf{T}^2$–equivariant bifurcation problem.

We begin by considering the Γ_s-equivariant bifurcation problem $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \lambda)$, $\mathbf{g} : \mathbf{C}^4 \times \mathbf{R} \to \mathbf{C}^4$. We assume that local to the bifurcation point, $\mathbf{z} = \mathbf{0}, \lambda = 0$, $\mathbf{g}$ can be expanded in a Taylor series about $\mathbf{z} = \mathbf{0}$. This series is determined to sufficient order that we are subsequently able to evaluate the sign of the real part of each eigenvalue.

The equivariance condition (2.16) for $\Gamma = \Gamma_s$ is satisfied if (see, for example, Appendix A.3 in [6])

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, z_2, z_3, z_4) \\ \dot{z}_2 &= g_1(z_2, \overline{z}_1, z_4, \overline{z}_3) \\ \dot{z}_3 &= g_1(z_3, \overline{z}_4, z_1, \overline{z}_2) \\ \dot{z}_4 &= g_1(z_4, z_3, z_2, z_1) \ , \end{aligned} \tag{4.1}$$

where

$$\overline{g_1(\mathbf{z})} = g_1(\bar{\mathbf{z}}) \ , \tag{4.2}$$

and

$$\Theta\Big(\bar{z}_1 g_1(\mathbf{z})\Big) = \bar{z}_1 g_1(\mathbf{z}) \ , \text{ for all } \Theta \in \mathbf{T}^2 \ . \tag{4.3}$$

Equivariance with respect to $\mathbf{D}_4 \subset \Gamma_s$ is guaranteed by conditions (4.1) and (4.2), while equivariance with respect to $\mathbf{T}^2 \subset \Gamma_s$ is equivalent to condition (4.3), *i.e.*, to $\overline{z}_1 g_1(\mathbf{z})$ being an invariant function of the $\mathbf{T}^2$-action. We proceed by finding the most general $\mathbf{T}^2$-invariant polynomial $h = \bar{z}_1 g_1(\mathbf{z})$; the details of this calculation are relegated to the appendix. The $\mathbf{T}^2$-invariant function h is then used to find the leading order terms in the equivariant bifurcation problem. We find that

$$\dot{z}_1 = z_1 f(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2) + b_1 \bar{z}_1^{\beta-1} \bar{z}_2^{\alpha} z_3^{\alpha} z_4^{\beta} + b_2 \bar{z}_1^{\alpha-1} z_2^{\beta} z_3^{\beta} \bar{z}_4^{\alpha} + \mathcal{O}(2(\alpha+\beta)), \tag{4.4}$$

where it follows from (4.2) that f is a real-valued function of its arguments and that $b_1, b_2 \in \mathbf{R}$. Condition (4.1) determines the remaining components of $\mathbf{g}$ from g_1. Note that the hidden symmetry (3.4) is automatically satisfied since $g_1(z_1, z_2, 0, 0) = g_1(z_1, \overline{z}_2, 0, 0)$.

The cubic truncation of (4.4) is given by

$$\dot{z}_1 \quad = \quad \lambda z_1 + z_1(a_1|z_1|^2 + a_2|z_2|^2 + a_3|z_3|^2 + a_4|z_4|^2) + \mathcal{O}(\| \mathbf{z} \|^5) \ , \tag{4.5}$$

where the coefficients are real, and we assume, without loss of generality, that time is scaled so that the linear term is $\lambda \mathbf{z}$.

4.2 Stability calculations.

To compute the eigenvalues that determine the orbital stability of the axial planforms we exploit the fact that their symmetry imposes severe restrictions on the 8×8 (real) Jacobian matrix D$\mathbf{g}$ evaluated on the solution branch. Described briefly below are the two approaches that we use for this computation, which are both standard (see [12]).

The first approach uses the observation that the Jacobian matrix evaluated on a solution branch $\mathbf{z}_\lambda$ commutes with each element $\sigma \in \Sigma_{\mathbf{z}_\lambda}$. For example, the Jacobian matrix D$\mathbf{g}$ evaluated on the rolls solution branch commutes with the linear transformations that generate $\Sigma = \mathbf{Z}_2^c \dot{+} \mathbf{S}^1$ in Table 3, namely

$$\mathbf{z} \mapsto \overline{\mathbf{z}} \tag{4.6}$$

and

$$\mathbf{z} \mapsto (z_1, e^{2\pi i(\alpha^2+\beta^2)s} z_2, e^{2\pi i(\alpha^2-\beta^2)s} z_3, e^{4\pi i\alpha\beta s} z_4), \quad s \in \mathbf{R}. \tag{4.7}$$

We choose an ordering of the coordinates on $\mathbf{R}^8$ to be $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$, where x_j and y_j are the real and imaginary parts of z_j, respectively. We let $g_j = g_j^r + i g_j^i$, where the r and i superscripts specify real and imaginary parts, so that $\dot{x}_j = g_j^r$ and $\dot{y}_j = g_j^i$, $j = 1, ..., 4$. It follows from the observation that D$\mathbf{g}$, evaluated on the rolls solution branch, must commute with the above transformations that D$\mathbf{g}$ is diagonal and three of the eigenvalues have multiplicity two. Moreover, the group orbit of rolls is one-dimensional so there is a zero eigenvalue associated with translation along the group orbit. The null direction is determined by computing the tangent vector to the group orbit, *i.e.*,

$$\frac{\partial}{\partial \theta_1}\Big|_{\Theta=\mathbf{0}} \Theta(x_1, 0, 0, 0) = (-2\pi i \alpha x_1, 0, 0, 0), \; x_1 \in \mathbf{R}\,, \tag{4.8}$$

where the action of Θ on $\mathbf{C}^4$ is given by (2.20). It follows that $(0, 0, 0, 0, 1, 0, 0, 0)$ is a null eigenvector of D$\mathbf{g}$ so that the eigenvalue $\frac{\partial g_1^i}{\partial y_1}$ is zero.

The second approach to computing the eigenvalues relies on forming the *isotypic decomposition* of $\mathbf{C}^4$ for the isotropy subgroup $\Sigma_{\mathbf{z}_\lambda}$ of a solution $\mathbf{z}_\lambda$ [12]. This decomposition determines coordinates that block-diagonalize D$\mathbf{g}$. The isotypic decomposition proceeds by first decomposing $\mathbf{C}^4$ into Σ-irreducible subspaces V_j so that $\mathbf{C}^4 = V_0 \oplus V_1 \oplus \cdots \oplus V_\ell$. (A representation is Σ-irreducible if the only Σ-invariant subspace of V_j, other than $\{\mathbf{0}\}$, is V_j itself.) The isotypic components W_j are then formed by combining the irreducible subspaces that are Σ-*isomorphic*. (Two Σ-irreducible subspaces are Σ-isomorphic if there exists a linear isomorphic mapping between them which commutes with the action of Σ.) The isotypic decomposition is $\mathbf{C}^4 = W_0 \oplus W_1 \oplus \cdots \oplus W_k$, $k \leq \ell$, where the W_j are uniquely determined.

For example the isotypic decomposition of $\mathbf{C}^4$ for $\Sigma = \mathbf{D}_4[R_{\pi/2}, \tau_{x_1}]$, which applies to the super squares state, is

$$\begin{aligned} \mathbf{C}^4 \;=\; & \mathbf{R}\{(1,1,1,1)\} \oplus \mathbf{R}\{(1,1,-1,-1)\} \oplus \mathbf{R}\{(1,-1,1,-1)\} \oplus \mathbf{R}\{(1,-1,-1,1)\} \\ & \oplus \mathbf{R}\{(i,i,i,i), (-i,i,-i,i), (i,0,0,i), (0,i,-i,0)\}\,. \end{aligned} \tag{4.9}$$

The one-dimensional isotypic components immediately determine four of the eigenvalues of D$\mathbf{g}$ evaluated on the super squares solution branch; these are

$$\begin{aligned} & \frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} + \frac{\partial g_1^r}{\partial x_4}\,, \quad \frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3} - \frac{\partial g_1^r}{\partial x_4}\,, \\ & \frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} - \frac{\partial g_1^r}{\partial x_4}\,, \quad \frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3} + \frac{\partial g_1^r}{\partial x_4}\,. \end{aligned} \tag{4.10}$$

Additional restrictions are placed on the matrix obtained by restricting $\mathrm{D}\mathbf{g}$ to the four-dimensional isotypic component W_4. Specifically, $\mathrm{D}\mathbf{g}|_{W_4}$ commutes with the action of $\mathbf{D}_4$ on $W_4 \cong \mathbf{R}^4$, where W_4 is the direct sum of two isomorphic $\mathbf{D}_4$-absolutely irreducible subspaces. (Recall that a representation of a group Γ acts *absolutely irreducibly* on a space V if the only linear maps on V commuting with Γ are multiples of the identity.) It follows that $\mathbf{A} = \mathrm{D}\mathbf{g}|_{W_4}$ has the form

$$\begin{pmatrix} a\mathbf{I}_2 & b\mathbf{I}_2 \\ c\mathbf{I}_2 & d\mathbf{I}_2 \end{pmatrix}, \tag{4.11}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix and $a, b, c, d \in \mathbf{R}$. Each eigenvalue in this matrix has multiplicity two. Moreover, two of the eigenvalues must be zero because the group orbit of super squares is two-dimensional. Thus the eigenvalues are determined by simply computing the trace of $\mathbf{A}$, which, in terms of real coordinates, is

$$\mathrm{Tr}(\mathbf{A}) = \frac{\partial g_1^i}{\partial y_1} + \frac{\partial g_2^i}{\partial y_2} + \frac{\partial g_3^i}{\partial y_3} + \frac{\partial g_4^i}{\partial y_4} \ . \tag{4.12}$$

This can be further simplified by noting that, since $\mathrm{D}\mathbf{g}$ commutes with the transformations $R_{\pi/2}$ and τ_{x_1},

$$\frac{\partial g_1^i}{\partial y_1} = \frac{\partial g_2^i}{\partial y_2} = \frac{\partial g_3^i}{\partial y_3} = \frac{\partial g_4^i}{\partial y_4} \tag{4.13}$$

on the super squares solution branch. Thus the repeated eigenvalue, $\frac{1}{2}\mathrm{Tr}(\mathbf{A})$, is simply $2\frac{\partial g_1^i}{\partial y_1}$.

The details of the computations of the eigenvalues for the remaining axial planforms are omitted but the results are summarised in the second column of Table 8. Note that symmetry considerations alone determine that the eigenvalues of $\mathrm{D}\mathbf{g}$ are real for all of the axial planforms.

From the leading order terms in the equivariant bifurcation problem (4.4), and the expressions for the eigenvalues given in the second column of Table 8, the signs of the eigenvalues of $\mathrm{D}\mathbf{g}$ at bifurcation may be determined. Provided the nondegeneracy condition,

$$b_1\beta + b_2\alpha \neq 0, \tag{4.14}$$

is satisfied, the sign of the second (repeated) eigenvalue for the super squares and anti-squares solution branches is evaluated by keeping all terms through $\mathcal{O}(2(\alpha+\beta)-1)$ in (4.4) and using the observation that

$$\frac{\partial g_1^i}{\partial y_j}\Big|_{z=\bar{z}} = \Big(\frac{\partial g_1}{\partial z_j} - \frac{\partial g_1}{\partial \bar{z}_j}\Big)\Big|_{z=\bar{z}} \ . \tag{4.15}$$

The signs of all remaining non-zero eigenvalues are determined by the cubic truncation (4.5) provided the nondegeneracy conditions,

$$a_1 \neq 0, \pm a_2, \pm a_3, \pm a_4 \ , \quad (a_1 + a_2) \neq \pm(a_3 + a_4) \ , \quad (a_1 - a_2) \neq \pm(a_3 - a_4) \ , \tag{4.16}$$

are satisfied. The results for all the axial planforms are summarized in the third column of Table 8. If the nondegeneracy conditions are satisfied we can draw a number of conclusions from this table.

1. Any one of the axial solution branches can bifurcate supercritically to produce a stable solution.

2. If the super squares and anti-squares are neutrally stable at cubic order, then one and only one of these two states bifurcates stably.

Table 8: Eigenvalues and their multiplicities for axial planforms associated with 8-dim. representations of Γ_s. The coefficients $a_1, a_2, a_3, a_4, b_1, b_2$ are defined by equations 4.4 and 4.5.

Axial Planform and Branching Equation	Eigenvalues	Signs of Non-zero Eigenvalues
Rolls $0 = \lambda x + a_1 x^3 + \cdots$	0, $\frac{\partial g_1^r}{\partial x_1}$, $\frac{\partial g_2^r}{\partial x_2}$ (twice), $\frac{\partial g_3^r}{\partial x_3}$ (twice), $\frac{\partial g_4^r}{\partial x_4}$ (twice)	$\mathrm{sgn}(a_1)$, $\mathrm{sgn}(a_2 - a_1)$, $\mathrm{sgn}(a_3 - a_1)$, $\mathrm{sgn}(a_4 - a_1)$
Simple Squares $0 = \lambda x + (a_1 + a_2) x^3 + \cdots$	0 (twice), $\frac{\partial g_3^r}{\partial x_3}$ (four times), $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2}$	$\mathrm{sgn}(a_3 + a_4 - a_1 - a_2)$, $\mathrm{sgn}(a_1 + a_2)$, $\mathrm{sgn}(a_1 - a_2)$
Rhombs ($\mathrm{Rh}_{s1,\alpha,\beta}$) $0 = \lambda x + (a_1 + a_3) x^3 + \cdots$	0 (twice), $\frac{\partial g_2^r}{\partial x_2}$ (four times), $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_3}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_3}$	$\mathrm{sgn}(a_2 + a_4 - a_1 - a_3)$, $\mathrm{sgn}(a_1 + a_3)$, $\mathrm{sgn}(a_1 - a_3)$
Rhombs ($\mathrm{Rh}_{s2,\alpha,\beta}$) $0 = \lambda x + (a_1 + a_4) x^3 + \cdots$	0 (twice), $\frac{\partial g_2^r}{\partial x_2}$ (four times), $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_4}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_4}$	$\mathrm{sgn}(a_2 + a_3 - a_1 - a_4)$, $\mathrm{sgn}(a_1 + a_4)$, $\mathrm{sgn}(a_1 - a_4)$
Super Squares $0 = \lambda x +$ $(a_1 + a_2 + a_3 + a_4) x^3 + \cdots$ $+(b_1 + b_2) x^{2(\alpha+\beta)-1} + \cdots$	0 (twice), $2\frac{\partial g_1^i}{\partial y_1}$ (twice), $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} + \frac{\partial g_1^r}{\partial x_4}$, $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3} - \frac{\partial g_1^r}{\partial x_4}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} - \frac{\partial g_1^r}{\partial x_4}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3} + \frac{\partial g_1^r}{\partial x_4}$	$\mathrm{sgn}(-b_1\beta - b_2\alpha)$, $\mathrm{sgn}(a_1 + a_2 + a_3 + a_4)$, $\mathrm{sgn}(a_1 + a_2 - a_3 - a_4)$, $\mathrm{sgn}(a_1 - a_2 + a_3 - a_4)$, $\mathrm{sgn}(a_1 - a_2 - a_3 + a_4)$
Anti-Squares $0 = \lambda x +$ $(a_1 + a_2 + a_3 + a_4) x^3 + \cdots$ $-(b_1 + b_2) x^{2(\alpha+\beta)-1} + \cdots$	Same as Super Squares	$\mathrm{sgn}(b_1\beta + b_2\alpha)$, $\mathrm{sgn}(a_1 + a_2 + a_3 + a_4)$, $\mathrm{sgn}(a_1 + a_2 - a_3 - a_4)$, $\mathrm{sgn}(a_1 - a_2 + a_3 - a_4)$, $\mathrm{sgn}(a_1 - a_2 - a_3 + a_4)$

3. If all the axial planforms bifurcate supercritically, then at least one of them is stable.

4. If *any* axial solution branch bifurcates subcritically, then rolls, super squares and anti-squares are all unstable.

5. If rolls, super squares, or anti-squares bifurcate subcritically, then *all* axial planforms are unstable at bifurcation.

6. If simple squares is the only axial solution branch to bifurcate subcritically, then one, but not both, of the rhombs solutions may be stable. Similarly, if one of the rhombs solutions bifurcates subcritically, then simple squares or the other rhombs solution branch may be stable, though they cannot both be stable in this case.

7. The only solution branches that can co-exist stably are simple squares SiS and the rhombs $\mathrm{Rh}_{s1,\alpha,\beta}$, $\mathrm{Rh}_{s2,\alpha,\beta}$. Any combination of two of these states can bifurcate stably, but not all three.

5 Stability Results: Hexagonal Lattice.

In this section we compute the linear stability, at bifurcation, of the axial planforms that are associated with the twelve-dimensional representations of Γ_h and $\Gamma_h \oplus \mathbf{Z}_2$ (see Tables 4 and 5). As with the square lattice case, we do this within the framework of a general Γ-equivariant bifurcation problem $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \lambda)$, where $\mathbf{g} : \mathbf{C}^6 \times \mathbf{R} \to \mathbf{C}^6$.

In the case of $\Gamma = \Gamma_h \oplus \mathbf{Z}_2$ there are only odd terms in the Taylor expansion of $\mathbf{g}$ due to the $\mathbf{Z}_2$ symmetry. However, if the $\mathbf{Z}_2$ symmetry is absent, then even terms are admissible. In particular, we find that the coefficients of most, but not all, quadratic terms in the Taylor expansion of $\mathbf{g}_1$ are zero; the exception is $\epsilon \equiv \frac{1}{2}\frac{\partial^2 g_1}{\partial \bar{z}_2 \partial \bar{z}_3}$, *i.e.*, the following vector is Γ_h-equivariant

$$(\bar{z}_2\bar{z}_3, \bar{z}_3\bar{z}_1, \bar{z}_1\bar{z}_2, \bar{z}_5\bar{z}_6, \bar{z}_6\bar{z}_4, \bar{z}_4\bar{z}_5) \tag{5.1}$$

The presence of such a quadratic term ensures that generically all of the axial planforms bifurcate unstably [15]. In order to obtain stable axial solution branches we focus on the degenerate bifurcation problem defined by $\epsilon = 0$. We then discuss briefly the unfolding of this bifurcation problem (*i.e.*, the case $0 < |\epsilon| \ll 1$), before analyzing the generic $\Gamma_h \oplus \mathbf{Z}_2$-equivariant bifurcation problem.

5.1 $\mathbf{D}_6 \dot{+} \mathbf{T}^2$–equivariant bifurcation problem.

In this section we give the Taylor expansion of the equivariant bifurcation problem (2.15) to sufficient order to determine the signs of the real part of eigenvalues for the axial planforms.

Our approach to determining the leading terms in the Γ_h-equivariant bifurcation problem is the same as that employed in Section 4.1 for the Γ_s-equivariant problem. The general Γ_h-equivariant vector field that satisfies the equivariance condition (2.16) is

$$\begin{aligned}
\dot{z}_1 &= g_1(z_1, z_2, z_3, z_4, z_5, z_6)\\
\dot{z}_2 &= g_1(z_2, z_3, z_1, z_5, z_6, z_4)\\
\dot{z}_3 &= g_1(z_3, z_1, z_2, z_6, z_4, z_5)\\
\dot{z}_4 &= g_1(z_4, z_6, z_5, z_1, z_3, z_2)\\
\dot{z}_5 &= g_1(z_5, z_4, z_6, z_2, z_1, z_3)\\
\dot{z}_6 &= g_1(z_6, z_5, z_4, z_3, z_2, z_1)\ ,
\end{aligned} \tag{5.2}$$

where

$$\overline{g_1(\mathbf{z})} = g_1(\bar{\mathbf{z}}) \ , \tag{5.3}$$

and

$$\Theta\Big(\bar{z}_1 g_1(\mathbf{z})\Big) = \bar{z}_1 g_1(\mathbf{z}) \ , \ \text{for all } \Theta \in \mathbf{T}^2 \ . \tag{5.4}$$

The hidden reflection (3.3) puts an additional restriction on the function $g_1(\mathbf{z})$, namely

$$g_1(z_1, z_2, z_3, 0, 0, 0) = g_1(z_1, z_3, z_2, 0, 0, 0) \ . \tag{5.5}$$

The details of the calculation of the $\mathbf{T}^2$-invariant function $\bar{z}_1 g_1$ are presented in the appendix. From this invariant function, we determine that the general form of the equivariant vector field through $\mathcal{O}(2\alpha - 1)$ is

$$\begin{aligned} \dot{z}_1 \ &= \ z_1 \ f_1(u_1, u_2, u_3, u_4, u_5, u_6, q_1, \bar{q}_1, q_4, \bar{q}_4) + \bar{z}_2 \bar{z}_3 \ f_2(u_1, u_2, u_3, u_4, u_5, u_6, q_1, \bar{q}_1, q_4, \bar{q}_4) \\ &+ e_1 \ \bar{z}_1^{\alpha-\beta-1} z_3^{\beta} z_4^{\beta} \bar{z}_6^{\alpha-\beta} \ + \ e_2 \ \bar{z}_1^{\beta-1} z_2^{\alpha-\beta} z_4^{\alpha-\beta} \bar{z}_5^{\beta} \ + \ \mathcal{O}(2\alpha) \ , \end{aligned} \tag{5.6}$$

where

$$u_j \equiv |z_j|^2 \ , \quad q_1 \equiv z_1 z_2 z_3 \ , \quad q_4 \equiv z_4 z_5 z_6 \ , \tag{5.7}$$

and $e_1, e_2 \in \mathbf{R}$ are constants. It follows from (5.3) and (5.5), respectively, that

$$\begin{aligned} \overline{f_j(u_1, u_2, u_3, u_4, u_5, u_6, q_1, \bar{q}_1, q_4, \bar{q}_4)} \ &= \ f_j(u_1, u_2, u_3, u_4, u_5, u_6, \bar{q}_1, q_1, \bar{q}_4, q_4) \ , \\ f_j(u_1, u_2, u_3, 0, 0, 0, q_1, \bar{q}_1, 0, 0) \ &= \ f_j(u_1, u_3, u_2, 0, 0, 0, q_1, \bar{q}_1, 0, 0) \ , \ j = 1, 2 \ . \end{aligned} \tag{5.8}$$

The other components of the bifurcation equations are determined from (5.6) using (5.2).

5.2 Stability in the degenerate case $\epsilon = 0$.

As in the square lattice case, we begin by determining the restrictions that symmetry places on the eigenvalues of the 12×12 real Jacobian matrix $D\mathbf{g}$ when it is evaluated on an axial solution branch. The results are summarized in column two of Table 9.

The calculations that lead to the results in Table 9 are similar to the corresponding calculations in the square lattice case. We omit the details. Note that rolls and simple hexagons lie in the six-dimensional fixed-point subspace $\{\mathbf{z} = (z_1, z_2, z_3, 0, 0, 0) : z_1, z_2, z_3 \in \mathbf{C}\}$ on which the hidden reflection $\widetilde{\tau}_{x_1}$ (3.3) acts; for these solutions the hidden symmetry is taken into account in determining the eigenvalues of $D\mathbf{g}$. In the case of super hexagons, we find the computation of the eigenvalues of $D\mathbf{g}$ is simplified by forming the $\mathbf{D}_6$-isotypic decomposition of $\mathbf{C}^6$. It is

$$\begin{aligned} \mathbf{C}^6 \ &= \ \mathbf{R}(1,1,1,1,1,1) \oplus \mathbf{R}(1,1,1,-1,-1,-1) \oplus \mathbf{R}(i,i,i,i,i,i) \oplus \mathbf{R}(i,i,i,-i,-i,-i) \\ &\oplus \ \mathbf{R}\{(1,-1,0,0,-1,1), (-1,0,1,-1,1,0), (0,1,-1,-1,1,0), (1,-1,0,1,0,-1)\} \\ &\oplus \ \mathbf{R}\{(-i,0,i,i,0,-i), (0,-i,i,0,i,-i), (0,i,-i,-i,i,0), (-i,i,0,-i,0,i)\} \ . \end{aligned} \tag{5.9}$$

From the expressions for the eigenvalues given in the second column of Table 9 and the equivariant bifurcation problem (5.6) the signs of the eigenvalues for the degenerate case $\epsilon = 0$ may be calculated. Provided the condition

$$\frac{e_1}{e_2} \neq -\frac{\alpha + \beta}{2\alpha - \beta} \tag{5.10}$$

is satisfied, the eigenvalue

$$\frac{\partial g_1^i}{\partial y_1} - \frac{1}{2}\Big(\frac{\partial g_1^i}{\partial y_2} + \frac{\partial g_1^i}{\partial y_3}\Big) \ , \tag{5.11}$$

degenerate case $\epsilon = 0$. The coefficients a_i, b_i, c_i, e_i are defined by equations 5.6 and 5.12 ($\epsilon = 0$). The amplitude x for simple and super hexagons is determined by the branching equation; see Table 10.

Planform	Eigenvalues	Signs of Non-zero Eigenvalues
Rolls	$\frac{\partial g_1^r}{\partial x_1}$, $\frac{\partial g_4^r}{\partial x_4}$ (twice), $\frac{\partial g_5^r}{\partial x_5}$ (twice), $\frac{\partial g_6^r}{\partial x_6}$ (twice), $\frac{\partial g_2^r}{\partial x_2} + \frac{\partial g_2^r}{\partial x_3}{}^{(a)}$ (twice), $\frac{\partial g_2^r}{\partial x_2} - \frac{\partial g_2^r}{\partial x_3}{}^{(a)}$ (twice), 0	$\mathrm{sgn}(a_1)$, $\mathrm{sgn}(a_4 - a_1)$, $\mathrm{sgn}(a_5 - a_1)$, $\mathrm{sgn}(a_6 - a_1)$, $\mathrm{sgn}(a_2 - a_1)$
Simple Hexagons	$\frac{\partial g_1^r}{\partial x_1} + 2\frac{\partial g_1^r}{\partial x_2}{}^{(a)}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2}{}^{(a)}$ (twice), $\frac{\partial g_4^r}{\partial x_4}$ (six times), $3\frac{\partial g_1^i}{\partial y_1}$, 0 (twice)	$\mathrm{sgn}(a_1 + 2a_2)$, $\mathrm{sgn}(a_1 - a_2)$, $\mathrm{sgn}(a_4 + a_5 + a_6 - a_1 - 2a_2)$, $\mathrm{sgn}[x(c_1 - b_1 - 2b_2)]$
Rhombs ($\mathrm{Rh}_{h1,\alpha,\beta}$)	$\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_4}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_4}$, 0 (twice) μ_1, μ_2 (four times) such that $\mu_1 + \mu_2 = \frac{\partial g_2^r}{\partial x_2} + \frac{\partial g_3^r}{\partial x_3}$, $\mu_1\mu_2 = \frac{\partial g_2^r}{\partial x_2}\frac{\partial g_3^r}{\partial x_3} - \frac{\partial g_2^r}{\partial x_3}\frac{\partial g_3^r}{\partial x_2}$	$\mathrm{sgn}(a_1 + a_4)$, $\mathrm{sgn}(a_1 - a_4)$, $\mathrm{sgn}(a_2 + a_5 - a_1 - a_4)$, $\mathrm{sgn}(a_2 + a_6 - a_1 - a_4)$
Rhombs ($\mathrm{Rh}_{h2,\alpha,\beta}$)	$\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_5}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_5}$, 0 (twice) μ_1, μ_2 (four times) such that $\mu_1 + \mu_2 = \frac{\partial g_2^r}{\partial x_2} + \frac{\partial g_3^r}{\partial x_3}$, $\mu_1\mu_2 = \frac{\partial g_2^r}{\partial x_2}\frac{\partial g_3^r}{\partial x_3} - \frac{\partial g_2^r}{\partial x_3}\frac{\partial g_3^r}{\partial x_2}$	$\mathrm{sgn}(a_1 + a_5)$, $\mathrm{sgn}(a_1 - a_5)$, $\mathrm{sgn}(a_2 + a_4 - a_1 - a_5)$, $\mathrm{sgn}(a_2 + a_6 - a_1 - a_5)$
Rhombs ($\mathrm{Rh}_{h3,\alpha,\beta}$)	$\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_6}$, $\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_6}$, 0 (twice) μ_1, μ_2 (four times) such that $\mu_1 + \mu_2 = \frac{\partial g_2^r}{\partial x_2} + \frac{\partial g_3^r}{\partial x_3}$, $\mu_1\mu_2 = \frac{\partial g_2^r}{\partial x_2}\frac{\partial g_3^r}{\partial x_3} - \frac{\partial g_2^r}{\partial x_3}\frac{\partial g_3^r}{\partial x_2}$	$\mathrm{sgn}(a_1 + a_6)$, $\mathrm{sgn}(a_1 - a_6)$, $\mathrm{sgn}(a_2 + a_4 - a_1 - a_6)$, $\mathrm{sgn}(a_2 + a_5 - a_1 - a_6)$
Super Hexagons	$\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} + \frac{\partial g_1^r}{\partial x_4} + \frac{\partial g_1^r}{\partial x_5} + \frac{\partial g_1^r}{\partial x_6}$, $\frac{\partial g_1^r}{\partial x_1} + \frac{\partial g_1^r}{\partial x_2} + \frac{\partial g_1^r}{\partial x_3} - \frac{\partial g_1^r}{\partial x_4} - \frac{\partial g_1^r}{\partial x_5} - \frac{\partial g_1^r}{\partial x_6}$, $\frac{\partial g_1^i}{\partial y_1} + \frac{\partial g_1^i}{\partial y_2} + \frac{\partial g_1^i}{\partial y_3} + \frac{\partial g_1^i}{\partial y_4} + \frac{\partial g_1^i}{\partial y_5} + \frac{\partial g_1^i}{\partial y_6}$, $\frac{\partial g_1^i}{\partial y_1} + \frac{\partial g_1^i}{\partial y_2} + \frac{\partial g_1^i}{\partial y_3} - \frac{\partial g_1^i}{\partial y_4} - \frac{\partial g_1^i}{\partial y_5} - \frac{\partial g_1^i}{\partial y_6}$, $2\frac{\partial g_1^i}{\partial y_1} - \frac{\partial g_1^i}{\partial y_2} - \frac{\partial g_1^i}{\partial y_3}$ (twice), 0 (twice), μ_1, μ_2 (twice), such that $\mu_1 + \mu_2 = 2\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3}$ $\mu_1\mu_2 = \frac{1}{2}\Big\{\Big(\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_2}\Big)^2 + \Big(\frac{\partial g_1^r}{\partial x_1} - \frac{\partial g_1^r}{\partial x_3}\Big)^2 + \Big(\frac{\partial g_1^r}{\partial x_2} - \frac{\partial g_1^r}{\partial x_3}\Big)^2$ $-\Big(\frac{\partial g_1^r}{\partial x_4} - \frac{\partial g_1^r}{\partial x_5}\Big)^2 - \Big(\frac{\partial g_1^r}{\partial x_4} - \frac{\partial g_1^r}{\partial x_6}\Big)^2 - \Big(\frac{\partial g_1^r}{\partial x_5} - \frac{\partial g_1^r}{\partial x_6}\Big)^2\Big\}$	$\mathrm{sgn}(a_1 + 2a_2 + a_4 + a_5 + a_6)$, $\mathrm{sgn}(a_1 + 2a_2 - a_4 - a_5 - a_6)$ $\mathrm{sgn}[x(-b_1 - 2b_2 - b_4 - b_5 - b_6 + c_1 + c_2 - c_3)]$, $\mathrm{sgn}[x(-b_1 - 2b_2 - b_4 - b_5 - b_6 + c_1 - c_2 + c_3)]$, $\mathrm{sgn}[-(2\alpha - \beta)e_1 - (\alpha + \beta)e_2]$, $\mathrm{sgn}(\mu_1 + \mu_2) = \mathrm{sgn}(a_1 - a_2)$, $\mathrm{sgn}(\mu_1\mu_2) = \mathrm{sgn}[2(a_1 - a_2)^2 - (a_4 - a_5)^2$ $-(a_4 - a_6)^2 - (a_5 - a_6)^2]$

$^{(a)}$ Here the effect on $\mathbf{D}\mathbf{g}$ of the hidden symmetry $\widetilde{\tau}_{x_1}$ (3.3) is included.

for super hexagons is determined by retaining all terms up to and including the leading order (α, β)-dependent terms in (5.6). The remaining eigenvalues are determined by a quartic truncation, that is

$$\begin{aligned}
\dot{z}_1 &= \lambda z_1 + \epsilon \bar{z}_2 \bar{z}_3 + z_1(a_1|z_1|^2 + a_2|z_2|^2 + a_2|z_3|^2 + a_4|z_4|^2 + a_5|z_5|^2 + a_6|z_6|^2) \\
&\quad + \bar{z}_2 \bar{z}_3(b_1|z_1|^2 + b_2|z_2|^2 + b_2|z_3|^2 + b_4|z_4|^2 + b_5|z_5|^2 + b_6|z_6|^2) \\
&\quad + z_1(c_1 z_1 z_2 z_3 + c_2 z_4 z_5 z_6 + c_3 \bar{z}_4 \bar{z}_5 \bar{z}_6) + \mathcal{O}(\| \mathbf{z} \|^5) \ ,
\end{aligned} \tag{5.12}$$

provided the additional nondegeneracy conditions

$$\begin{aligned}
a_1 &\neq 0, \ a_2, \ \pm a_4, \ \pm a_5, \ \pm a_6 \ , \\
(a_1 + 2a_2) &\neq 0, \ \pm(a_4 + a_5 + a_6) \ , \\
a_1 - a_2 &\neq \pm(a_4 - a_5), \ \pm(a_4 - a_6), \ \pm(a_5 - a_6) \ , \\
2(a_1 - a_2)^2 &\neq (a_4 - a_5)^2 + (a_4 - a_6)^2 + (a_5 - a_6)^2 \ , \\
c_1 - b_1 - 2b_2 &\neq 0, \ b_4 + b_5 + b_6 \pm (c_2 - c_3) \ ,
\end{aligned} \tag{5.13}$$

are satisfied. Again, all coefficients are real and we assume time has been scaled so that the linear term in $\mathbf{g}(\mathbf{z})$ is $\lambda \mathbf{z}$. The third column of Table 9 summarises the results in the case $\epsilon = 0$.

From the signs of the eigenvalues given in Table 9 we can draw a number of conclusions:

1. While all axial solution branches bifurcate unstably when $\epsilon \neq 0$, we find, in the degenerate case $\epsilon = 0$, that any one of the axial solution branches can bifurcate supercritically to produce a stable solution.

2. There are two distinct branches of simple and super hexagons, denoted $\mathrm{SiH}^\pm$ and $\mathrm{SuH}^\pm$, respectively, associated with $x > 0$ and $x < 0$ in the branching equation. If simple hexagons are neutrally stable at cubic order, then one and only one of the two branches $\mathrm{SiH}^\pm$ is stable. If super hexagons are neutrally stable at cubic order, then one and only one of the two branches $\mathrm{SuH}^\pm$ will be stable if $(2\alpha - \beta)e_1 + (\alpha + \beta)e_2 > 0$, while they are both unstable if $(2\alpha - \beta)e_1 + (\alpha + \beta)e_2 < 0$.

3. If $(2\alpha - \beta)e_1 + (\alpha + \beta)e_2 < 0$ then it is possible for all of the axial planforms to bifurcate supercritically, but none be stable. On the other hand, if $(2\alpha - \beta)e_1 + (\alpha + \beta)e_2 > 0$ and all axial planforms bifurcate supercritically, then at least one of them must be stable.

4. If *any* axial solution branch bifurcates subcritically, then rolls and super hexagons are unstable.

5. If rolls or super hexagons bifurcate subcritically, then *all* axial planforms are unstable at bifurcation.

6. If simple hexagons is the only axial solution branch to bifurcate subcritically, then it is still possible that one, but not more, of the rhombs solutions is stable. Similarly, if rhombs is the only axial solution branch to bifurcate subcritically, then it is possible for simple hexagons to be stable, or for one or more of the remaining rhombs solutions to be stable. However, if simple hexagons *and* one of the rhombs bifurcate subcritically, then all axial solution branches are unstable.

7. If two of the rhombs solution branches bifurcate subcritically, then it is possible that the remaining rhombs solution or simple hexagons is stable, but not both. However, if all three rhombs solution branches are subcritical, then all axial planforms are unstable.

Table 10: Stability results for the hexagonal lattice bifurcation problem in the case $|\epsilon| \ll 1$, from column two of Table 9, and equations 5.2, 5.6 and 5.12. Also see column three of Table 9; only the eigenvalues that depend on ϵ are given here.

Planform	ϵ-Dependent Eigenvalues	Branching Equation
Rolls	$\epsilon x + (a_2 - a_1)x^2 + \cdots$ $-\epsilon x + (a_2 - a_1)x^2 + \cdots$	$0 = \lambda x + a_1 x^3$ $+\mathcal{O}(x^5)$
Simple Hexagons	$\epsilon x + 2(a_1 + 2a_2)x^2 + \cdots$ $-2\epsilon x + 2(a_1 - a_2)x^2 + \cdots$ $-\epsilon x + (a_4 + a_5 + a_6 - a_1 - 2a_2)x^2 + \cdots$ $-3\epsilon x + 3(c_1 - b_1 - 2b_2)x^3 + \cdots$	$0 = \lambda x + \epsilon x^2$ $+(a_1 + 2a_2)x^3$ $+\mathcal{O}(x^4)$
Rhombs ($Rh_{h1,\alpha,\beta}$ [(a)])	$\mu_1, \mu_2;\ \mu_1 + \mu_2 = (-2a_1 - 2a_4 + 2a_2 + a_5 + a_6)x^2 + \cdots$ $\mu_1\mu_2 = -\epsilon^2 x^2$ $+(a_1 + a_4 - a_2 - a_5)(a_1 + a_4 - a_2 - a_6)x^4 + \cdots$	$0 = \lambda x + (a_1 + a_4)x^3$ $+\mathcal{O}(x^5)$
Super Hexagons	$\epsilon x + 2(a_1 + 2a_2 + a_4 + a_5 + a_6)x^2 + \cdots$ $\epsilon x + 2(a_1 + 2a_2 - a_4 - a_5 - a_6)x^2 + \cdots$ $-3[\epsilon x + (b_1 + 2b_2 + b_4 + b_5 + b_6 - c_1 - c_2 + c_3)x^3] + \cdots$ $-3[\epsilon x + (b_1 + 2b_2 + b_4 + b_5 + b_6 - c_1 + c_2 - c_3)x^3] + \cdots$ $\mu_1, \mu_2;\ \mu_1 + \mu_2 = -4\epsilon x + 4(a_1 - a_2)x^2 + \cdots$ $\mu_1\mu_2 = 4\epsilon^2 x^2 - 8(a_1 - a_2)\epsilon x^3 + 4(a_1 - a_2)^2 x^4$ $-2[(a_4 - a_5)^2 + (a_4 - a_6)^2 + (a_5 - a_6)^2]x^4 + \cdots$	$0 = \lambda x + \epsilon x^2$ $+(a_1 + 2a_2)x^3$ $+(a_4 + a_5 + a_6)x^3$ $+\mathcal{O}(x^4)$

[(a)] The results for Rhombs ($\mathrm{Rh}_{h2,\alpha,\beta}$), or Rhombs ($\mathrm{Rh}_{h3,\alpha,\beta}$), are obtained from those for Rhombs ($\mathrm{Rh}_{h1,\alpha,\beta}$) by interchanging the 4 and 5, or 4 and 6, subscripts, respectively.

8. The only solution branches that can co-exist stably are simple hexagons SiH and the rhombs $\mathrm{Rh}_{h1,\alpha,\beta}$, $\mathrm{Rh}_{h2,\alpha,\beta}$, $\mathrm{Rh}_{h3,\alpha,\beta}$. Any combination of two of these states can bifurcate stably. It is also possible for all three types of rhombs to be stable simultaneously. However, if two or more of the rhombs are stable, then simple hexagons are unstable.

5.3 Secondary Bifurcations for $0 < |\epsilon| \ll 1$.

In this section we address briefly the unfolding of the degenerate bifurcation problem $\epsilon = 0$ analyzed in the previous section. Specifically, we indicate how the stability of the axial solutions change along the solution branch in the case that $|\epsilon| \ll 1$. While a complete analysis of the unfolding is beyond the scope of the present paper, we do present an example in which part of a bifurcation diagram is computed. This example indicates the wealth of secondary transitions that occur close to $\lambda = 0$ when $|\epsilon| \ll 1$. For $\epsilon \neq 0$, certain eigenvalues given in Table 9 are modified to those given in Table 10. Note that, as discussed above, the presence of the quadratic term in the bifurcation problem ensures that at least one of the eigenvalues for each axial planform is positive for $(\lambda, \mathbf{z})$ sufficiently close to the origin.

As a specific example, we consider the bifurcation problem

$$\begin{aligned}\dot{z}_1 &= \lambda\ z_1 + \epsilon\ \bar{z}_2\bar{z}_3 + z_1\ (a_1|z_1|^2 + a_2|z_2|^2 + a_2|z_3|^2 + a_4|z_4|^2 + a_5|z_5|^2 + a_6|z_6|^2) \\ &\quad + b_2\ \bar{z}_2\bar{z}_3\ (|z_2|^2 + |z_3|^2)\ +\ e_1\ \bar{z}_1^{\alpha-\beta-1} z_3^{\beta} z_4^{\beta} \bar{z}_6^{\alpha-\beta}\ +\ e_2\ \bar{z}_1^{\beta-1} z_2^{\alpha-\beta} z_4^{\alpha-\beta} \bar{z}_5^{\beta}\ ,\end{aligned} \tag{5.14}$$

where

$$a_1 = -1.5,\ \ a_2 = -3.5,\ \ a_4 = 0.5,\ \ a_5 = 0.6,\ \ a_6 = 0.7,\ \ b_2 = 0.6,\ \ e_1 = 1.0,\ \ e_2 = 0.5. \tag{5.15}$$

This choice satisfies the non-degeneracy conditions (5.13). It follows from Table 9 that all three rhomb states are stable when $\epsilon = 0$. We show two bifurcation diagrams for $0 < \epsilon \ll 1$. The bifurcation diagrams indicate, schematically, the amplitude $\| \mathbf{z} \|$ as a function of the bifurcation parameter λ for each axial planform. Solutions on the same group orbit are identified and bifurcation points are indicated by solid circles. We follow the convention that solid lines indicate stable solutions and dotted lines indicate unstable solutions. Figure 6 is a well-known bifurcation diagram that applies to the six-dimensional representation of Γ_h; here it is obtained by restricting our analysis to the six-dimensional subspace where $\mathbf{z} = (z_1, z_2, z_3, 0, 0, 0)$. Figure 7 gives the bifurcation diagram that applies, for the same coefficient values (5.15), in the full twelve-dimensional space.

In the six-dimensional subspace, where $z_4 = z_5 = z_6 = 0$, only two axial planforms exist, rolls and simple hexagons. In this subspace, and for the choice of coefficients (5.15), Figure 6 indicates that as λ increases through 0, the trivial solution becomes unstable and there is a transition to stable simple hexagons. On further increase of λ the hexagons become unstable and there is a transition to rolls. Both the transition to hexagons and that to rolls exhibit hysteresis. The bifurcation scenario of Figure 6 has been investigated in a wide variety of hydrodynamic systems [3, 10, 18, 24], in solidification problems [2, 22, 30], and in chemical reaction-diffusion systems [9, 21].

Figure 7 indicates how the familiar bifurcation diagram in Figure 6 is modified when we consider stability within the full twelve-dimensional space. In this case, rolls are always unstable to rhombs, and the range of stability of simple hexagons is greatly decreased. Indeed simple hexagons are stable only in a subcritical regime where super hexagons are also stable. In this case, on increasing λ, there is first a jump at $\lambda = 0$ to stable super hexagons and then a transition to one of the three stable rhombs states. All of the transitions exhibit hysteresis. While the bifurcations to simple and super hexagons are transcritical, all other primary bifurcations are pitchforks. All of the secondary bifurcation points indicated in the diagram approach $\lambda = \| \mathbf{z} \| = 0$ as $\epsilon \to 0$. The paths of the secondary branches have not been computed.

5.4 Stability results: $\Gamma = \Gamma_h \oplus \mathbf{Z}_2$.

In this section we consider the consequences of the additional $\mathbf{Z}_2$ symmetry, $\kappa(\mathbf{z}) = -\mathbf{z}$, for the generic bifurcation problem on the hexagonal lattice, $\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, \lambda)$, $\mathbf{g} : \mathbf{C}^6 \times \mathbf{R} \to \mathbf{C}^6$. Specifically we consider the branching and stability assignments for the axial planforms listed in Tables 4 and 5.

The $\mathbf{Z}_2$ symmetry places some additional restrictions on the eigenvalues of rolls and rhombs listed in Table 9; specifically, it ensures that $\frac{\partial g_2^r}{\partial x_3} = \frac{\partial g_3^r}{\partial x_2} = 0$ on these solution branches. The eigenvalues for simple and super hexagons, listed in Table 9, are unchanged. The eigenvalues of D$\mathbf{g}$ for the remaining axial planforms are listed in Table 11. We note that the $\mathbf{D}_6$-isotypic decomposition of $\mathbf{C}^6$ is the same for the super hexagons, anti-hexagons, super-triangles and anti-triangles planforms; it is given by (5.9). Indeed, the only difference between the eigenvalue structure for the triangle states and the hexagon states is that the null vectors lie in different isotypic components in the two cases.

The additional $\mathbf{Z}_2$ symmetry forces the coefficients of all even order terms in the Taylor expansion of the Γ_h-equivariant bifurcation problem (5.6) to be zero. Hence there are no quadratic terms; the differences between the degenerate bifurcation problem with Γ_h-symmetry ($\epsilon = 0$) and the generic bifurcation problem with $\Gamma_h \oplus \mathbf{Z}_2$-symmetry arise at $\mathcal{O}(\| \mathbf{z} \|^4)$. Thus the eigenvalues for the rolls and the rhombs in Table 9, which are determined by a cubic truncation, are unchanged by the extra $\mathbf{Z}_2$ symmetry. Note that certain eigenvalues of simple and super hexagons for the

Planform	Eigenvalues	Signs of Non-zero Eigenvalues
Rhombs (Rh_{h0})	$\frac{\partial g_1^r}{\partial x_1}+\frac{\partial g_1^r}{\partial x_2}$ [a], $\frac{\partial g_1^r}{\partial x_1}-\frac{\partial g_1^r}{\partial x_2}$ [a], $\frac{\partial g_3^r}{\partial x_3}$, $\frac{\partial g_3^i}{\partial y_3^i}$, $\frac{\partial g_4^r}{\partial x_4}$ (twice), $\frac{\partial g_5^r}{\partial x_5}$ (twice), $\frac{\partial g_6^r}{\partial x_6}$ (twice), 0 (twice)	$\mathrm{sgn}(a_1+a_2)$, $\mathrm{sgn}(a_1-a_2)$, $-\mathrm{sgn}(a_1-a_2)$, $\mathrm{sgn}(a_4+a_6-a_1-a_2)$, $\mathrm{sgn}(a_4+a_5-a_1-a_2)$, $\mathrm{sgn}(a_5+a_6-a_1-a_2)$
Simple Hexagons	See Table 9.	$\mathrm{sgn}(a_1+2a_2), \mathrm{sgn}(a_1-a_2), \mathrm{sgn}(a_4+a_5+a_6-a_1-2a_2), -\mathrm{sgn}(d_2)$
Simple Triangles	$\frac{\partial g_1^i}{\partial y_1}+2\frac{\partial g_1^i}{\partial y_2}$ [a], $\frac{\partial g_1^i}{\partial y_1}-\frac{\partial g_1^i}{\partial y_2}$ [a] (twice), $\frac{\partial g_4^r}{\partial x_4}$ (six times), $3\frac{\partial g_1^r}{\partial x_1}$, 0 (twice)	$\mathrm{sgn}(a_1+2a_2)$, $\mathrm{sgn}(a_1-a_2)$, $\mathrm{sgn}(a_4+a_5+a_6-a_1-2a_2), \mathrm{sgn}(d_2)$
Super Hexagons	See Table 9.	Signs of $\lambda_1, \lambda_2, \mu_1$ and μ_2 as for super triangles. $-\mathrm{sgn}(d_2+d_3)$, $-\mathrm{sgn}[(2\alpha-\beta)e_1+(\alpha+\beta)e_2]$, $-\mathrm{sgn}[3(d_2+d_4)-(2\beta-\alpha)(e_1-e_2)x^{2(\alpha-3)}]$ [b]
Anti-Hexagons	Same as Super Hexagons. See Table 9.	Signs of $\lambda_1, \lambda_2, \mu_1$ and μ_2 as for super triangles. $(-1)^{\alpha+1}\mathrm{sgn}[(2\alpha-\beta)e_1+(\alpha+\beta)e_2]$, $-\mathrm{sgn}(d_2-d_3)$ $-\mathrm{sgn}[3(d_2-d_4)-(-1)^{\alpha}(2\beta-\alpha)(e_1-e_2)x^{2(\alpha-3)}]$ [b]
Super Triangles	$\frac{\partial g_1^i}{\partial y_1}+\frac{\partial g_1^i}{\partial y_2}+\frac{\partial g_1^i}{\partial y_3}+\frac{\partial g_1^i}{\partial y_4}+\frac{\partial g_1^i}{\partial y_5}+\frac{\partial g_1^i}{\partial y_6}$ $\frac{\partial g_1^i}{\partial y_1}+\frac{\partial g_1^i}{\partial y_2}+\frac{\partial g_1^i}{\partial y_3}-\frac{\partial g_1^i}{\partial y_4}-\frac{\partial g_1^i}{\partial y_5}-\frac{\partial g_1^i}{\partial y_6}$, 0 (twice), μ_1, μ_2 (twice) such that $\mu_1+\mu_2=2\frac{\partial g_1^i}{\partial y_1}-\frac{\partial g_1^i}{\partial y_2}-\frac{\partial g_1^i}{\partial y_3}$ $\mu_1\mu_2=\frac{1}{2}\Big\{\Big(\frac{\partial g_1^i}{\partial y_1}-\frac{\partial g_1^i}{\partial y_2}\Big)^2+\Big(\frac{\partial g_1^i}{\partial y_1}-\frac{\partial g_1^i}{\partial y_3}\Big)^2+\Big(\frac{\partial g_1^i}{\partial y_2}-\frac{\partial g_1^i}{\partial y_3}\Big)^2-\Big(\frac{\partial g_1^i}{\partial y_4}-\frac{\partial g_1^i}{\partial y_5}\Big)^2$ $-\Big(\frac{\partial g_1^i}{\partial y_4}-\frac{\partial g_1^i}{\partial y_6}\Big)^2-\Big(\frac{\partial g_1^i}{\partial y_5}-\frac{\partial g_1^i}{\partial y_6}\Big)^2\Big\}$, $2\frac{\partial g_1^r}{\partial x_1}-\frac{\partial g_1^r}{\partial x_2}-\frac{\partial g_1^r}{\partial x_3}$ (twice), $\frac{\partial g_1^r}{\partial x_1}+\frac{\partial g_1^r}{\partial x_2}+\frac{\partial g_1^r}{\partial x_3}-\frac{\partial g_1^r}{\partial x_4}-\frac{\partial g_1^r}{\partial x_5}-\frac{\partial g_1^r}{\partial x_6}$, $\frac{\partial g_1^r}{\partial x_1}+\frac{\partial g_1^r}{\partial x_2}+\frac{\partial g_1^r}{\partial x_3}+\frac{\partial g_1^r}{\partial x_4}+\frac{\partial g_1^r}{\partial x_5}+\frac{\partial g_1^r}{\partial x_6}$	$\mathrm{sgn}(\lambda_1)=\mathrm{sgn}(a_1+2a_2+a_4+a_5+a_6)$, $\mathrm{sgn}(\lambda_2)=\mathrm{sgn}(a_1+2a_2-a_4-a_5-a_6)$, $\mathrm{sgn}(\mu_1+\mu_2)=\mathrm{sgn}(a_1-a_2)$, $\mathrm{sgn}(\mu_1\mu_2)=\mathrm{sgn}[2(a_1-a_2)^2-(a_4-a_5)^2-(a_4-a_6)^2-(a_5-a_6)^2]$ $(-1)^{\alpha+1}\mathrm{sgn}[(2\alpha-\beta)e_1+(\alpha+\beta)e_2]$, $\mathrm{sgn}(d_2-d_3)$, $\mathrm{sgn}[3(d_2+d_4)+(-1)^{\alpha}(2\beta-\alpha)(e_1-e_2)x^{2(\alpha-3)}]$ [b]
Anti-Triangles	Same as Super Triangles.	Signs of $\lambda_1, \lambda_2, \mu_1$ and μ_2 as for super triangles. $-\mathrm{sgn}[(2\alpha-\beta)e_1+(\alpha+\beta)e_2]$, $\mathrm{sgn}(d_2+d_3)$, $\mathrm{sgn}[3(d_2-d_4)+(2\beta-\alpha)(e_1-e_2)x^{2(\alpha-3)}]$ [b]

[a] Here the effect on $\mathbf{Dg}$ of the hidden symmetry $\widetilde{\tau}_{x_1}$ (3.3) is included. [b] The (α,β)-dependent terms can be neglected here for all cases except $(\alpha,\beta)=(3,2)$.

The coefficients a_i, d_i, e_i are defined by equation 5.16.

degenerate Γ_h bifurcation problem depend on the coefficients of quartic terms (see Table 9). These eigenvalues are now determined at quintic order, that is by the truncation

$$\begin{aligned}
\dot{z}_1 &= \lambda z_1 + z_1(a_1|z_1|^2 + a_2|z_2|^2 + a_2|z_3|^2 + a_4|z_4|^2 + a_5|z_5|^2 + a_6|z_6|^2) \\
&\quad + z_1(f_{11}|z_1|^4 + f_{12}|z_1|^2|z_2|^2 + \cdots + f_{56}|z_5|^2|z_6|^2 + f_{66}|z_6|^4) \\
&\quad + \bar{z}_2\bar{z}_3(d_1 z_1 z_2 z_3 + d_2 \bar{z}_1\bar{z}_2\bar{z}_3 + d_3 z_4 z_5 z_6 + d_4 \bar{z}_4\bar{z}_5\bar{z}_6) \\
&\quad + e_1 \bar{z}_1^{\alpha-\beta-1} z_3^{\beta} z_4^{\beta} \bar{z}_6^{\alpha-\beta} + e_2 \bar{z}_1^{\beta-1} z_2^{\alpha-\beta} z_4^{\alpha-\beta} \bar{z}_5^{\beta} + \mathcal{O}(\| \mathbf{z} \|^7) \ .
\end{aligned} \tag{5.16}$$

The stability results for the axial planforms are summarized in Table 11. Note that the leading order α, β dependent terms are $\mathcal{O}(2\alpha - 1)$, where $2\alpha - 1 \geq 5$, with $2\alpha - 1 = 5$ only in the case of $(\alpha, \beta) = (3, 2)$.

We assume that the following nondegeneracy conditions are satisfied:

$$\begin{aligned}
a_1 &\neq 0,\ \pm a_2,\ \pm a_4,\ \pm a_5,\ \pm a_6 \ , \\
(a_1 + a_2) &\neq (a_4 + a_5),\ (a_4 + a_6),\ (a_5 + a_6) \ , \\
(a_1 + 2a_2) &\neq 0,\ \pm(a_4 + a_5 + a_6), \\
(a_1 - a_2) &\neq \pm(a_4 - a_5),\ \pm(a_4 - a_6),\ \pm(a_5 - a_6) \ , \\
2(a_1 - a_2)^2 &\neq (a_4 - a_5)^2 + (a_4 - a_6)^2 + (a_5 - a_6)^2 \ , \\
\frac{e_1}{e_2} &\neq -\frac{\alpha + \beta}{2\alpha - \beta} \ , \\
d_2 &\neq 0,\ \pm d_3, \\
d_2 \pm d_4 &\neq \begin{cases} 0 & \text{if } (\alpha, \beta) \neq (3, 2) \\ \pm\left(\frac{e_1 - e_2}{3}\right) & \text{if } (\alpha, \beta) = (3, 2) \end{cases}
\end{aligned} \tag{5.17}$$

In this case we can draw a number of conclusions from Tables 9 and 11.

1. The rhombs Rh_{h0} always bifurcate unstably. This observation was made in [13].
2. If simple triangles and simple hexagons are neutrally stable at cubic order, then one and only one of the two branches is stable. The relative stability properties of these two solutions is determined at quintic order.
3. It is possible for super hexagons, anti-hexagons, super triangles and anti-triangles to be unstable, even if they are all neutrally stable at cubic order.
4. It is possible for all of the axial planforms to bifurcate supercritically, but none be stable.
5. If *any* axial solution branch bifurcates subcritically, then rolls, super hexagons, super triangles, anti-hexagons and anti-triangles are all unstable.
6. If rolls or super hexagons bifurcate subcritically, then *all* axial planforms are unstable at bifurcation.
7. If simple hexagons and simple triangles are the only axial solution branches to bifurcate subcritically, then it is still possible that one, but not more, of the rhombs solutions is stable. Similarly, if rhombs $\mathrm{Rh}_{hj,\alpha,\beta}$ ($j = 1, 2,$ or 3) is the only axial solution branch to bifurcate subcritically, then it is possible for simple hexagons or simple triangles to be stable, or for one or more of the remaining rhombs solutions to be stable.

8. If simple hexagons and the rhombs Rh_{h0} bifurcate subcritically, then it is possible for one, but not more, of the other rhombs to be stable. However, if simple hexagons and one of the rhombs other than Rh_{h0} bifurcate subcritically, then all axial solution branches are unstable.

9. If two of the rhombs $\mathrm{Rh}_{hj,\alpha,\beta}$ ($j = 1, 2, 3$) solution branches bifurcate subcritically, then it is possible that one, and only one, of the following solutions is stable: the remaining rhombs $\mathrm{Rh}_{hj,\alpha,\beta}$, simple hexagons or simple triangles. However, if Rh_{h0} and one of the other rhombs bifurcate subcritically, or if all three of the rhombs $\mathrm{Rh}_{hj,\alpha,\beta}$ solution branches are subcritical, then all of the axial solutions are unstable.

10. If α is odd, then the only solution branches that can co-exist stably are simple hexagons or simple triangles and the rhombs $\mathrm{Rh}_{h1,\alpha,\beta}$, $\mathrm{Rh}_{h2,\alpha,\beta}$, $\mathrm{Rh}_{h3,\alpha,\beta}$. Any combination of two of these states can bifurcate stably. It is also possible for all three types of rhombs to be stable simultaneously. However, if two or more of the rhombs are stable, then simple hexagons and simple triangles are unstable. If α is even, then it is also possible for super hexagons and anti-hexagons to be stable simultaneously, or for super triangles and anti-triangles to both be stable.

6 Conclusions.

We have investigated steady, spatially-periodic planforms which bifurcate from a spatially-uniform time-independent solution of E(2)-equivariant and E(2)$\oplus\mathbf{Z}_2$-equivariant PDEs. We considered separately the cases where the solutions are doubly-periodic on a square lattice and on a hexagonal lattice. The fundamental period ℓ of the lattice was chosen so that the dual lattice contains wavevectors of critical length k_c. For the square lattice $\ell = \sqrt{\alpha^2 + \beta^2}/k_c$, while for the hexagonal lattice $\ell = \sqrt{\alpha^2 + \beta^2 - \alpha\beta}/k_c$, where $\alpha > \beta > 0$ are integers.

For each lattice we determined the relative stability of the planforms which are guaranteed to bifurcate from the trivial solution by the equivariant branching lemma [12, 29]. Our analysis proceeded by first determining the general form of the equivariant bifurcation problems, and then using these equations to compute the linear orbital stability of the axial planforms. For the square lattice, an order $2(\alpha + \beta) - 1$ truncation of the equivariant bifurcation problem on $\mathbf{C}^4$ is required to completely determine the signs of the eigenvalues. In the case of the hexagonal lattice, an order $(2\alpha - 1)$ truncation is necessary. However, an important practical consideration, in both cases, is that much is already determined at cubic order.

Previous bifurcation studies of the stability of spatially-periodic planforms have focused on the "small box" limit for which the size of the periodic domain coincides with the wavelength of the instability, *i.e.*, $\ell = 1/k_c$. This leads to a bifurcation problem on $\mathbf{C}^2$ for the square lattice and a bifurcation problem on $\mathbf{C}^3$ for the hexagonal lattice. These bifurcation problems allow one to investigate the relative stability of rolls and simple squares in the case of the square lattice, and rolls and simple hexagons in the hexagonal case. In this paper we considered the countable set of irreducible representations, on $\mathbf{C}^4$ and $\mathbf{C}^6$ respectively, of the symmetry groups associated with the square and hexagonal lattices. This analysis extends the results of the earlier $\mathbf{C}^2$ and $\mathbf{C}^3$ bifurcation studies, both enlarging the number of planforms which are supported by the lattice and allowing for a wider class of disturbances in the stability analysis. In particular, by considering all of the irreducible representations the stability of rolls, simple squares, simple hexagons, and simple triangles, to a countable set of perturbations, can be determined. For example, our bifurcation analysis provides a framework for addressing the relative stability of simple hexagons (or simple squares) and a countable set of rhombs. This is of special interest in light of recent laboratory experiments on chemical Turing patterns in which both hexagonal and rhombic patterns are observed [14]. Our unfolding of the degenerate bifurcation problem on the hexagonal lattice provides a simple mathematical setting for investigating a transition between these states.

Our stability analysis in the case of the Euclidean group is incomplete since, for a given periodic pattern, we consider only those perturbations that are periodic on the same type of lattice. For example, there is no simple bifurcation theoretic framework for computing the relative stability of squares and hexagons since no lattice supports them both. However, we are able to compute the stability of hexagons relative to rhombs, which are "almost square", *i.e.*, which are composed of rectangles with aspect ratio that is close to 1 (see Table 7). For example, for the representation of $\mathbf{D}_6\dot{+}\mathbf{T}^2$ with $(\alpha,\beta)=(4,3)$, the rhombs $\mathrm{Rh}_{h3,4,3}$ are made up of rectangles with aspect ratio approximately 0.96; the angle between the wave vectors $\mathbf{K}_1$ and $\mathbf{K}_6$ in this case is about 92°.

By not requiring the periodicity ℓ of the lattice to coincide with the wavelength of the instability $1/k_c$, we were able to investigate axial solution branches with periodicity $1/k_c$ and simultaneously solution branches that have fundamental periodicity $\ell \gg 1/k_c$. In this paper we called the latter states super squares, super hexagons, super triangles, and anti-squares, anti-hexagons, anti-triangles. For $\mathrm{E}(2)(\oplus\mathbf{Z}_2)$-equivariant PDEs, there is an infinite family of these solution branches that is parameterized by the integer pair (α,β). The wavelength of the instability $1/k_c$ determines the size of the small scale structure of these super and anti-state patterns, while ℓ determines their periodicity. By increasing α and β we obtain axial planforms that are periodic on larger and larger scales relative to $1/k_c$, all of which bifurcate from the trivial solution at $\lambda=0$. This is perhaps interesting in light of recent hydrodynamic experiments on quasi-patterns [10]. We emphasize, however, that the existence of a center manifold in the case where the critical wavevectors do not generate a periodic lattice, has not been established, *e.g.*, in the case that there are twelve critical wavevectors equally spaced on the critical circle.

The symmetry groups assumed in our analysis arise naturally when periodic boundary conditions are used. Thus our results apply directly to certain numerical studies that incorporate such boundary conditions. The "small box" limit is obtained by restricting the computational domain so that it contains only one hexagon or square and applying periodic boundary conditions. Our analysis applies to computations carried out on larger periodic domains. For example, computations done on a square domain of side length $\sqrt{\alpha^2+\beta^2}/k_c$ admit, local to the bifurcation point, not only steady rolls and simple squares, but also two different rhombic patterns, as well as the super and anti-square states. This paper has shown that each of these states has the possibility of being stable.

Acknowledgements

We would like to thank I. Melbourne for helpful discussions, and a referee for a thorough reading of the original manuscript and many helpful suggestions. This work was begun during a visit by the authors to the Fields Institute, and we are grateful to the Fields Institute for support. The research of BD was supported by the Natural Sciences and Engineering Research Council of Canada. The research of MS was supported by NSF grants DMS-9410115 and DMS-9404266, and by an NSF CAREER award DMS-9502266. The research of ACS was supported by the EPSRC under grant GR/K41311.

7 Appendix.

In this appendix we sketch the calculations of the equivariant bifurcation equations presented in sections 4.1 and 5.1.

$\mathbf{D}_4\dot{+}\mathbf{T}^2$–equivariant bifurcation problem:

Let $h(\mathbf{z}) = \bar{z}_1 g_1(\mathbf{z})$ be a $\mathbf{T}^2$-invariant function that has Taylor expansion

$$h(\mathbf{z}) = \sum_{k_i \geq 0} a_{\mathbf{k}}\, z_1^{k_1}\, \bar{z}_1^{k_2}\, z_2^{k_3}\, \bar{z}_2^{k_3}\, z_3^{k_4}\, \bar{z}_3^{k_5}\, z_4^{k_7}\, \bar{z}_4^{k_8}\ , \tag{7.1}$$

where $k_2 > 0$. Given the action (2.20) of $\mathbf{T}^2$ on $\mathbf{C}^4$, the $\mathbf{T}^2$-invariance of h determines that $a_{\mathbf{k}} = 0$ unless

$$\begin{aligned}(k_1 - k_2)\,(\alpha\theta_1 + \beta\theta_2) + (k_3 - k_4)\,(-\beta\theta_1 + \alpha\theta_2) & \\ + (k_5 - k_6)\,(\beta\theta_1 + \alpha\theta_2) + (k_7 - k_8)\,(-\alpha\theta_1 + \beta\theta_2) &= 0\end{aligned} \tag{7.2}$$

for all $(\theta_1, \theta_2) \in \mathbf{T}^2$. Clearly, $k_1 = k_2$, $k_3 = k_4$, $k_5 = k_6$, and $k_7 = k_8$ is a solution to equation (7.2), which yields the $\mathbf{T}^2$-invariants $|z_j|^2$, $j = 1, 2, 3, 4$. In the following we factor out all powers of $|z_j|^2$ from the translation invariant monomials $P_{\mathbf{k}}(z) = z_1^{k_1}\, \bar{z}_1^{k_2}\, z_2^{k_3}\, \bar{z}_2^{k_4}\, z_3^{k_5}\, \bar{z}_3^{k_6}\, z_4^{k_7}\, \bar{z}_4^{k_8}$, and only consider monomials of the form $z_1^m z_2^n z_3^p z_4^q$, where we adopt the convention that $z_j^m \equiv \bar{z}_j^{|m|}$ if $m < 0$. In this way we reduce the problem of finding all $\mathbf{T}^2$-invariant monomials to one of finding $m, n, p, q \in \mathbf{Z}$ such that

$$m(\alpha\theta_1 + \beta\theta_2) + n(-\beta\theta_1 + \alpha\theta_2) + p(\beta\theta_1 + \alpha\theta_2) + q(-\alpha\theta_1 + \beta\theta_2) = 0. \tag{7.3}$$

Since θ_1 and θ_2 are independent this requires

$$(m - q)\alpha - (n - p)\beta = 0\ , \qquad (n + p)\alpha + (m + q)\beta = 0\ . \tag{7.4}$$

Furthermore, since α and β are relatively prime, this implies

$$\begin{aligned} m - q &= j\beta\ , & n - p &= j\alpha\ , \\ n + p &= k\beta\ , & m + q &= -k\alpha\ , \end{aligned} \tag{7.5}$$

where $j, k \in \mathbf{Z}$. Solving for m, n, p and q gives

$$\begin{aligned} m &= -\frac{1}{2}(k\alpha - j\beta)\ , & n &= \frac{1}{2}(j\alpha + k\beta)\ , \\ p &= -\frac{1}{2}(j\alpha - k\beta)\ , & q &= -\frac{1}{2}(k\alpha + j\beta), \end{aligned} \tag{7.6}$$

Since α and β are relatively prime and not both odd, the system of equations (7.6) has no (nontrivial) solution if any two of m, n, p or q are zero. It then follows that both k and j are even and equations (7.6) may be replaced by

$$\begin{aligned} m &= -(k'\alpha - j'\beta)\ , & n &= (j'\alpha + k'\beta)\ , \\ p &= -(j'\alpha - k'\beta)\ , & q &= -(k'\alpha + j'\beta)\ , \end{aligned} \tag{7.7}$$

where $j', k' \in \mathbf{Z}$. If $j' = 0, k' \neq 0$ or $j' \neq 0, k' = 0$, then (7.7) yields the translation invariant monomials

$$\bar{z}_1^{\alpha} z_2^{\beta} z_3^{\beta} \bar{z}_4^{\alpha}\ , \quad z_1^{\beta} z_2^{\alpha} \bar{z}_3^{\alpha} \bar{z}_4^{\beta}\ , \tag{7.8}$$

and their complex conjugates.

The monomials (7.8) are order $2(\alpha+\beta)$. We now show that these are the lowest order (nontrivial) translation invariant monomials of the form $z_1^m z_2^n z_3^p z_4^q$. To do this we consider the remaining cases for which $j'k' \neq 0$ in 7.7:

1. $j' > 0, k' > 0$
2. $j' > 0, k' < 0$
3. $j' < 0, k' > 0$
4. $j' < 0, k' < 0$.

In case 1, since no solution to (7.7) exists if two of m, n, p and q are zero, the order of the invariant is

$$|m| + |n| + |p| + |q| > |n| + |q| \geq (j' + k')(\alpha+\beta) \geq 2(\alpha+\beta) \tag{7.9}$$

since $j', k' > 0$.

Similarly, in case 2 the order of the invariant is

$$|m| + |n| + |p| + |q| > |m| + |p| \geq (j' + |k'|)(\alpha+\beta) \geq 2(\alpha+\beta). \tag{7.10}$$

The argument that $|m|+|n|+|p|+|q| > 2(\alpha+\beta)$ is similar for cases 3 and 4. Hence no invariants of order less than or equal to $2(\alpha+\beta)$ occur for $j'k' \neq 0$.

$\mathbf{D}_6\dot{+}\mathbf{T}^2$–equivariant bifurcation problem:

As in the square lattice case, we assume that local to the bifurcation point we can Taylor expand the function $g_1(\mathbf{z})$. We proceed by determining the form of $\mathbf{T}^2$-invariant monomials. In the following, we assume that overall factors of $|z_j|^2$, $j = 1, ..., 6$, have been removed from the monomials. Then, following the convention that $z_j^n \equiv \bar{z}_j^{|n|}$ if $n < 0$, we find that the $\mathbf{T}^2$-invariant monomials have the form

$$z_1^m z_2^n z_3^p z_4^q z_5^r z_6^s, \tag{7.11}$$

where $m, n, p, q, r, s \in \mathbf{Z}$ satisfy

$$\begin{aligned} m - n + q - s &= j\beta\ , & n - p - r + s &= -j\alpha\ , \\ n - p - q + r &= k\beta\ , & m - p - q + s &= k\alpha\ , \end{aligned} \tag{7.12}$$

where $j, k \in \mathbf{Z}$.

There are no nontrivial solutions of (7.12) with more than three of m, n, p, q, r, s zero. In the case that $j = k = 0$ in (7.12), then $m = n = p$ and $q = r = s$, which yield the invariants

$$z_1 z_2 z_3 \quad \text{and} \quad z_4 z_5 z_6\ , \tag{7.13}$$

and their complex conjugates.

We claim that the lowest order (α, β)-dependent invariants $(|j| + |k| \neq 0)$ are

$$z_1^\beta \bar{z}_2^{\alpha-\beta} \bar{z}_4^{\alpha-\beta} z_5^\beta\ , \qquad z_2^\beta \bar{z}_3^{\alpha-\beta} \bar{z}_5^{\alpha-\beta} z_6^\beta\ , \qquad z_3^\beta \bar{z}_1^{\alpha-\beta} \bar{z}_6^{\alpha-\beta} z_4^\beta\ , \tag{7.14}$$

and their complex conjugates, which are order 2α. Note that the set of the three invariants (7.14) together with their complex conjugates is invariant under the action of $\mathbf{D}_6$. We justify the above assertion by the following steps:

1. Show that we can assume that one of m, n, p is zero and that one of q, r, s is zero. We then focus on the specific case with $p = 0$ since the invariants with $m = 0$ and $n = 0$ can be transformed to $p = 0$ by the action of $R_{\pi/3} \in \mathbf{D}_6$ on the invariant.

2. Consider the cases where $p = 0$ and exactly two of m, n, q, r, s are zero.

3. Consider the cases where $p = 0$ and only one of q, r, s is zero.

Step 1. Consider $m, n, p \geq 0$. If m, n, p are all nonzero, then we can construct a lower degree $\mathbf{T}^2$-invariant monomial from (7.11) by factoring out the invariant $z_1 z_2 z_3$. Thus the lowest degree monomials cannot have $m, n, p > 0$. Similarly, if $m, n, p < 0$, then we can lower the degree by factoring out the invariant $\bar{z}_1 \bar{z}_2 \bar{z}_3$.

Suppose now that m, n, p do not all have the same sign; for example, consider the case $m \geq n \geq 0 \geq p$. Then the order of the monomial is $m + n + |p| + |q| + |r| + |s|$. However, if $z_1^m z_2^n \bar{z}_3^{|p|} z_4^q z_5^r z_6^s$ is invariant, then so is $z_1^{m-n} \bar{z}_3^{|p|+n} z_4^q z_5^r z_6^s$, which is of lower order $m+|p|+|q|+|r|+|s|$ unless $n = 0$, in which case it has the same order. Since we only aim to find the lowest order (α, β)-dependent monomials, we can assume $n = 0$ in (7.11). The argument is the same for the other orderings of m, n, p, and 0; in each case we can find a lower degree invariant monomial unless one of m, n, p is zero. Hence, the lowest order (α, β)-dependent monomial has at least one of m, n, p equal to zero, and by a similar argument we can assume that one of q, r, s is zero. In the following steps, we assume that $p = 0$.

Step 2. Let $p = 0$ and exactly two of m, n, q, r, s be zero. From step 1, we can assume that at least one of q, r, s is zero. There are nine combinations to consider; in each case we obtain an invariant of degree greater than 2α. As an example, consider $p = r = s = 0$. Then equations (7.12) can be solved provided we choose j and k such that

$$(k - j)\alpha = (j + 2k)\beta \ . \tag{7.15}$$

Thus

$$k - j = l\beta \ , \qquad j + 2k = l\alpha \ , \tag{7.16}$$

where $l \in \mathbf{Z}$. Solving for j and k gives

$$j = \frac{1}{3}l(\alpha - 2\beta) \ , \qquad k = \frac{1}{3}l(\alpha + \beta) \ , \tag{7.17}$$

and, since $\alpha + \beta$ is not a multiple of 3, l must be divisible by 3. Let $l = 3l', l' \in \mathbf{Z}$, then we find

$$\begin{aligned} m &= l'\beta(2\alpha - \beta) \ , \\ n &= l'\alpha(2\beta - \alpha) \ , \\ q &= -l'(\alpha^2 - \alpha\beta + \beta^2) \ . \end{aligned} \tag{7.18}$$

The order of the invariant $z_1^m z_2^n z_4^q$ is $3l'\alpha\beta$, which is greater than the order of the invariants (7.14). The other eight combinations, with three non-zero exponents, also give invariants of order either $3\alpha\beta$, $\alpha^2 + \alpha\beta$, or $2\alpha^2 + \alpha\beta - \beta^2$, all of which are greater than 2α.

Step 3. If $p = 0$, then it follows from (7.12) that

$$\begin{aligned} m &= j\beta - \frac{1}{3}(j - k)(\alpha + \beta) \ , \\ n &= k\beta - j\alpha + \frac{1}{3}(j - k)(\alpha + \beta) \ , \\ q - s &= j\beta - k\alpha - \frac{1}{3}(j - k)(\alpha + \beta) \ , \\ r - s &= k\beta + \frac{1}{3}(j - k)(\alpha + \beta) \ . \end{aligned} \tag{7.19}$$

We set $(k-j)=3l$, $l\in\mathbf{Z}$, because $\alpha+\beta$ in not a multiple of 3. Hence,

$$\begin{aligned} m &= j\beta + l(\alpha+\beta) \ , \\ n &= -j(\alpha-\beta)+l(2\beta-\alpha) \ , \\ q-s &= -j(\alpha-\beta)-l(2\alpha-\beta) \ , \\ r-s &= j\beta + l(2\beta-\alpha) \ , \end{aligned} \tag{7.20}$$

where $j,l\in\mathbf{Z}$.

We consider the three cases $q=0$, $r=0$, and $s=0$ separately. The invariant $z_1^{\beta}\bar{z}_2^{\alpha-\beta}\bar{z}_4^{\alpha-\beta}z_5^{\beta}$ in (7.14) is obtained in the case $p=s=0$ for $l=0$, $j=1$ in (7.20). The cases $p=q=0$ and $p=r=0$ lead to nontrivial invariant monomials of degree greater than 2α. For example, if $p=q=0$, the degree of the monomial is $|m|+|n|+|r|+|s|$. It follows from the restriction $\alpha>\beta>\alpha/2$ in Table 2 that

$$|m|+|n|+|r|+|s| > |m+n+r| = |l|(\alpha+4\beta) > 2|l|\alpha \ . \tag{7.21}$$

Hence, the degree of the monomial is greater than 2α unless $l=0$. However, if $l=0$, then

$$|m|+|n|+|r|+|s| = |j|(3\alpha-\beta) > 2|j|\alpha \ . \tag{7.22}$$

This proves that the monomials associated with $p=q=0$, $mnrs\neq 0$, are all of degree greater than 2α. The argument in the case $p=r=0$ is similar.

References

[1] I. Bosch Vivancos, P. Chossat and I. Melbourne. New planforms in systems of partial differential equations with Euclidean symmetry. Institut Non Linéaire de Nice preprint (1994).

[2] K. Brattkus and S.H. Davis. Cellular growth near absolute stability. *Phys. Rev. B 38* (1988) 11452–11460.

[3] F.H. Busse. Non-linear properties of thermal convection. *Rep. Prog. Phys.* **41** (1978) 1929–1967.

[4] E. Buzano and M. Golubitsky. Bifurcation on the hexagonal lattice and the planar Bénard problem. *Phil. Trans. R. Soc. Lond.* **A 308** (1983) 617–667.

[5] S.R. Coriell and G.B. McFadden. Morphological Stability. *Handbook of Crystal Growth* **1**. D.T.J. Hurle ed. (1993) 785–857.

[6] J.D. Crawford. $\mathbf{D}_4\dot{+}\mathbf{T}^2$ mode interactions and hidden symmetry. *Nonlinearity* **7** (1994) 697–739.

[7] B. Dionne. Spatially Periodic Patterns in Two and Three Dimensions. Ph.D. Thesis, University of Houston (1990).

[8] B. Dionne and M. Golubitsky. Planforms in two and three dimensions. *Z. Angew. Math. Phys.* **43** (1992) 36–62.

[9] V. Dufiet and J. Boissonade. Conventional and unconventional Turing patterns. *J. Chem. Phys.* **96** (1992) 664–673.

[10] W.S. Edwards and S. Fauve. Patterns and quasi-patterns in the Faraday experiment. *J. Fluid Mech.* **278** (1994) 123–148.

[11] M. Golubitsky, B. Dionne and I. Stewart. Coupled-cells, wreath products and direct products. *Dynamics, Bifurcations, Symmetry.* P. Chossat ed. (1994) 127–138.

[12] M. Golubitsky, I.N. Stewart and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory: Vol. II*. Appl. Math. Sci. Ser. **69**. Springer-Verlag, New York (1988).

[13] M. Golubitsky, J.W. Swift, and E. Knobloch. Symmetries and pattern selection in Rayleigh-Bénard convection. *Physica D* **10** (1984) 249–276.

[14] G.H. Gunaratne, Q. Ouyang, and H.L. Swinney. Pattern formation in the presence of symmetries. *Phys. Rev. E* **50**. (1994) 2802–2820.

[15] E. Ihrig and M. Golubitsky. Pattern selection with O(3) symmetry. *Physica D* **12** (1984) 1–33.

[16] K. Kirchgässner. Exotische Lösungen des Bénardschen Problems. *Math. Meth. in the Appl. Sci.* **1** (1979) 453–467.

[17] G.H. Knightly and D. Sather. Stability of cellular convection. *Arch. Rational Mech. Anal.* **97** (1987) 271–297.

[18] E. Knobloch. Pattern selection in long-wavelength convection. *Physica D* **41** (1990) 450–479.

[19] J.E. Marsden and M. McCracken. *The Hopf Bifurcation and its Applications.* Appl. Math. Sci. Ser. **19**. Springer, Berlin (1976).

[20] I. Melbourne. Steady-state bifurcation with Euclidean symmetry. University of Houston Research Report UH/MD-214 (1996).

[21] Q. Ouyang and H.L. Swinney. Transition from a uniform state to hexagonal and striped Turing patterns. *Nature* **352** (1991) 610–612.

[22] D.C. Sarocka and A.J. Bernoff. An intrinsic equation of interfacial motion for the solidification of a pure hypercooled melt. *Physica D* **85** (1995) 348–374.

[23] A. Schlüter, D. Lortz and F.H. Busse. On the stability of steady finite amplitude convection. *J. Fluid Mech.* **23** (1965) 129–144.

[24] M. Silber and E. Knobloch. Pattern Selection in Ferrofluids. *Physica D* **30** (1988) 83–98.

[25] M. Silber and E. Knobloch. Pattern selection in steady binary–fluid convection. *Phys. Rev. A* **38** (1988) 1468–1477.

[26] G.I. Sivashinsky. Instabilities, pattern-formation, and turbulence in flames. *Ann. Rev. Fluid Mech.* **15** (1983) 179–199.

[27] J.W. Swift. Bifurcation and Symmetry in Convection. Ph.D. thesis, U.C. Berkeley (1984).

[28] A.M. Turing. Chemical basis of morphogenesis. *Phil. Trans. R. Soc. London, Ser. B* **327** (1952) 37–72.

[29] A. Vanderbauwhede. Local Bifurcation and symmetry. Habilitation thesis. Rijksuniversiteit Gent (1980).

[30] D.J. Wollkind, R. Sriranganathan, and D.B. Oulton. Interfacial patterns during plane front alloy solidification. *Physica D* **12** (1984) 215–240.

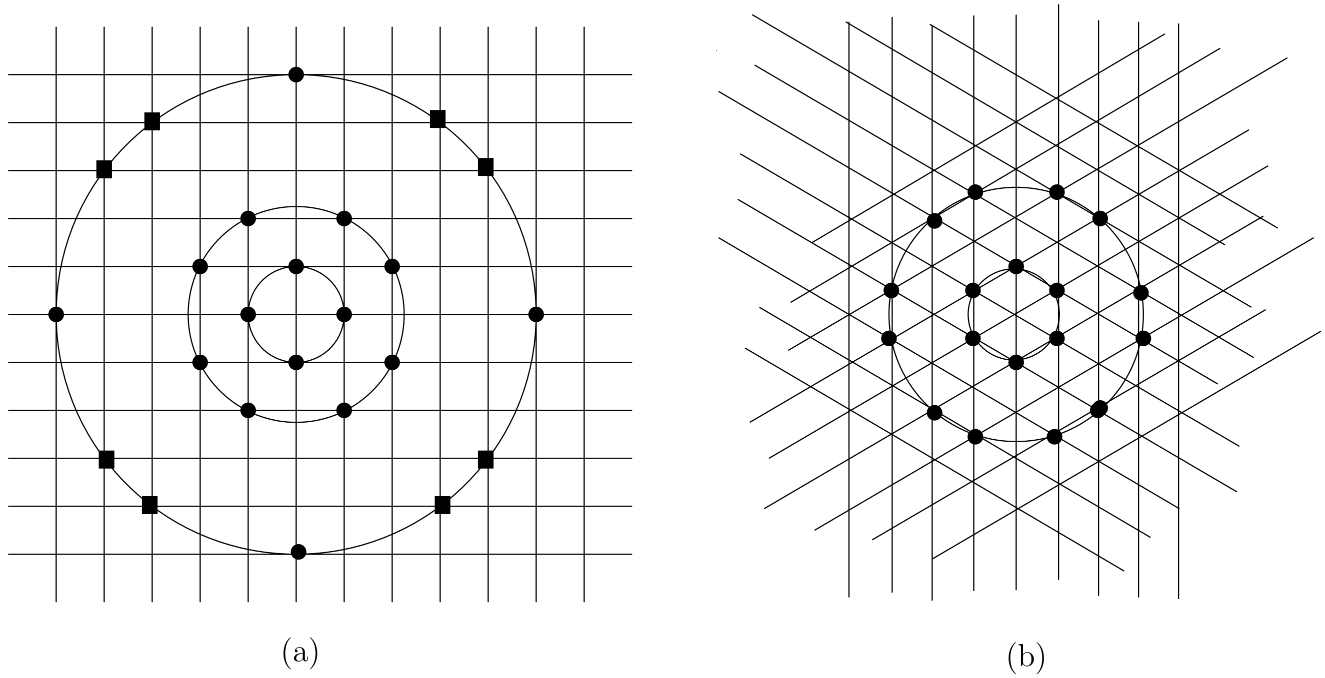

Figure 1: (a) Critical circles for the square lattice when $\mathbf{k}_c = 1$, $\sqrt{3}$ and 5. (b) Critical circles for the hexagonal lattice when $\mathbf{k}_c = 1$ and $\sqrt{7}$.

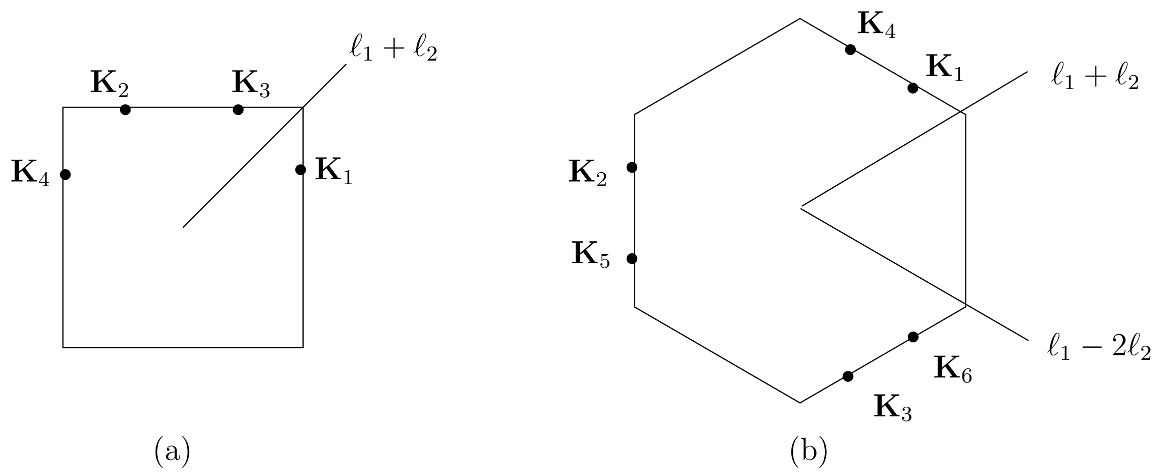

Figure 2: (a) Square lattice wave vectors $\mathbf{K}_j$ for the eight-dimensional representations of Γ_s in Table 1. (b) Hexagonal lattice wave vectors $\mathbf{K}_j$ for the twelve-dimensional representations of Γ_h in Table 2.

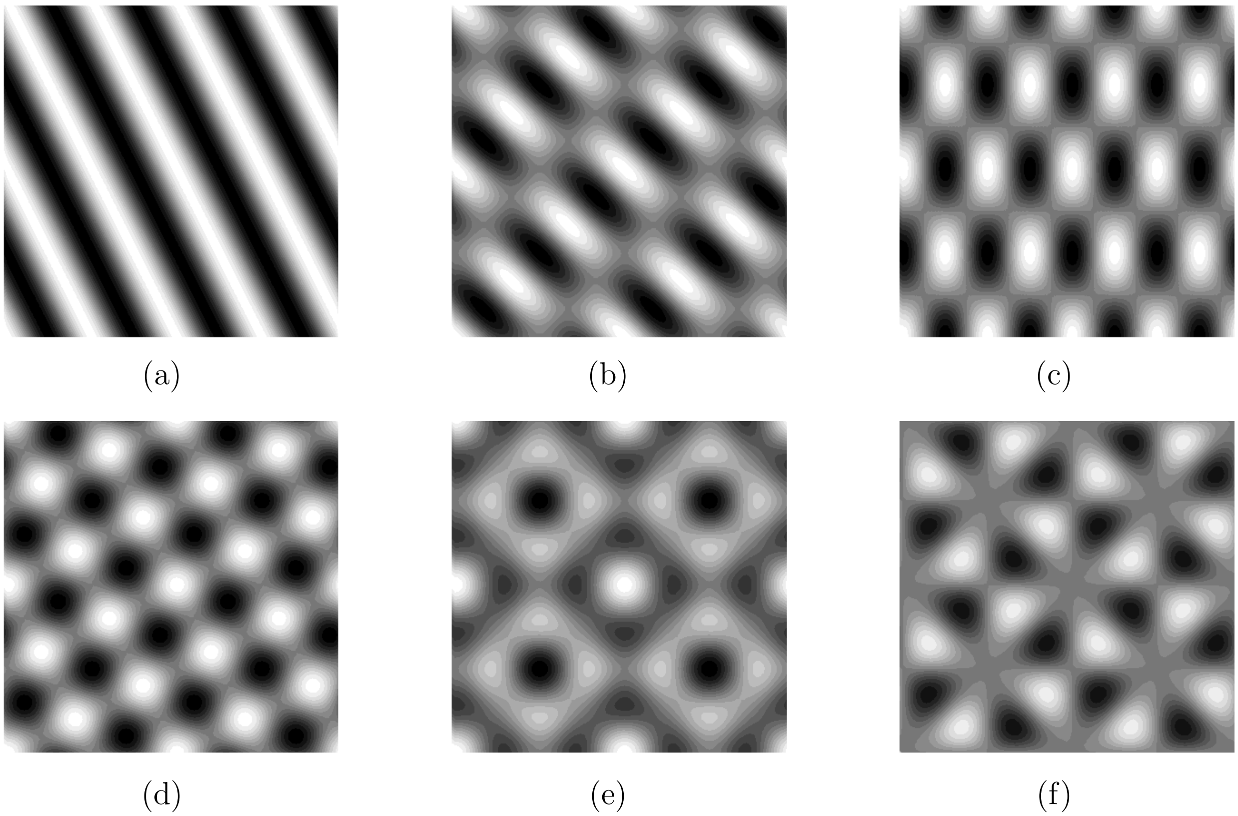

Figure 3: Axial planforms associated with 8-dimensional representation of Γ_s with $(\alpha, \beta) = (2, 1)$ and $x_1, x_2 \in [-1, 1]$ (*i.e.*, four copies of the fundamental domain are shown); (a) rolls, (b) rhombs ($\mathrm{Rh}_{s1,2,1}$), (c) rhombs ($\mathrm{Rh}_{s2,2,1}$), (d) simple squares, (e) super squares, and (f) anti-squares.

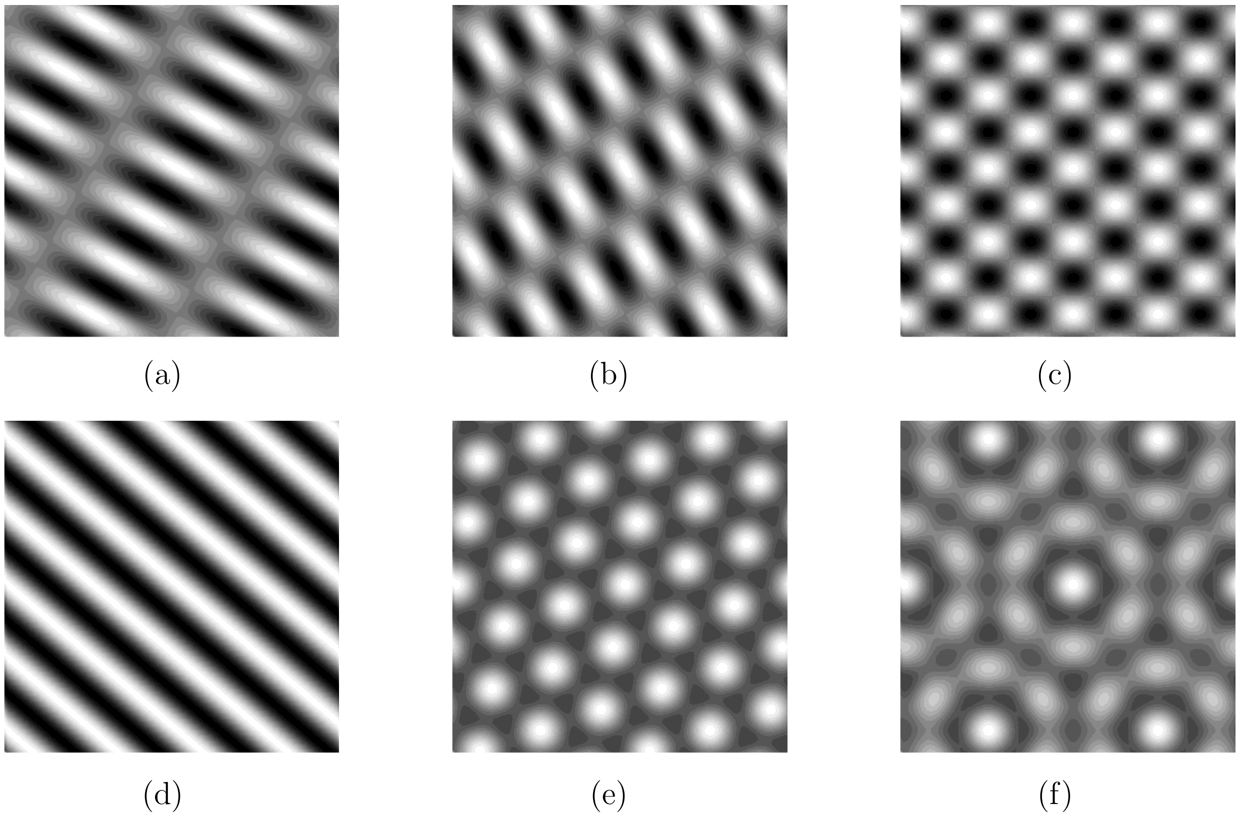

Figure 4: Axial planforms associated with 12-dimensional representation of Γ_h with $(\alpha, \beta) = (3, 2)$, $x_1, x_2 \in [-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}]$; (a) rhombs ($\text{Rh}_{h1,3,2}$), (b) rhombs ($\text{Rh}_{h2,3,2}$), (c) rhombs ($\text{Rh}_{h3,3,2}$), (d) rolls, (e) simple hexagons (SiH^+), and (f) super hexagons (SuH^+).

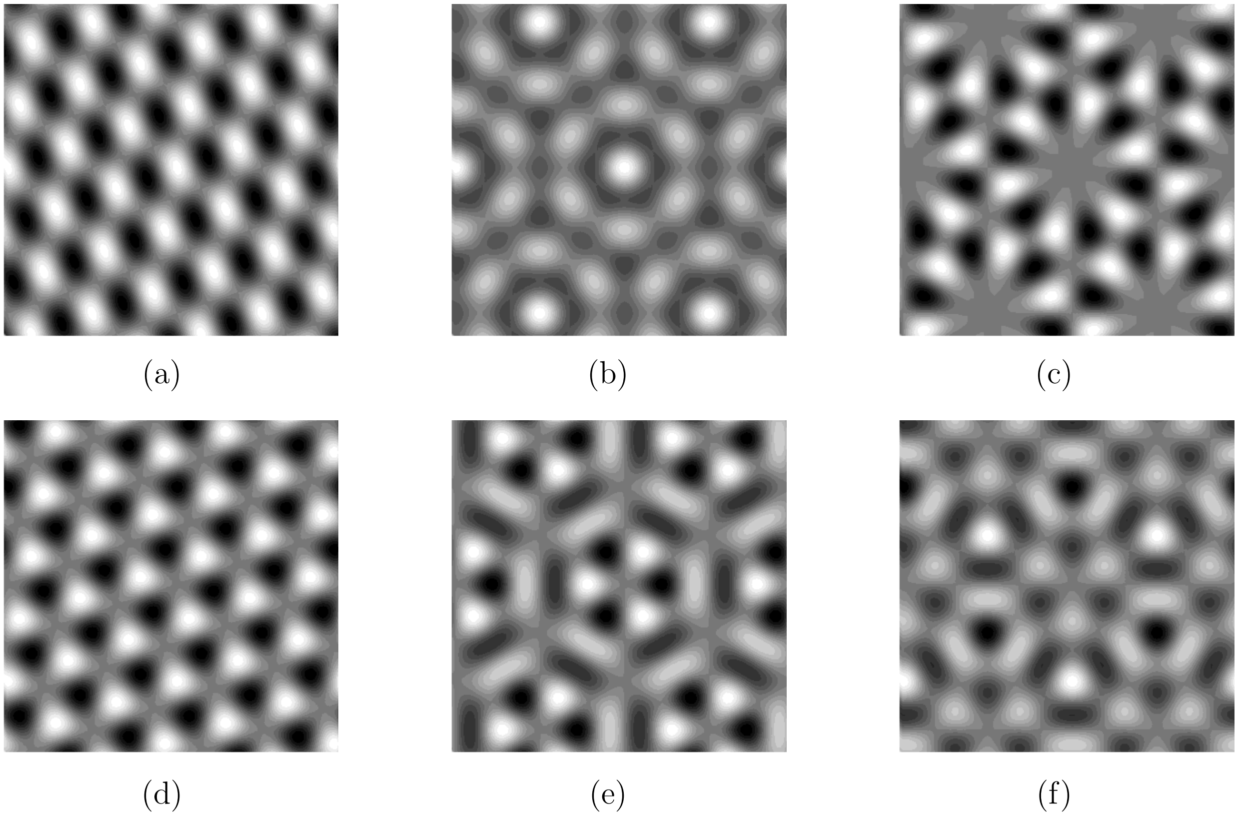

Figure 5: Axial planforms associated with 12-dimensional representation of $\Gamma_h \oplus \mathbf{Z}_2$ with $(\alpha, \beta) = (3, 2)$, $x_1, x_2 \in [-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}]$; (a) rhombs ($\mathrm{Rh}_{h0}$), (b) super hexagons, (c) anti-hexagons, (d) simple triangles, (e) super triangles, and (f) anti-triangles. (See Figure 4 for the additional axial planforms: rolls, simple hexagons, $\mathrm{Rh}_{h1,3,2}$, $\mathrm{Rh}_{h2,3,2}$, and $\mathrm{Rh}_{h3,3,2}$.)

Figure 6: Example of an hexagonal lattice bifurcation diagram for solutions in the six-dimensional subspace, $\mathbf{z} = (z_1, z_2, z_3, 0, 0, 0)$. Solid (dotted) lines indicate stable (unstable) solutions. The secondary solution branch has the form $\mathbf{z} = (x_1, x_2, x_2, 0, 0, 0)$, where $x_1, x_2 \in \mathbf{R}$.

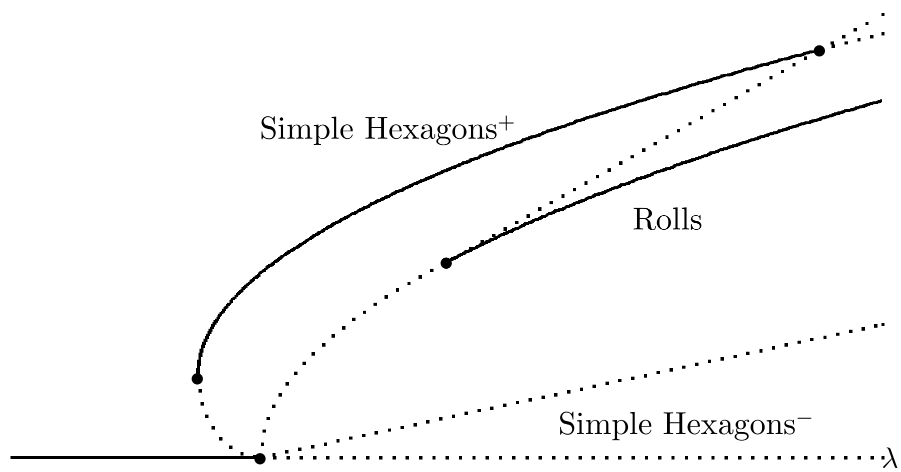

Figure 7: Example of an hexagonal lattice bifurcation diagram for the twelve-dimensional representations of Γ_h. Here $0 < \epsilon \ll 1$ in equation 5.14; see equation 5.15 for the other coefficients. Secondary bifurcation points are indicated by a solid circle; no secondary solution branches are shown.