Stability Transitions for Axisymmetric Relative Equilibria of Euclidean Symmetric Hamiltonian Systems

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December 2006[†]

Abstract

In the presence of noncompact symmetry, the stability of relative equilibria under momentum-preserving perturbations does not generally imply robust stability under momentum-changing perturbations. For axisymmetric relative equilibria of Hamiltonian systems with Euclidean symmetry, we investigate different mechanisms of stability: stability by energy-momentum confinement, KAM, and Nekhoroshev stability, and we explain the transitions between these. We apply our results to the Kirchhoff model for the motion of an axisymmetric underwater vehicle, and we numerically study dissipation induced instability of KAM stable relative equilibria for this system.

1 Introduction

Relative equilibria of Hamiltonian systems with symmetry are special solutions which are equilibria of the symmetry reduced dynamics. Relative equilibria are called stable if they are stable equilibria for the symmetry reduced dynamics. In other words, in a Hamiltonian system with symmetry group $\mathcal{G}$, a relative equilibrium is called stable (more precisely $\mathcal{G}$-stable [13,14,16]) if every time-orbit close to the relative equilibrium stays close the the $\mathcal{G}$-orbit of the relative equilibrium. Since the $\mathcal{G}$-orbit of the relative equilibrium usually strictly contains its time-orbit, $\mathcal{G}$-stability is usually weaker than orbital stability.

[†]\today: January 5, 2008

The stability of relative equilibria is delicate for Hamiltonian systems with noncompact symmetry. We have established [14] exactly why it is erroneous in general to conclude stability of relative equilibria under momentum-changing perturbations from stability under momentum-preserving perturbations. In the presence of noncompact symmetry, there is a gap between these. In that gap, energy-momentum confinement fails, meaning that stability under momentum-changing perturbations cannot be established by energy-momentum Lyapunov functions.

Here we investigate the stability of axisymmetric relative equilibria of Hamiltonian systems which admit the *Euclidean symmetry* $\mathcal{G} = \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)$. Our work bears on some results of Leonard and Marsden [7] concerning a class of relative equilibria in the 12-dimensional Kirchhoff model for an axially symmetric underwater vehicle, in which the vehicle falls and spins. They derive a condition for energy-momentum confinement under arbitrary perturbations ([7], Theorem 4.4), and they calculate that, at the boundary of this stability region, a Hamiltonian Hopf bifurcation occurs. We show that energy-momentum confinement under arbitrary perturbations actually occurs in a smaller region, but that, in the intervening gap, stability can be established by KAM methods. We also show that the coincidence of the gap boundary and the Hamiltonian-Hopf bifurcation is to be generally expected when the phase space is 12-dimensional.

In the underwater vehicle system, we provide numerical evidence that the KAM stability in the gap is destroyed by small dissipation, whereas the stability by energy-momentum confinement is preserved. In this system, the transition from energy-momentum region to the gap is spin independent, whereas the transition from the gap to spectral instability, at which the Hopf bifurcation occurs, does depend on spin. All the axisymmetric relative equilibria that are spin-stabilized are in the gap, and their KAM stability is destroyed by small momentum-preserving dissipation. The implication for gyroscopically stabilized devices is startling. *In the presence of noncompact symmetry, robust stability may not be achievable by the use of spin: dissipation induced loss of stability of the relative equilibrium will occur even in absence of dissipation of spin.* When the symmetry is noncompact, a general understanding of the kinds of stability which generically occur, and the transitions between them, is necessary for the determination of robust stability criteria.

This paper is structured as follows: We begin in Section 2 with a description of axisymmetric relative equilibria of Hamiltonian systems with Euclidean symmetry. We introduce coordinates related to the reduction of the system by its largest abelian subgroup $\mathcal{K} = \mathbb{R}^3 \times \mathrm{SO}(2)$ of $\mathcal{G}$. We obtain a family of Hamiltonian systems parametrized by the corresponding momenta. This lays plain the essential difficulty, because it shows that *perturbations to arbitrary momentum are* $\mathrm{SO}(2)$ *symmetry-breaking*, for an additional $\mathrm{SO}(2)$ symmetry present in the system which does not commute with the abelian subgroup $\mathcal{K}$. The symmetry-breaking occurs because the additional $\mathrm{SO}(2)$ symmetry acts on the whole family of Hamiltonian systems, including the parameterizing momenta, and acts on one single Hamiltonian system of the parameterized family only for certain *vertical momenta*; only those Hamiltonian systems are $\mathrm{SO}(2)$-symmetric. In Section 3 we study the stability of axisymmetric relative equilibria. In general,

we establish the presence of the gap and discuss different mechanisms of stability, and for Hamiltonian systems with 12-dimensional phase space, we establish the Hopf eigenvalue collision at the gap boundary. In Section 4 we begin with a brief summary of the Kirchhoff model. We recover the stability criteria of [7] for the falling, spinning relative equilibria, and prove stability of the relative equilibria within the gap by verifying the Moser twist condition for the corresponding equilibrium on the reduced space. In Section 5 we numerically demonstrate that addition of small dissipation distinguishes energy-momentum from KAM stability, by observing that stability is maintained in the former and destroyed in the latter by the addition of momentum conserving dissipation.

Acknowledgments


This work benefited from research stays at the Banff Research Station and the Bernoulli Center of EPFL. CW was supported by a grant from the Nuffield Foundation and by the EPSRC First Grant Scheme. GWP is supported by the Natural Sciences and Engineering Research Council of Canada.

2 Dynamics near axisymmetric relative equilibria


In this section we introduce axisymmetric relative equilibria of Hamiltonian systems, and introduce suitable coordinates near them for use in the stability analysis in Section 3. We consider a general context that includes the underwater vehicle example in Section 4. Let

$$\dot{x} = f_H(x) \tag{2.1}$$

be a Hamiltonian system defined by an energy $H\colon \mathcal{M} \to \mathbb{R}$ on a connected symplectic manifold $(\mathcal{M}, \omega)$, i.e.

$$\omega(x)\big(f_H(x), w\big) = \mathrm{D}H(x)w \quad \text{for all } x \in \mathcal{M}, w \in \mathcal{T}_x\mathcal{M}.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the unit vectors along the x, y, z axis of $\mathbb{R}^3$, and for any $v \in \mathbb{R}^3$ denote

$$v^\wedge = \widehat{v} = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix}, \tag{2.2}$$

so that, for example,

$$\exp(\widehat{\mathbf{e}}_3\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define the group

$$\mathcal{G} = \mathrm{SE}(2) \times \mathbb{R} \times \mathrm{SO}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2),$$

where $\ltimes$ stands for a semidirect product in that the multiplication for $g = (\phi, a, \theta) \in \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)$ is

$$g_1 g_2 = (\phi_1, a_1, \theta_1)(\phi_2, a_2, \theta_2) = \big(\phi_1 + \phi_2, a_1 + \exp(\widehat{\mathbf{e}}_3 \phi_1) a_2, \theta_1 + \theta_2\big).$$

Assume $\mathcal{G}$ acts symplectically on $\mathcal{M}$, i.e. ω is $\mathcal{G}$-invariant, and suppose that H is $\mathcal{G}$-invariant. This implies that the vector field f_H commutes with $\mathcal{G}$, as does its flow, i.e. f_H and its flow are $\mathcal{G}$-equivariant. Also, assume that $\mathcal{G}$ acts properly and faithfully.

In the following, $\mathrm{SO}(2)_R$ is the copy of $\mathrm{SO}(2)$ in $\mathcal{G}$ which lies in the center of $\mathcal{G}$, so that $\theta \in \mathrm{SO}(2)_R$, and $\mathrm{SO}(2)_L$ is such that $\phi \in \mathrm{SO}(2)_L$. Here "$R$" stands for right and "$L$" for left. In the application to axisymmetric underwater vehicles, see Section 4 below, the $\mathrm{SO}(2)_L \ltimes \mathbb{R}^3$ is related to spatial isotropy, it corresponds to a left multiplication, and it translates and rotates the body in space. The $\mathrm{SO}(2)_R$ action is related to a material symmetry, it corresponds to a right multiplication, and it spins the body around a symmetry axis.

By Noether's theorem there are $\dim \mathcal{G} = 5$ locally defined conserved quantities of (2.1). We assume these exist globally and organize them in the momentum map $\boldsymbol{J} \colon \mathcal{M} \to \mathfrak{g}^*$ such that $\omega(x)(\xi x, w) = \mathrm{D}\boldsymbol{J}_\xi(x) w$ for all $x \in \mathcal{M}$, $w \in \mathcal{T}_x\mathcal{M}$, $\xi \in \mathfrak{g}$. Here $\mathfrak{g} = \mathfrak{so}(2)_L \ltimes \mathbb{R}^3 \times \mathfrak{so}(2)_R = \mathcal{T}_{\mathrm{id}}\mathcal{G}$ is the Lie algebra of $\mathcal{G}$. Denote the components of $\boldsymbol{J}$ by

$$\boldsymbol{J}(x) = (\boldsymbol{J}^\phi, \boldsymbol{J}^a, \boldsymbol{J}^\theta)(x), \quad \boldsymbol{J}^\phi(x), \boldsymbol{J}^\theta(x) \in \mathfrak{so}(2)^*, \quad \boldsymbol{J}^a(x) \in (\mathbb{R}^3)^* \tag{2.3}$$

and the elements of $\mathfrak{g}^*$ by

$$\mu = (\mu^\phi, \mu^a, \mu^\theta) \in \mathfrak{g}^* = \mathfrak{so}(2)_L^* \ltimes \mathbb{R}^3 \times \mathfrak{so}(2)_R^*.$$

In the underwater vehicle example of Section 4, the component $\boldsymbol{J}^a$ is the linear momentum, $\boldsymbol{J}^\phi$ the angular momentum, and $\boldsymbol{J}^\theta$ the momentum of the spin of the body. As in that system, we will assume that the momentum mapping $\boldsymbol{J}$ transforms by the coadjoint action on $\mathfrak{g}^*$, i.e. [8]

$$\boldsymbol{J}(gx) = (\mathrm{Ad}_g^*)^{-1} \boldsymbol{J}(x) \quad \text{for any } g \in \mathcal{G} \text{ and } x \in \mathcal{M}.$$

Here Ad_g^* is defined by the condition $(\mathrm{Ad}_g^* \mu)(\xi) = \mu(\mathrm{Ad}_g \xi)$, $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, $g \in \mathcal{G}$, and $\mathrm{Ad}_g \xi = g \xi g^{-1}$. A standard computation (see, for example, [8]) gives

$$\begin{aligned}
&\boldsymbol{J}^\phi\big((\phi, a, \theta)x\big) = \boldsymbol{J}^\phi(x) - (\exp(\widehat{\mathbf{e}}_3 \phi) a)_2 \boldsymbol{J}^{a_1}(x) + (\exp(\widehat{\mathbf{e}}_3 \phi) a)_1 \boldsymbol{J}^{a_2}(x), \\
&\boldsymbol{J}^{a_1}\big((\phi, a, \theta)x\big) = \boldsymbol{J}^{a_1}(x) \cos\phi - \boldsymbol{J}^{a_2}(x) \sin\phi, \\
&\boldsymbol{J}^{a_2}\big((\phi, a, \theta)x\big) = \boldsymbol{J}^{a_2}(x) \cos\phi + \boldsymbol{J}^{a_1}(x) \sin\phi, \\
&\boldsymbol{J}^{a_3}\big((\phi, a, \theta)x\big) = \boldsymbol{J}^{a_3}(x), \\
&\boldsymbol{J}^\theta\big((\phi, a, \theta)x\big) = \boldsymbol{J}^\theta(x).
\end{aligned} \tag{2.4}$$

2.1 Axisymmetric relative equilibria of Euclidean group actions

Let x_e be a relative equilibrium of (2.1) i.e. let

$$\xi_e = (\xi_e^\phi, \xi_e^a, \xi_e^\theta) \in \mathfrak{so}(2) \oplus \mathbb{R}^3 \oplus \mathfrak{so}(2) = \mathfrak{g},$$

and suppose $\exp(\xi_e t)x_e$ is a solution of the differential equations $\dot{x} = f_H(x)$. We assume x_e is *axisymmetric*, meaning that its isotropy subgroup $\mathcal{G}_{x_e} = \{g \in \mathcal{G} : gx_e = x_e\}$ is the diagonal subgroup

$$\mathrm{SO}(2)_D = \big\{(\phi, 0, \phi) \in \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)\big\} \subseteq \mathcal{G}.$$

As is general for Hamiltonian systems with symmetry, if η_e is in the isotropy algebra $\mathfrak{g}_{x_e}$ then $(\xi_e + \eta_e)x_e = \xi_e x_e = f_H(x_e)$ because $\eta_e x_e = 0$, hence $\exp\big(t(\xi_e + \eta_e)\big)x_e$ is also a solution of the Hamiltonian system. Thus ξ_e is only determined up to addition of elements in

$$\mathfrak{g}_{x_e} = \mathfrak{so}(2)_D = \big\{(\xi^\phi, 0, \xi^\phi) \in \mathfrak{so}(2) \ltimes \mathbb{R}^3 \times \mathfrak{so}(2)\big\} \subseteq \mathfrak{g},$$

and so we can choose $\xi_e^\phi = 0$. In the underwater vehicle, x_e is an equilibrium in a frame that co-moves with the action of $\exp(\xi_e t)$, $\omega^{\mathrm{rot}} = \xi_e^\phi - \xi_e^\theta = -\xi_e^\theta$ is the angular velocity, and ξ_e^a is the translational velocity. The momentum value $\mu_e = \boldsymbol{J}(x_e)$ of the relative equilibrium is of the form $\mu_e = (\mu_e^\phi, \mu_e^a, \mu_e^\theta)$ with $\mu_e^a \parallel \mathbf{e}_3$, because it is fixed under the action of $\mathrm{SO}(2)_D$ that occurs in the transformation rule (2.4). The momentum map $\boldsymbol{J}^{\mathrm{SO}(2)_D}$ of the action of the symmetry group $\mathrm{SO}(2)_D$ is

$$\boldsymbol{J}^{\mathrm{SO}(2)_D}(x) = \boldsymbol{J}^\phi(x) + \boldsymbol{J}^\theta(x).$$

It follows from $\xi_e^\phi = 0$ that x_e is a relative equilibrium for the abelian symmetry group

$$\mathcal{K} = \mathbb{R}^3 \times \mathrm{SO}(2)_R = \big\{(0, a, \phi) \in \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)\big\} \subseteq \mathcal{G}, \tag{2.5}$$

and x_e becomes an equilibrium after reduction by this group. The stability of this equilibrium implies stability of the relative equilibrium, see Section 2.2 below. We will study the stability of such an axisymmetric relative equilibrium, which in the underwater vehicle example of Section 4, corresponds to a vehicle spinning about its symmetry axis with angular velocity ω^{rot}, and translating along its symmetry axis with translational velocity ξ_e^a.

2.2 Reduction by $\mathcal{K} = \mathbb{R}^3 \times \mathrm{SO}(2)_R$

The following theorem provides the coordinates which we require for the subsequent stability analysis.

Theorem 2.1. *In a $\mathcal{G}$-invariant neighborhood of an axisymmetric relative equilibrium x_e of (2.1) there are coordinates $x = (a, \theta, \nu^a, \nu^\theta, w)$ such that*

(a) $\boldsymbol{J}^a = \nu^a$, $\boldsymbol{J}^\theta = \nu^\theta$, *and the differential equations* (2.1) *are*

$$\dot{a} = \mathrm{D}_{\nu^a} H(\nu, w), \quad \dot{\theta} = \mathrm{D}_{\nu^\theta} H(\nu, w), \quad \dot{\nu}^a = 0, \quad \dot{\nu}^\theta = 0, \quad \dot{w} = \mathbb{J}\mathrm{D}_w H(\nu, w), \quad (2.6)$$

where $\nu = (\nu^a, \nu^\theta)$, $\mathbb{J}$ *is the standard symplectic structure matrix on the linear symplectic space* $W = \mathbb{R}^{\dim \mathcal{M}-8}$, *and* $w \in W$.

(b) The Hamiltonian $H(\nu, w)$ *is invariant under the action of* $\mathrm{SO}(2)_D$, *which takes the form*

$$\nu^a \to \exp(\phi \widehat{\mathbf{e}}_3)\nu^a, \quad \nu^\theta \to \nu^\theta, \quad w \to R_\phi(\nu, w),$$

i.e. the action on w *generally depends on* ν *and this equation defines* R_ϕ.

(c) The coordinates of the axisymmetric relative equilibrium x_e *are*

$$a = 0, \quad \theta = 0, \quad \nu = \nu_e = \big(\boldsymbol{J}^a(x_e), \boldsymbol{J}^\theta(x_e)\big), \quad w = 0,$$

and $w = 0$ *is an equilibrium of the* $\dot{w}$*-equation of* (2.6) *at* $\nu = \nu_e$.

The $a = 0, \theta = 0$ plane is locally a slice at x_e through the action of $\mathcal{K} = \mathbb{R}^3 \times \mathrm{SO}(2)_R$ and models the Poisson reduced space $\mathcal{M}/\mathcal{K}$. The symplectic leaves of this Poisson space are given by fixing ν. At fixed ν, the system

$$\dot{w} = \mathbb{J}\mathrm{D}_w H(\nu, w) \quad (2.7)$$

is the (Marsden-Weinstein) symplectic reduced system [8] corresponding to the abelian subgroup $\mathcal{K}$ at momentum ν, obtained simply by "ignoring cyclic coordinates". The additional 1-dimensional symmetry is expected because $\mathcal{K}$ is a codimension 1 subgroup of $\mathcal{G}$. Most of the proof follows from the general theory developed in [15], but since the symmetry group $\mathcal{K}$ is abelian there is also the following elementary proof.

Proof of Theorem 2.1. The four momenta $\boldsymbol{J}^a_1$, $\boldsymbol{J}^a_2$, $\boldsymbol{J}^a_3$, $\boldsymbol{J}^\theta$ Poisson commute since $\mathcal{K}$ is abelian, and their derivatives are linearly independent at x_e since the isotropy group of x_e is not bigger than $\mathrm{SO}(2)_D$. Define $\nu^a = \boldsymbol{J}^a, \nu^\theta = \boldsymbol{J}^\theta$, and using the Darboux theorem [18], choose four conjugate functions, i.e. four functions a_1, a_2, a_3, θ, such that

$$\{a_i, \nu^a_j\} = \delta_{ij}, \quad \{\theta, \nu^\theta\} = 1, \quad \{a_i, \nu^\theta\} = 0.$$

Find functions $q_1, \ldots, q_k, p_1, \ldots, p_k$, $k = \frac{1}{2}(\dim \mathcal{M} - 8)$, that Poisson commute with the original 8 and satisfy

$$\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0.$$

Set $w = (q, p)$, and translate so that x_e is at $w = 0$. The group $\mathcal{K}$ acts by addition on the conjugate coordinates a_1, a_2, a_3, θ since this action is generated by the momenta $\nu^{a_1}, \nu^{a_2}, \nu^{a_3}, \nu^\theta$. So H does not depend on the coordinates a_1, a_2, a_3, θ, and we have the $\nu = (\nu^a, \nu^\theta)$ parametrized canonical system on the linear symplectic space W given by the system (2.7) of (2.6). The other equations of (2.6) follow immediately.

The symmetry group $\mathrm{SO}(2)_D$ acts on the variables (w, ν) independently of a, θ. Indeed, if f is a function which does not depend on a, θ, if x is one of a_1, a_2, a_3, θ and the conjugate of x is $\bar{x}$, then

$$\begin{aligned}\frac{\partial}{\partial x}\{\boldsymbol{J}^\theta + \boldsymbol{J}^\phi, f\} &= \{\bar{x}, \{\boldsymbol{J}^\theta + \boldsymbol{J}^\phi, f\}\} \\ &= -\{\boldsymbol{J}^\theta + \boldsymbol{J}^\phi, \{f, \bar{x}\}\} - \{f, \{\bar{x}, \boldsymbol{J}^\theta + \boldsymbol{J}^\phi\}\}. \end{aligned} \tag{2.8}$$

The first term of (2.8) is zero because

$$\{f, \bar{x}\} = -\frac{\partial f}{\partial x} = 0.$$

For the second term of (2.8), note that $\bar{x}$ is one of $\nu^{a_1}, \nu^{a_2}, \nu^{a_3}, \nu^\theta$, which are momenta of $\mathcal{K}$. Since $\mathcal{K}$ is a normal subgroup of $\mathcal{G}$, its momenta are a Poisson ideal of the momenta $\boldsymbol{J}_\xi$, $\xi \in \mathfrak{g}$. This can also be seen directly from (2.4) because the Poisson brackets of $\boldsymbol{J}^\phi$ with the other components of the momentum map $\boldsymbol{J}$ are found by differentiation in ϕ at $\phi = 0$ of the right hand sides of (2.4). The results are linear combinations of $\boldsymbol{J}^\theta$ and $\boldsymbol{J}^a_i$, $i = 1, 2, 3$. In particular, $\{\bar{x}, \boldsymbol{J}^\theta + \boldsymbol{J}^\phi\}$ is a linear combination of $\nu^{a_1}, \nu^{a_2}, \nu^{a_3}, \nu^\theta$, so the second term of (2.8) is zero for the same reason as the first. Thus there is an action of $\mathrm{SO}(2)_D$ on the variables (ν, w). Since $\boldsymbol{J}$ is equivariant and $\boldsymbol{J} = (\boldsymbol{J}^\phi, \nu)$, the resulting action on the ν variables is independent of w and equal to the coadjoint action of $(\theta, 0, \theta)$ on ν, which by (2.4) is a rotation of (ν^a_1, ν^a_2) by θ. The coordinates above can be restricted to an $\mathrm{SO}(2)_D$ invariant neighborhood because x_e is fixed by the action of this compact group and are therefore coordinates in a $\mathcal{G}$-invariant neighbourhood of $\mathcal{G}x_e$.

Definition 2.2. *A momentum $\nu \in \left(\mathbb{R}^3 \times \mathfrak{so}(2)\right)^*$ is vertical if it is fixed by the coadjoint action of $\mathrm{SO}(2)_D$, i.e. if $\nu^a \parallel \mathbf{e}_3 \in \mathbb{R}^3$.*

For vertical ν, the system (2.7) inherits an $\mathrm{SO}(2)_D$ symmetry from the full phase space $\mathcal{M}$. It does not have this additional $\mathrm{SO}(2)_D$ symmetry for nonvertical ν. Hence a perturbation of the $\mathrm{SO}(2)_D$-symmetric Hamiltonian system (2.7) at a vertical momentum value ν to a non-vertical momentum value is an $\mathrm{SO}(2)_D$-symmetry breaking perturbation.

Proposition 2.3. *For vertical momenta ν, the $\mathrm{SO}(2)_D$ action on W is symplectic with momentum map*

$$\boldsymbol{J}_W = \boldsymbol{J}^{\mathrm{SO}(2)_D}(a, \theta, \nu, w)\Big|_{a=0,\,\theta=0}.$$

In particular, at vertical momentum ν, the system (2.7) is $\mathrm{SO}(2)_D$ symmetric and conserves $\boldsymbol{J}_W$.

Proof. The action of $\mathrm{SO}(2)_D$ in the coordinates $(a, \theta, \nu^a, \nu^\theta, w)$ of Theorem 2.1 is symplectic since it is the action of a subgroup of $\mathcal{G}$, which acts symplectically by assumption. It follows that $\mathrm{SO}(2)_D$ acts symplectically on W, since the symplectic

form on W at fixed ν is the restriction of the symplectic form on $\mathcal{M}$. Using the symmetry properties (2.4) of the momentum map,

$$\begin{aligned}\boldsymbol{J}_W(\nu,w) &= \boldsymbol{J}^{\mathrm{SO}(2)_D}(0,0,\nu^a,\nu^\theta,w)\\ &= (\boldsymbol{J}^\phi+\boldsymbol{J}^\theta)\big((-a,-\theta)(a,\theta,\nu^a,\nu^\theta,w)\big)\\ &= (\boldsymbol{J}^\phi+\boldsymbol{J}^\theta)(a,\theta,\nu^a,\nu^\theta,w)+a_2\nu_1^a-a_1\nu_2^a\\ &= \boldsymbol{J}^{\mathrm{SO}(2)_D}(a,\theta,\nu^a,\nu^\theta,w)+a_2\nu_1^a-a_1\nu_2^a,\end{aligned} \tag{2.9}$$

so $\boldsymbol{J}_W = \boldsymbol{J}^{\mathrm{SO}(2)_D}$ if $\nu_1^a = \nu_2^a = 0$. Thus for fixed vertical ν, $\boldsymbol{J}_W$ is equal to the momentum generating the $\mathrm{SO}(2)_D$ action on W.

Remark 2.4. Equation (2.9) and the differential equation for $\dot{a}$ in the system (2.6) implies that the conservation of $\boldsymbol{J}_W$ is not typical for nonvertical momenta. Indeed, if $\boldsymbol{J}_W$ is conserved at some fixed nonvertical $\nu = \nu_0$, then adding any function $\epsilon\tilde{H}(\nu,w)$ to H results in

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_2\nu_{01}^a - a_1\nu_{02}^a) = \epsilon\left(\frac{\partial\tilde{H}}{\partial\nu_2^a}\nu_{01}^a - \frac{\partial\tilde{H}}{\partial\nu_1^a}\nu_{02}^a\right). \tag{2.10}$$

We can choose an $\mathrm{SO}(2)_D$ invariant function $\tilde{H}$ such that the right side of (2.10) is not everywhere zero for ϵ arbitrarily small.

Remark 2.5. Locally, the symmetry reduced space takes the form $(\nu,w) \in \mathfrak{k}^* \oplus W$ where $\mathfrak{k}$ is the Lie algebra of $\mathcal{K}$. The space W is called the *symplectic normal space* at x_e with respect to the symmetry group $\mathcal{K}$, see [15], since it is transversal to the group orbit $\mathcal{K}x_e$ at x_e and is a symplectic space. Moreover it is the largest symplectic subspace of the normal space $\mathcal{N}$ to the group orbit $\mathcal{G}x_e$ at x_e. In the notation of [15], we have $\mathcal{N}_0 = \mathfrak{k}^*$, $\mathcal{N}_1 = W$. An application of the results of [15] would mean a reduction by the group $\mathcal{H} = \mathrm{SO}(2)_{AD} \ltimes \mathbb{R}^3$ where

$$\mathrm{SO}(2)_{AD} = \big\{(\phi,0,-\phi) \in \mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)\big\}$$

is the antidiagonal embedding of $\mathrm{SO}(2)$ in $\mathrm{SO}(2)_L \times \mathrm{SO}(2)_R$. But the resulting Poisson structure on $\mathcal{N}_0 = \mathfrak{h}^* \simeq \mathfrak{se}(2)^* \oplus \mathbb{R}^*$ (where $\mathfrak{h}$ is the Lie algebra of $\mathcal{H}$) is nontrivial since the symmetry group $\mathcal{H}$ is nonabelian, so that $\mathcal{N} \simeq \mathcal{N}_0 \oplus \mathcal{N}_1$. In particular the symplectic leaves have nonconstant dimension at the axisymmetric relative equilibrium. For this reduction the conserved quantity $\boldsymbol{J}^{\mathrm{SO}(2)_D}$ is a Poisson momentum map for the $\mathrm{SO}(2)_D$ action on $\mathcal{N}_0 \oplus \mathcal{N}_1$, and this momentum map is a conserved quantity for the reduced system on $\mathcal{N}_0 \oplus \mathcal{N}_1$. But the nontrivial Poisson structure on $\mathcal{N}_0$ would make the stability analysis more difficult. We would have to employ the general methods developed in [14] to study stability by energy-momentum confinement and we would have to desingularize the Poisson structure using blow up methods as in [13] to study the KAM stability of the axisymmetric relative equilibrium x_e.

Remark 2.6. In Theorem 2.12 we could also construct an $\mathrm{SO}(2)_D$ invariant Witt decomposition of the tangent space $\mathcal{T}_{x_e}\mathcal{M}$ and then use the equivariant Darboux theorem to obtain a model $\mathcal{K} \times \mathfrak{k}^* \times W$ of a $\mathcal{G}$-invariant neighborhood of $\mathcal{G}x_e$. In this

construction the $\mathrm{SO}(2)_D$ action on W would be linear and ν-independent, see [15]. Note however that in this case $\mathfrak{k}^*$ is not the annihilator of $\mathfrak{g}_{x_e} = \mathfrak{so}(2)_D$, so the situation here is not exactly the setting of [15].

3 Stability of axisymmetric relative equilibria

In this section we use the coordinates from Theorem 2.1 to study the stability of the axisymmetric relative equilibrium x_e. We show that definiteness of the Hessian of $H(\nu_e, w)$ at the equilibrium $w = 0$ of (2.7) implies stability of x_e under arbitrary perturbations. When this Hessian is not definite, for momentum-preserving perturbations, a constant times the momentum $\boldsymbol{J}_W$ can be added to the energy to obtain a Lyapunov function of the system (2.7). However, for nonvertical linear momentum, $\boldsymbol{J}_W$ is not a conserved quantity of (2.7), so it *cannot* be used in a Lyapunov function to establish stability for perturbations to nonvertical linear momentum. The *gap* (Sections 1 and 3.1) is defined to be those relative equilibria for which a constant times the momentum $\boldsymbol{J}_W$ *must* be added to obtain a Lyapunov function. If x_e is in the gap then the equilibrium $w = 0$ of (2.7) has imaginary spectrum and indefinite Hessian, and is amenable to classical KAM and Nekhoroshev results (Section 3.1) to obtain stability.

In the case of a 12-dimensional phase space $\mathcal{M}$, the system (2.7) has 2 degrees of freedom, and Propositions 3.9–3.11 below show that formal stability (Section 3.1) of the axisymmetric relative equilibrium x_e fails exactly at an eigenvalue collision. Moreover we show that this is exactly where a Hamiltonian Hopf bifurcation typically occurs and a loss of linear stability is expected. Thus, in the case of a 12-dimensional phase space, formal stability and linearized stability of axisymmetric relative equilibria are typically equivalent, explaining the observation of this by Leonard and Marsden in the Kirchhoff model of axisymmetric underwater vehicles [7].

3.1 Nonlinear Stability

The relative equilibrium x_e is called *A stable* for some subset A of $\mathcal{G}$, if initial data starting sufficiently close to x_e stay arbitrarily close to Ax_e for all times. It is *formally stable* if $\mathrm{D}^2(H + \lambda \boldsymbol{J}_W)(x_e)|_W$ is definite for some $\lambda \in \mathbb{R}$. Recall from (2.5) that $\mathcal{K} = \mathbb{R}^3 \times \mathrm{SO}(2)_R$.

Proposition 3.1. *The axisymmetric relative equilibrium x_e of* (2.1) *is $\mathcal{K}$-stable if the Hessian $\mathrm{D}^2_w H(\nu_e, 0)$ is definite.*

Proof. If the Hessian $\mathrm{D}^2_w H(\nu_e, 0)$ of the Hamiltonian system (2.7) is definite then, using energy as a Lyapunov function, the equilibrium $w = 0$ of (2.7) is stable under perturbations in w, i.e. within $\nu = \nu_e$. The equilibrium $w = 0$ persists to nearby momentum values ν, and the corresponding equilibria of (2.7) at those momentum values are also stable. Since the momentum $\nu(t)$ is a conserved quantity of (2.6) we conclude that, for initial data close to $\nu = \nu_e, w = 0$, the solution $\big(\nu(t), w(t)\big)$ of (2.6) stays close to $(\nu_e, 0)$ for all times. Since the only remaining variables are a, θ, and these correspond to the action of $\mathcal{K}$, the proof is complete. 

Remark 3.2. Better stability criteria cannot be obtained by relaxing to the weaker stability modulo the whole group $\mathcal{G}$, instead of just the subgroup $\mathcal{K}$, because the isotropy of x_e implies that these two stabilities are equivalent.

Let $\boldsymbol{J}^{\mathcal{K}} = (\boldsymbol{J}^a, \boldsymbol{J}^\theta)$ denote the momentum map for the symplectic action of $\mathcal{K}$ on $\mathcal{M}$.

Proposition 3.3. *The axisymmetric relative equilibrium x_e of (2.1) is $\mathcal{K}$-stable under perturbations that preserve the momentum $\boldsymbol{J}^{\mathcal{K}}$ of $\mathcal{K}$ if it is formally stable.*

Proof. A perturbation which preserves $\boldsymbol{J}^{\mathcal{K}}$ does not change ν and therefore is a perturbation of the stable equilibrium $w = 0$ of the system (2.7). Since $\boldsymbol{J}_W$ is a conserved quantity of (2.7) at $\nu = \nu_e$ by Proposition 2.3, the equilibrium $w = 0$ of (2.7) is stable if there is some $\lambda \in \mathbb{R}$ such that $\mathrm{D}^2 H(0) + \lambda \mathrm{D}^2 \boldsymbol{J}_W(0)$ is definite. 

So we see that the presence of the $\mathrm{SO}(2)_D$ symmetry and its momentum $\boldsymbol{J}_W(w)$ at vertical momenta $\nu \parallel \mathbf{e}_3$ causes a gap between energy-momentum confinement under momentum-preserving and general perturbations of x_e.

Definition 3.4. *Let $\mathcal{A}_e$ be the set of axially symmetric relative equilibria. The EM-region is the subset $\mathcal{A}_e^{\mathrm{EM}} \subseteq \mathcal{A}_e$ such that $\mathrm{D}_w^2 H(\nu_e, 0)$ is definite. The gap is the subset $\mathcal{A}_e^{\mathrm{gap}} \subseteq \mathcal{A}_e$ that is formally stable but not in the EM-region.*

Remark 3.5. If the symmetry group $\mathcal{G}$ is compact then the gap is absent since formal stability implies $\mathcal{G}$-stability, see e.g. [12, 14].

We will say that the axisymmetric relative equilibrium x_e is *nondegenerate* if the Hessian $\mathrm{D}_w^2 H(\nu_e, 0)$ of the equilibrium $w = 0$ of (2.7) is invertible. Suppose that $x_e \in \mathcal{A}_e^{\mathrm{gap}}$ is nondegenerate. Then $w = 0$ is a stable equilibrium of (2.7), so the linearization $\mathbb{J}\mathrm{D}_w^2 H(\nu_e, 0)$ of (2.7) has nonzero purely imaginary eigenvalues, i.e. it is elliptic. We call the relative equilibrium x_e of (2.1) *elliptic* if the corresponding equilibrium $w = 0$ of (2.7) is elliptic. Depending on the dimension of the phase space $\mathcal{M}$, different scenarios are possible.

(a) If $\dim \mathcal{M} = 10$ then (2.7) is a 1 degree of freedom Hamiltonian system. Equilibria of such Hamiltonian systems have definite Hessians if and only if they are elliptic. So, the Hessian $\mathrm{D}_w^2 H(\nu_e, 0)$ is definite, contradicting $x_e \in \mathcal{A}_e^{\mathrm{gap}}$. In particular, at this dimension there are no nondegenerate relative equilibria in the gap.

(b) If $\dim \mathcal{M} = 12$ then, for each ν, (2.7) is a 2 degree of freedom Hamiltonian system. The equilibrium $w = 0$ persists as elliptic equilibrium of (2.7) to nearby ν. Elliptic equilibria of such Hamiltonian systems are expected to be KAM stable [2], and so the axisymmetric relative equilibrium x_e is expected to be $\mathcal{K}$-stable.

(c) If $\dim \mathcal{M} \geq 14$ then the equilibrium $w = 0$ persists as in case (b), but for each ν the system (2.7) has at least three degrees of freedom. The equilibrium $w = 0$ of (2.7) is therefore not expected to be stable because it will exhibit Arnold diffusion. However, Nekhoroshev stability [5, 11] is expected, which implies $\mathcal{K}$ stability of the axisymmetric relative equilibrium x_e over exponentially long times.

Remark 3.6. In case (b), in order to prove KAM-stability of the axisymmetric relative equilibrium x_e, one can verify the twist condition (see [9]) for the equilibrium $w = 0$ of the 4-dimensional SO(2) symmetric Hamiltonian system (2.7) at $\nu = \nu_e$. If the twist condition holds at $\nu = \nu_e$, then it holds for the equilibria persisting to nearby ν and x_e is therefore $\mathcal{K}$-stable. The additional SO(2) symmetry of (2.7) at $\nu = \nu_e$ can be used to express the reduced Hamiltonian of (2.7) in terms of $\mathrm{SO}(2)_D$-invariant functions. This simplifies the verification of the twist condition, see Section 4.4. But, as already emphasized, the conserved quantity $\boldsymbol{J}_W$ of (2.7) at $\nu = \nu_e$ cannot be used because it is only conserved for vertical ν. The system (2.7) for which the twist condition needs to be verified, can be obtained in a concrete example by choosing any realization of the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ Marsden-Weinstein reduced space.

Remark 3.7. If $x_e \in \mathcal{A}_e^{\mathrm{gap}}$ then, as already noted above, x_e is a spectrally stable equilibrium with indefinite Hessian of the Hamiltonian system (2.7). As shown in [3, 4], adding arbitrarily small dissipation to such Hamiltonian systems results in spectral instability, because the negative eigendirection of the Hessian forces instability in the presence of decreasing energy. Consequently, adding a small $\mathrm{SO}(2)_D$ invariant dissipation for each vertical ν, and extending that to nonvertical ν, will result in dissipation induced instability at each persisting equilibrium of the Hamiltonian systems parameterized by ν. If the dissipation preserves also the $\mathrm{SO}(2)_D$ momentum $\boldsymbol{J}_W$, then it preserves all momenta resulting from the symmetries of the original system on $\mathcal{M}$. In this case the persisting equilibria of (2.7) remain spectrally stable for nearby vertical ν while for nearby nonvertical ν there is again spectral instability. In Section 5 we verify this by numerical simulation of an $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduction of the underwater vehicle system.

3.2 Hamiltonian Hopf bifurcation

In this section we assume that the phase space $\mathcal{M}$ is 12-dimensional, so that the system (2.7) is 4-dimensional. The linearization of the action of $\mathrm{SO}(2)_D$ on W at the equilibrium $w = 0$ is a linear symplectic representation, about which we need a few elementary facts [10]. Consider a symplectic representation of SO(2) on a $2d$-dimensional linear symplectic space $\mathbb{W}$. These representations are classified by tuples of integers $n_1 \le n_2 \le \cdots \le n_d$. We say that the *action is of type* $(n_1, n_2, \ldots, n_d)$, in which case, there is a linear splitting of $\mathbb{W}$ into the sum of two 2-dimensional invariant subspaces on which the action is isomorphic to $z \mapsto e^{in_j\theta}z$ on the vector space $\mathbb{R}^2 \cong \mathbb{C}$ with its standard symplectic structure. With respect to such a splitting, the associated momentum mapping is

$$\frac{1}{2}n_1|z_1|^2 + \cdots + \frac{1}{2}n_d|z_d|^2,$$

and particularly, the quadratic momentum map for type $(n_1, n_2, \ldots, n_d)$ actions has both positive and negative definite eigendirections if both positive and negative n_j occur in its type.

Suppose $\mathbb{W}$ is a linear symplectic space and $A\colon \mathbb{W} \to \mathbb{W}$ is infinitesimally symplectic. Let $\pm i\omega$ be a simple complex conjugate pair of purely imaginary eigenvalues of A.

The corresponding (2-dimensional) real eigenspace is invariant for the symplectic flow $\exp(At)$, which defines a symplectic representation of $\mathrm{SO}(2)$ of either type $+1$ or type -1. The *Krein sign* of $\pm i\omega$ is the sign of its type. We adopt the convention that the frequency ω has the same sign as the Krein sign of the eigenvalue $\pm i\omega$, so that the corresponding quadratic Hamiltonian on the real eigenspace, with respect to a basis with the standard symplectic structure, is $+\frac{1}{2}\omega(q^2+p^2)$. Since the frequencies are signed, resonances between two frequencies are also signed e.g. there can be $-1:1$ and $1:1$ resonances, and these are distinct cases.

Remark 3.8. Actions of $\mathrm{SO}(2)$ of type $(-1,1)$ occur frequently in dimension 4, and this case occurs in the underwater vehicle example, see (4.25). This case can be realized as $q \mapsto R_\phi q$, $p \mapsto R_\phi p$ where R_ϕ is the action of $\mathrm{SO}(2)$ by counterclockwise rotations (see the proof of Proposition 3.11). This is the standard $\mathrm{SO}(2)$ action on $\mathbb{R}^4$ obtained by lifting the standard action of $\mathrm{SO}(2)$ on $\mathbb{R}^2 = \{q\}$ to the cotangent bundle $\mathcal{T}^*\mathbb{R}^2 = \{(q,p)\}$.

Proposition 3.9. *Let x_e be an axisymmetric relative equilibrium of the $\mathrm{SO}(2) \ltimes \mathbb{R}^3 \times \mathrm{SO}(2)$ symmetric Hamiltonian system (2.1) on a 12-dimensional phase space $\mathcal{M}$. Assume that x_e is semisimple and elliptic, and let the action of $\mathrm{SO}(2)$ be of type n_j on the real $\pm i\omega_j$ eigenspace, where ω_1, ω_2 are the two normal frequencies of the equilibrium $w = 0$. Then x_e is formally stable if and only if there is not an $n_1 : n_2$-resonance between ω_1, ω_2, i.e. if and only if $\omega_1 n_2 - \omega_2 n_1 \neq 0$.*

Proposition 3.9 implies that, in 12 dimensions, elliptic axisymmetric relative equilibria are typically formally stable because resonance is atypical in the linearization of the corresponding equilibrium in the reduction by $\mathcal{K}$. The proof of this Proposition follows by applying the following result to the linearized $\mathrm{SO}(2)_D$ action at x_e, restricted to W.

Proposition 3.10. *Let $\mathbb{W}$ be a 4-dimensional linear symplectic space and suppose $\mathrm{SO}(2)$ acts linearly and symplectically on $\mathbb{W}$ with momentum map $\boldsymbol{J}$. Suppose that $H\colon \mathbb{W} \to \mathbb{R}$ is $\mathrm{SO}(2)$ invariant and that $0 \in \mathbb{W}$ is an elliptic semisimple equilibrium where the linearization $\mathbb{J}\mathrm{D}^2 H(0)$ has eigenvalues $i\omega_j$, $j = 1, 2$. Suppose that $\omega_1 < 0$ and $\omega_2 > 0$, and let the action of $\mathrm{SO}(2)$ be of type n_j on the real $\pm i\omega_j$ eigenspace. Then there is a λ such that $\mathrm{D}_w^2 H(0) + \lambda \mathrm{D}_w^2 \boldsymbol{J}(0)$ is definite (i.e. $0 \in \mathbb{W}$ is formally stable) if and only if*

$$\omega_1 n_2 - \omega_2 n_1 \neq 0. \tag{3.1}$$

Proof. Split $\mathbb{W} = \mathbb{R}^2 \oplus \mathbb{R}^2$ into the real eigenspaces of $\mathbb{J}\mathrm{D}^2 H(0)$ to the eigenvalues $i\omega_j$, choosing a basis giving the standard symplectic structure matrix on each factor. Then for $w = (w_1, w_2) \in \mathbb{R}^2 \oplus \mathbb{R}^2$

$$\begin{aligned} \big\langle w, \big(\mathrm{D}^2 H(0) + \lambda \mathrm{D}^2 \boldsymbol{J}(0)\big) w \big\rangle &= \frac{1}{2}(\omega_1 |w_1|^2 + \omega_2 |w_2|^2) + \frac{\lambda}{2}(n_1 |w_1|^2 + n_2 |w_2|^2) \\ &= \frac{1}{2}\big((\omega_1 + \lambda n_1)|w_1|^2 + (\omega_2 + \lambda n_2)|w_2|^2\big). \end{aligned} \tag{3.2}$$

Since the action of SO(2) is nontrivial, one of n_1, n_2 is not zero, and

$$\lambda \mapsto (\omega_1, \omega_2) + \lambda(n_1, n_2)$$

is a line. To show that (3.2) is definite, its is sufficient that this line meets the first or third quadrant. But such a line is contained in the second and fourth quadrant only if it contains the origin, which is excluded by (3.1).

Conversely, suppose that $\omega_1 n_2 - \omega_2 n_1 = 0$. Then there is a κ such that $(n_1, n_2) = \kappa(\omega_1, \omega_2)$ and (3.2) becomes

$$\big\langle w, \big(\mathrm{D}^2 H(0) + \lambda \mathrm{D}^2 \boldsymbol{J}(0)\big) w \big\rangle = \frac{1}{2}(\kappa + \lambda)\big(\omega_1 |w_1|^2 + \omega_2 |w_2|^2\big),$$

and this is not definite for any λ because $\omega_1 < 0 < \omega_2$. 

We now consider the case when condition (3.1) is violated.

Proposition 3.11. *Assume the setting of Proposition 3.10 again, but let $H(\nu, w)$ depend on a parameter $\nu \in \mathbb{R}$ and assume, as before, that the* SO(2) *action is of type (n_1, n_2). Assume that*

(i) the equilibrium is elliptic for $\nu < 0$, $\nu \approx 0$;

(ii) condition (3.1) *holds for $\nu < 0$, $\nu \approx 0$, but is violated at $\nu = 0$.*

Then:

(a) if $n_1 \neq -n_2$ then the equilibrium is linearly stable for $\nu > 0$, $\nu \approx 0$; and

(b) if $n_1 = -n_2$ and if the transversality condition (3.3) *holds then a Hamiltonian Hopf bifurcation occurs and the equilibrium* 0 *becomes spectrally unstable for $\nu > 0$.*

Proof. For part (a), if $n_1 \neq \pm n_2$ then the condition $\omega_1 n_2 - \omega_2 n_1 = 0$ at $w = 0$, $\nu = 0$ implies that the eigenvalues $i\omega_j$ do not collide at $\nu = 0$. Therefore the symplectic eigenvalue theorem [1] implies that the eigenvalues are imaginary in a neighborhood of $\nu = 0$. If $n_1 = n_2$ then $0 = \omega_1 n_2 - \omega_2 n_1 = n_1(\omega_1 - \omega_2)$. Since the SO(2) action is assumed to be nontrivial, we have $n_1 \neq 0$, and so $\omega_1 = \omega_2$. This contradicts the assumption $\omega_1 < 0 < \omega_2$.

For part (b), so assuming $n \equiv n_2 = -n_1$, there is a equivariant Lagrangian splitting $\mathbb{W} = \mathbb{R}^2 \oplus \mathbb{R}^2$ such that the SO(2) action is $z \mapsto e^{in\theta} z$ on each factor $\mathbb{R}^2 \simeq \mathbb{C}$. The standard symplectic structure matrix blocks with respect to this splitting, is the identity on the $(1,2)$-block, and minus the identity on the $(2,1)$-block. Indeed, with respect to the symplectic splitting of Proposition 3.10, the basis

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix},$$
$$e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \qquad e_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix},$$

accomplishes this, as is easily verified. With respect to this basis, write the linearization at $w = 0$ as

$$L = \mathbb{J}\mathrm{D}^2 H(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are $\mathrm{SO}(2)$-equivariant 2×2 matrices, which can therefore be identified with complex numbers. Since L is infinitesimally symplectic, $d = -a^t$ and b, c are symmetric, i.e. $d = -\bar{a}$ and $b, c \in \mathbb{R}$, and the eigenvalues of L are

$$\lambda = i\,\mathrm{Im}(a) \pm \sqrt{-\mathrm{Im}(a)^2 - (ad - bc)} = i\,\mathrm{Im}(a) \pm \sqrt{\mathrm{Re}(a)^2 + bc}.$$

So L is spectrally stable if the discriminant $D = \mathrm{Re}(a)^2 + bc$ is negative, unstable if D is positive, and an eigenvalue collision occurs if $D = 0$. Since $D = 0$ when $\nu = 0$ and $D < 0$ for $\nu < 0$, $\nu \approx 0$, D changes sign under the transversality condition

$$\left.\frac{\mathrm{d}D}{\mathrm{d}\nu}\right|_{\nu=0} \neq 0, \tag{3.3}$$

whereupon the equilibrium becomes spectrally unstable for $\nu > 0$, $\nu \approx 0$, and undergoes a Hamiltonian Hopf bifurcation at $\nu = 0$ see, e.g. [17].

4 Application to the Kirchhoff model

In this section we use the theory from Section 3 to analyze the stability of vertically falling, spinning relative equilibria of a neutrally buoyant submerged axisymmetric body. In the Kirchhoff approximation (see e.g. [6, 7]), this is a Lagrangian system with configuration space $\mathrm{SE}(3)$, such that the position and orientation of the body in configuration $(A, b) \in \mathrm{SE}(3)$ are obtained relative to a reference body, by the Euclidean transformation $x \mapsto Ax + b$. The reference body is chosen with a vertical axis of symmetry, so that in the configuration (A, b), the submerged body has axis of symmetry in the direction $A\mathbf{e}_3$. The Lagrangian is

$$L(A, b, \Omega, v) = \frac{1}{2}\Omega^t I \Omega - ml\mathbf{e}_3 \cdot \Omega \times v + \frac{1}{2} v^t M v + mgl(\mathbf{e}_3 \cdot A\mathbf{e}_3). \tag{4.1}$$

Here $(\Omega, v) \in \mathcal{T}\mathrm{SE}(3)$ are the body-referenced angular and translational velocities, obtained by left translation, i.e. $(\widehat{\Omega}, v) = (A, b)^{-1}(\dot{A}, \dot{b})$, or

$$\widehat{\Omega} = A^{-1}\dot{A}, \qquad v = A^{-1}\dot{b}. \tag{4.2}$$

The *added inertia matrix* I and *added mass matrix* M are the 3×3 diagonal matrices

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \qquad M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_3 \end{pmatrix},$$

and are determined from the shape and mass distribution of the body [6]. The reference body is such that the center of buoyancy is at the origin and the center of mass is at distance l below the center of buoyancy, so the vehicle is bottom heavy when $l > 0$ and top heavy when $l < 0$. We will assume, as in the physical situation, that I, M, m and l satisfy

$$I_1 > 0, \quad I_3 > 0, \quad M_1 > 0, \quad M_3 > 0, \quad I_1 M_1 - m^2 l^2 > 0. \tag{4.3}$$

The Lagrangian is positive definite quadratic in Ω, v under assumptions (4.3).

The system admits the symmetries of

(a) *spatial isotropy:* the left action of the subgroup $\mathrm{SE}(2) \times \mathbb{R}$, corresponding to translations in any direction and rotations about the vertical; and

(b) *material symmetry:* the action of the subgroup $\mathrm{SO}(2)$ by multiplication of the inverse on the right, corresponding to rotating the body about its axis.

Thus the system has the symmetry $\mathcal{G}$ considered in Section 3. The point x_e defined by

$$x_e : \quad A = \mathrm{id}, \quad b = 0, \quad \Omega = \frac{S_e}{I_3}\mathbf{e}_3, \quad v = \frac{P_e}{M_3}\mathbf{e}_3,$$

is an axisymmetric relative equilibrium, corresponding to the motion where the vehicle spins at angular velocity S_e/I_3 and angular momentum S_e about its (vertical) symmetry axis, and translates along that axis with vertical velocity P_e/M_3 and vertical momentum P_e. We will apply the general stability theory developed in Section 3 to this family of relative equilibria.

4.1 Hamiltonian formulation of the Kirchhoff model

We briefly review the derivation of the Hamiltonian and the equation of motion for the Kirchhoff model. See [6] for more details. The conjugate variables Π, P to A, b are

$$\Pi = \left.\frac{\partial L}{\partial \Omega}\right|_{(A,b)=(\mathrm{id},0)} = I\Omega + ml\mathbf{e}_3 \times v, \quad P = \left.\frac{\partial L}{\partial v}\right|_{(A,b)=(\mathrm{id},0)} = Mv - ml\mathbf{e}_3 \times \Omega,$$

and these equations can be inverted to give

$$\begin{aligned} \Omega_1 &= \frac{M_1\Pi_1 + mlP_2}{I_1M_1 - m^2l^2}, & \Omega_2 &= \frac{M_1\Pi_2 - mlP_1}{I_1M_1 - m^2l^2}, & \Omega_3 &= \frac{\Pi_3}{I_3}, \\ v_1 &= \frac{I_1P_1 - ml\Pi_2}{I_1M_1 - m^2l^2}, & v_2 &= \frac{I_1P_2 + ml\Pi_1}{I_1M_1 - m^2l^2}, & v_3 &= \frac{P_3}{M_3}. \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H(A, b, \Pi, P) &= \langle \Omega, \Pi \rangle + \langle v, P \rangle - L \\ &= \frac{1}{2}\langle \Omega, \Pi \rangle + \frac{1}{2}\langle v, P \rangle - mgl(\mathbf{e}_3 \cdot A\mathbf{e}_3) \end{aligned}$$

$$= \frac{M_1}{2(I_1M_1 - m^2l^2)}\big(\Pi_1^2 + \Pi_2^2\big) + \frac{1}{2I_3}\Pi_3^2 + \frac{I_1}{2(I_1M_1 - m^2l^2)}\big(P_1^2 + P_2^2\big)$$
$$+ \frac{1}{2M_3}P_3^2 + \frac{ml}{I_1M_1 - m^2l^2}(\Pi_1P_2 - \Pi_2P_1) - mgl\Gamma_3 \tag{4.4}$$

where $\Gamma = A^T\mathbf{e}_3$. The equations of motion are

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \Pi \times \Omega + P \times v - mgl\,\Gamma \times \mathbf{e}_3, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = P \times \Omega, \quad \frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \Gamma \times \Omega. \tag{4.5}$$

This system is Poisson with bracket

$$\{f, k\} = \nabla f^t\, \mathbb{J}(\Pi, P, \Gamma)\, \nabla k \quad \text{where} \quad \mathbb{J}(\Pi, P, \Gamma) = \begin{pmatrix} \Pi^\wedge & P^\wedge & \Gamma^\wedge \\ P^\wedge & 0 & 0 \\ \Gamma^\wedge & 0 & 0 \end{pmatrix}. \tag{4.6}$$

The momentum map $\boldsymbol{J} = (\boldsymbol{J}^\phi, \boldsymbol{J}^a, \boldsymbol{J}^\theta)$, c.f. (2.3), is given by

$$\begin{aligned} \boldsymbol{J}^\phi(A, b, \Pi, P) &= (A\Pi + b \times AP) \cdot \mathbf{e}_3, \\ \boldsymbol{J}^a(A, b, \Pi, P) &= AP, \\ \boldsymbol{J}^\theta(A, b, \Pi, P) &= -\Pi \cdot \mathbf{e}_3. \end{aligned} \tag{4.7}$$

4.2 Reduction by $\mathbb{R}^3 \times \mathrm{SO}(2)_R$

As shown in Section 3, the stability of the relative equilibrium x_e can be established by a study of the stability of the equilibrium of (2.7) at the parameter value $\nu = (\mu_e^a, \mu_e^\theta) = (P_e\mathbf{e}_3, S_e)$. The system (2.7) is the Marsden-Weinstein reduction (see [8]) of the full Hamiltonian system by the group $\mathbb{R}^3 \times \mathrm{SO}(2)_R$, at the momentum level (μ_e^a, μ_e^θ), i.e. (2.7) is obtained from the full Hamiltonian system (2.1) by fixing the values of the conserved quantities $\boldsymbol{J}^a$ and $\boldsymbol{J}^\theta$ to their respective values μ_e^a and μ_e^ϕ, and eliminating variables along the group $\mathbb{R}^3 \times \mathrm{SO}(2)_R$. For the Kirchhoff model, equations (4.5) cannot be directly used because they are the result of a reduction by the group $\mathrm{SO}(2)_L \ltimes \mathbb{R}^3$. In this section we compute the reduced spaces near x_e, in a way that relates to the variables Π, P, Γ of (4.5), by comoving with the action of $\mathrm{SO}(2)_R$.

Given q_1, q_2, define $q = q_1\mathbf{e}_1 + q_2\mathbf{e}_2$ and

$$A_{q_1,q_2} = \mathrm{id} + \widehat{x} + f\widehat{x}^2\Big|_{x=-\mathbf{e}_3\times q} \quad \text{where} \quad f = \frac{1}{1 + \sqrt{1 - \|q\|^2}}, \tag{4.8}$$

where $\widehat{x}$ is defined in (2.2). The matrix A_{q_1,q_2} in (4.8) is orthogonal because f satisfies the equation $\|q\|^2f^2 - 2f + 1 = 0$ and f is the solution of this quadratic that is smooth at $q = 0$. The choice $\boldsymbol{x} = -\mathbf{e}_3 \times q$ is so that $A^t_{q_1,q_2}\mathbf{e}_3 = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + \sqrt{1 - q_1^2 - q_2^2}\,\mathbf{e}_3$. The map $(q_1, q_2, \theta) \mapsto A_{q_1,q_2}\exp(-\widehat{\mathbf{e}}_3\theta)$ coordinatizes SO(3) near the identity and provides an angle $\theta(A)$, which may be used to define the coordinates (a, θ, q_1, q_2) by

$$(A, b) = (a, \theta) \cdot (A_{q_1,q_2}, mlM^{-1}q) = (A_{q_1,q_2}\exp(-\theta\widehat{\mathbf{e}}_3), mlM^{-1}q + a).$$

The choice of b is such that the two sections $a = 0, \theta = 0$ and $q_1 = q_2 = 0$ are orthogonal (at their intersection point) in the kinetic energy metric. This is an example of a general procedure advocated in [16]. The group $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ acts by addition in a, θ, and

$$\Gamma = \exp(\theta \widehat{\mathbf{e}}_3)(q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2) + \Gamma_3 \mathbf{e}_3 \tag{4.9}$$

so (q_1, q_2) comoves with (Γ_1, Γ_2).

Let the conjugate momenta to a, θ, q_1, q_2 be $\nu^a, \nu^\theta, p_1, p_2$. The coordinates a and θ are cyclic and so ν^a, ν^θ are conserved. Thus the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduction results in a Hamiltonian in terms of canonical coordinates q_i, p_i, which is parametrized by ν^a, ν^θ. This is computed in the following proposition.

Proposition 4.1. *The Hamiltonian of the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduction of the Kirchhoff model* (4.4) *in the canonical coordinates q_i, p_i is*

$$\begin{aligned} H(q_1, q_2, \tilde{p}_1, \tilde{p}_2) = & \frac{M_1}{2(I_1 M_1 - m^2 l^2)} \left(\tilde{p}_1^2 + \tilde{p}_2^2 - (q_2 \tilde{p}_1 - q_1 \tilde{p}_2)^2 \right) \\ & + \frac{1}{2} \left(\frac{1}{M_3} - \frac{I_1}{(I_1 M_1 - m^2 l^2)} \right) (\nu^a \cdot \Gamma^-)^2 \\ & + \frac{ml}{I_1 M_1 - m^2 l^2} \left((q_1 \tilde{p}_2 - q_2 \tilde{p}_1)(\nu^a \cdot \Gamma^-) + \nu_2^a \tilde{p}_1 - \nu_1^a \tilde{p}_2 \right) \\ & - mgl \Gamma_3, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} & \tilde{p}_1 = p_2 - f q_1 \nu^\theta - \frac{ml}{M_1} \nu_2^a, \qquad \tilde{p}_2 = -p_1 - f q_2 \nu^\theta + \frac{ml}{M_1} \nu_1^a, \\ & \Gamma^- = A^t_{q_1, q_2} \mathbf{e}_3 = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 - \Gamma_3 \mathbf{e}_3, \qquad \Gamma_3 = \sqrt{1 - q_1^2 - q_2^2}. \end{aligned} \tag{4.11}$$

Proof. The $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduced Hamiltonian can be computed by computing p_i in terms of Π and q_1, q_2 and eliminating P and Π_3 using (see (4.7)) $P = A^t \nu^a$ and $\Pi_3 = -\nu^\theta$. Since the Hamiltonian is invariant, it does not depend on a or θ, so we set $a = 0$ and $\theta = 0$. By definition of canonical coordinates,

$$p_i = \Omega_{q_i} \cdot \Pi + v_{q_i} \cdot P \quad \text{where} \quad \widehat{\Omega}_{q_i} = A^t \frac{\partial A}{\partial q_i}, \quad v_{q_i} = A^t \frac{\partial b}{\partial q_i}.$$

Then

$$\begin{aligned} A^t_{q_1, q_2} \frac{\partial A_{q_1, q_2}}{\partial q_1} &= \left(-\mathbf{e}_2 + \frac{f^2 q_1}{f - 1} \, \mathbf{e}_3 \times q + f q_2 \, \mathbf{e}_3 \right)^\wedge, \\ A^t_{q_1, q_2} \frac{\partial A_{q_1, q_2}}{\partial q_2} &= \left(\mathbf{e}_1 - \frac{f^2 q_2}{f - 1} \, \mathbf{e}_3 \times q - f q_1 \, \mathbf{e}_3 \right)^\wedge, \end{aligned}$$

from which

$$p_1 = -\frac{f^2 q_1 q_2}{f - 1} \Pi_1 + \left(-1 + \frac{f^2 q_1^2}{f - 1} \right) \Pi_2 - f q_2 \nu^\theta + \frac{ml}{M_1} \nu_1^a,$$

$$p_2 = \left(1 - \frac{f^2 q_2^2}{f-1}\right)\Pi_1 + \frac{f^2 q_1 q_2}{f-1}\Pi_2 + f q_1 \nu^\theta + \frac{ml}{M_1}\nu_2^a.$$

These linear equations may be solved for Π_1, Π_2, with the result

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} 1 - f q_2^2 & f q_1 q_2 \\ f q_1 q_2 & 1 - f q_1^2 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} \tag{4.12}$$

where $\tilde{p}_i$ are defined by (4.11). The matrix in (4.12) is the same as the upper 2×2 submatrix of $-\widehat{\mathbf{e}}_3 A^t_{q_1,q_2} \widehat{\mathbf{e}}_3$, which is otherwise sparse, so setting $\tilde{p} = \tilde{p}_1 \boldsymbol{e}_1 + \tilde{p}_2 \mathbf{e}_2$,

$$\Pi = -\mathbf{e}_3 \times \left(A^t_{q_1,q_2}(\mathbf{e}_3 \times \tilde{p})\right) - \nu^\theta \mathbf{e}_3, \tag{4.13}$$

and

$$\begin{aligned}
&\Pi_1^2 + \Pi_2^2 = \left\| -\mathbf{e}_3 \times \left(A^t_{q_1,q_2}(\mathbf{e}_3 \times \tilde{p})\right)\right\| = \tilde{p}_1^2 + \tilde{p}_2^2 - (q_2\tilde{p}_1 - q_1\tilde{p}_2)^2, \\
&\Pi_1 P_2 - \Pi_2 P_1 = \mathbf{e}_3 \cdot (\Pi \times A^t_{q_1,q_2}\nu^a) = (q_1\tilde{p}_2 - q_2\tilde{p}_1)(\nu^a \cdot \Gamma) + \nu_2^a \tilde{p}_1 - \nu_1^a \tilde{p}_2, \\
&P_3^2 = (\nu^a \cdot A^t_{q_1,q_2}\mathbf{e}_3)^2.
\end{aligned}$$

The Hamiltonian (4.4) is obtained from (4.10) after substitution these and deleting inessential constants.

As already noted, the case of ν^a vertical is important for the stability and KAM analysis, and we specialize to this now.

Proposition 4.2. *At vertical momentum $\nu_1^a = \nu_2^a = 0$, the reduced Hamiltonian system* (4.4) *is*

$$\begin{aligned}
H(q_1,q_2,p_1,p_2) = \frac{F_p}{2}\big((p_1 + F_l\Gamma_3 q_1 + \nu^\theta f q_2)^2 - (q_1 p_1 + q_2 p_2)^2 \\
+ (p_2 + F_l\Gamma_3 q_2 - \nu^\theta f q_1)^2\big) \\
+ \frac{F_q}{2}\|q\|^2 + mgl\left(f - \frac{1}{2}\right)\|q\|^2 + \frac{1}{2}F_p F_l^2 \|q\|^4.
\end{aligned} \tag{4.14}$$

where

$$F_p = \frac{M_1}{I_1 M_1 - m^2 l^2}, \qquad F_q = mgl - (\nu_3^a)^2\left(\frac{1}{M_3} - \frac{1}{M_1}\right), \qquad F_l = \frac{ml\nu_3^a}{M_1}. \tag{4.15}$$

Proof. Using (4.11), the Hamiltonian (4.10) can be written in the canonical coordinates q_1, q_2, p_1, p_2 as follows:

$$\begin{aligned}
&H(q_1,q_2,p_1,p_2) \\
&\quad = \frac{M_1}{2(I_1 M_1 - m^2 l^2)}\left(\tilde{p}_1^2 + \tilde{p}_2^2 - (q_2\tilde{p}_1 - q_1\tilde{p}_2)^2\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{M_3} - \frac{I_1}{(I_1 M_1 - m^2 l^2)}\right)(\nu_3^a \Gamma_3)^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{ml}{I_1M_1 - m^2l^2}(q_1\tilde{p}_2 - q_2\tilde{p}_1)(\nu_3^a\Gamma_3) - mgl\Gamma_3 \\
&= \frac{M_1}{2(I_1M_1 - m^2l^2)}\left(\left(\tilde{p}_1 + \frac{ml\nu_3^a}{M_1}q_2\Gamma_3\right)^2 + \left(\tilde{p}_2 - \frac{ml\nu_3^a}{M_1}q_1\Gamma_3\right)^2 - (q_2\tilde{p}_1 - q_1\tilde{p}_2)^2\right) \\
&\quad - \frac{(\nu_3^a)^2}{2}\left(\frac{1}{M_3} - \frac{I_1}{(I_1M_1 - m^2l^2)} + \frac{m^2l^2(1-\|q\|^2)}{M_1(I_1M_1 - m^2l^2)}\right)\|q\|^2 - mgl\Gamma_3 \\
&= \frac{M_1}{2(I_1M_1 - m^2l^2)}\left(\left(p_1 + \frac{ml\nu_3^a}{M_1}q_1\Gamma_3 + \nu^\theta q_2 f\right)^2 - (q_1p_1 + q_2p_2)^2\right. \\
&\quad \left. + \left(p_2 + \frac{ml\nu_3^a}{M_1}q_2\Gamma_3 - \nu^\theta q_1 f\right)^2\right) \\
&\quad + \frac{1}{2}\left(2mglf - (\nu_3^a)^2\left(\frac{1}{M_3} - \frac{1}{M_1} - \frac{m^2l^2\|q\|^2}{M_1(I_1M_1 - m^2l^2)}\right)\right)\|q\|^2,
\end{aligned}
$$

where a constant has been deleted at the final equality. Substitution of (4.15) then gives the Hamiltonian (4.14).

Remark 4.3. In accord with Proposition 2.3, the Hamiltonian (4.14) admits an additional SO(2) symmetry because ν^a is vertical, which we can take as the diagonal action of SO(2) on $(q_1, q_2), (p_1, p_2)$, with the standard conserved momentum $q_1p_2 - q_2p_1$. From (4.12),

$$
\begin{aligned}
\Pi\cdot\Gamma &= q_1\tilde{p}_1 + q_2\tilde{p}_2 \\
&= q_1(p_2 - fq_1\nu^\theta) + q_2(-p_1 - fq_2\nu^\theta) + \Pi_3\Gamma_3 \\
&= q_1p_2 - q_2p_1 - f\nu^\theta(q_1^2 + q_2^2) - \nu^\theta(1 - \|q\|^2 f) \\
&= (q_1p_2 - q_2p_1) - \nu^\theta.
\end{aligned}
$$

Consequently, the additional conserved momentum (*conserved for vertical ν only*) is equivalent to the subcasimir $\Pi\cdot\Gamma$ of [7].

4.3 Energy-momentum confinement, spectral stability

Theorem 4.4. *The axisymmetric relative equilibrium x_e, with linear momentum P_e and spin momentum S_e, of the Kirchhoff model* (4.5) *is $\mathcal{G}$-stable by energy-momentum confinement if*

$$
mgl > \left(\frac{1}{M_3} - \frac{1}{M_1}\right)P_e^2 \tag{4.16}
$$

and spectrally stable if

$$
mgl > \left(\frac{1}{M_3} - \frac{1}{M_1}\right)P_e^2 - \frac{M_1}{4(I_1M_1 - m^2l^2)}S_e^2. \tag{4.17}
$$

Proof. By Proposition 3.1, it is sufficient to consider the equilibrium $q = 0, p = 0$ of the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduced Kirchhoff Hamiltonian at the parameter values $\nu_3^a = P_e$ and $\nu^\theta = S_e$. The relevant Hessian may be obtained from (4.14) by deleting terms of higher than order two, resulting in

$$H = \frac{F_p}{2}|p + Fq|^2 + \frac{F_q}{2}|q|^2 \tag{4.18}$$

where, for convenience, we use the notations

$$q = q_1 + iq_2, \quad p = p_1 + ip_2, \quad F = F_l - \frac{i}{2}S_e.$$

By assumption $F_p > 0$, so the Hamiltonian (4.18) is positive definite when $F_q > 0$, a condition which, in view of (4.15), is equivalent to (4.16).

To show (4.17), note that the linearized equations at $q = 0, p = 0$ are obtained from (4.18) by Hamilton's equations:

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{\mathrm{d}q_1}{\mathrm{d}t} + i\frac{\mathrm{d}q_2}{\mathrm{d}t} = \frac{\partial H}{\partial p_1} + i\frac{\partial H}{\partial p_2} = F_p F\, q + F_p\, p, \\
\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}p_1}{\mathrm{d}t} + i\frac{\mathrm{d}p_2}{\mathrm{d}t} = -\frac{\partial H}{\partial q_1} - i\frac{\partial H}{\partial q_2} = -(F_p|F|^2 + F_q)\, q - F_p\bar{F}\, p.
\end{aligned}$$

Due to SO(2)-equivariance, computing the eigenvalues of the linearization reduces to solving the equation

$$0 = \det \begin{pmatrix} F_pF - \lambda & F_p \\ -(F_p|F|^2 + F_q) & -F_p\bar{F} - \lambda \end{pmatrix} = \lambda^2 + iF_pS_e\lambda + F_pF_q.$$

This has roots

$$\lambda = \frac{1}{2}\left(-iF_pS_e \pm \sqrt{-F_p^2S_e^2 - 4F_pF_q}\right), \tag{4.19}$$

so spectral stability holds if and only if

$$F_p^2S_e^2 + 4F_pF_q = 4F_p\left(mgl - \left(\frac{1}{M_3} - \frac{1}{M_1}\right)P_e^2 + \frac{M_1}{4(I_1M_1 - m^2l^2)}S_e^2\right) > 0, \tag{4.20}$$

which is equivalent to (4.17). 

Remark 4.5. Energy-momentum confinement under momentum-preserving perturbations, and spectral stability, fail coincidentally as predicted in general by Proposition 3.11, and as observed in this example by [7]. We can check this by determining the conditions such that $H + F_p\lambda(q_1p_2 - q_2p_1)$ is positive definite for some $\lambda \in \mathbb{R}$, where H is the Hessian in (4.18). One computes

$$\begin{aligned}
H + \lambda(q_1p_2 - q_2p_1) &= \frac{1}{2}\big(F_p|p + Fq|^2 + 2\lambda(q_1p_2 - q_2p_1)\big) + \frac{F_q}{2}|q|^2 \\
&= \frac{1}{2F_p}|F_p(p + Fq) + i\lambda q|^2 + \frac{1}{2F_p}\big(F_pF_q + \lambda F_pS_e - \lambda^2\big)|q|^2,
\end{aligned} \tag{4.21}$$

where we have used

$$|F_p(p+Fq)+i\lambda q|^2 = F_p^2|p+Fq|^2+\lambda^2|q|^2+2\,\mathrm{Im}\big(F_p(p+Fq)(\lambda q)^-\big)$$
$$= F_p^2|p+Fq|^2+\lambda^2|q|^2+2\lambda F_p\big((q_1p_2-q_2p_1)+|q|^2\,\mathrm{Im}\,F\big).$$

In (4.21), the coefficient of $|q|^2$ is positive for some λ if and only if $F_pF_q+\lambda F_pS_e-\lambda^2$ has a real root, i.e. if and only if

$$F_p^2S_e^2-4(F_pF_q)(-1)=F_p^2S_e^2+4F_pF_q>0,$$

which is exactly the condition for spectral stability as in (4.20).

4.4 KAM stability

In this section we verify that (at vertical ν^a) the equilibrium $q=0, p=0$ of the $\mathbb{R}^3\times\mathrm{SO}(2)$ reduced Hamiltonian (4.10) satisfies the twist condition of the Arnold stability theorem (see [9]) in the gap between (4.16) and (4.17):

Theorem 4.6. *The axisymmetric relative equilibrium x_e of the Kirchhoff model* (4.5) *(with linear momentum P_e and spin momentum S_e) is stable if*

$$P_e^2\left(\frac{1}{M_3}-\frac{1}{M_1}\right)>mgl>P_e^2\left(\frac{1}{M_3}-\frac{1}{M_1}\right)-\frac{M_1}{4(I_1M_1-m^2l^2)}S_e^2.$$

For the proof we use Arnold's Theorem, see e.g. [9, Chapter IX.E]. This requires us to bring the Hamiltonian into the normal form

$$H=H_2+H_4\cdots+H_{2N}+H^\dagger \tag{4.22}$$

where

(a) H_{2k}, $1\le k\le 2N$ are homogeneous degree k in the polynomials $\mathcal{I}_1=\frac{1}{2}(q_1^2+p_1^2)$ and $\mathcal{I}_2=\frac{1}{2}(q_2^2+p_2^2)$;

(b) $H^\dagger$ is at least order $2N+1$ in q_1,q_2,p_1,p_2;

(c) $H_2=\omega_1\mathcal{I}_1-\omega_2\mathcal{I}_2$, where $\omega_1,\omega_2\neq 0$.

Theorem (Arnold Stability Theorem)**.** *The origin is stable for the Hamiltonian system* (4.22)*, provided for some k, $2\le k\le N$, the twist condition $D_{2k}=H_{2k}(\omega_2,\omega_1)\neq 0$ is satisfied.*

Proof of Theorem 4.6. Necessarily $\nu^\theta=S_e\neq 0$, or else there is no gap. By choice of time scale, we can assume that $S_e=1$. Also, $F_q<0$ in view of (4.15) and (4.17). The proof requires four steps.

1. Consolidate parameters. The conditions $D_{2k}\neq 0$ of the Arnold Stability Theorem are algebraically complicated in the parameters. The number of these conditions that is expected to be required is one more than the number of free parameters in the Hamiltonian, so as to have the reasonable expectation that there will be no value

of the free parameters for which all conditions vanish. To facilitate the symbolic computation of the required normal form, so that the computation to be tractable, it is necessary to reduce as much as possible the number of parameters, and to cast the remaining free parameters so that they appear as simple powers.

First, the parameter F_l of the Hamiltonian (4.14) may be eliminated by the substitution

$$p_1 = u_1 - \frac{F_l q_1}{\Gamma_3}, \qquad p_2 = u_2 - \frac{F_l q_2}{\Gamma_3}.$$

This is a symplectic transformation because it is a shift by the exact form

$$-F_l \, \mathrm{d}\Gamma_3 = \frac{F_l q_1}{\Gamma_3} \, \mathrm{d}q_1 + \frac{F_l q_2}{\Gamma_3} \, \mathrm{d}q_2.$$

By a simple direct computation,

$$\begin{aligned}(p_1 + F_l\Gamma_3 q_1 + f q_2)^2 - (q_1 p_1 + q_2 p_2)^2 + (p_2 + F_l\Gamma_3 q_2 - f q_1)^2 \\ = (u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2 - F_l^2 |q|^4\end{aligned}$$

so (4.10) is transformed to

$$\begin{aligned}H(q_1, q_2, u_1, u_2) = \frac{F_p}{2}\big((u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2\big) \\ + \frac{F_q}{2}\|q\|^2 + mgl\left(f - \frac{1}{2}\right)\|q\|^2,\end{aligned}$$

where, evidently, F_l is absent.

The frequencies of the linearization of $q = 0, p = 0$ are related to F_p, F_q by (4.19):

$$f_1 = \frac{1}{2}(F_p + \sqrt{F_p^2 + 4F_p F_q}), \qquad f_2 = \frac{1}{2}(F_p - \sqrt{F_p^2 + 4F_p F_q}),$$

and these can be inverted to give

$$F_p = f_1 + f_2, \qquad F_q = -\frac{f_1 f_2}{f_1 + f_2}. \tag{4.23}$$

Note that $f_1, f_2 > 0$ and $f_1 - f_2 = \sqrt{F_p^2 + 4F_p F_q} > 0$ so that $f_1 > f_2 > 0$. Replacing mgl with $-\mu$, and deleting an inessential constant, it is sufficient to consider the Hamiltonian

$$\begin{aligned}H(q_1, q_2, u_1, u_2) = \frac{f_1 + f_2}{2}\big((u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2\big) \\ - \frac{f_1 f_2}{2(f_1 + f_2)}\|q\|^2 + \mu\left(\sqrt{1 - \|q\|^2} + \frac{1}{2}\|q\|^2\right)\end{aligned} \tag{4.24}$$

under the assumptions that

$$f_1 > f_2 > 0.$$

The number of free parameters is 2, because an inessential multiplier of the Hamiltonian can reduce the triple f_1, f_2, μ by one. Thus, three twist conditions are expected to be required.

2. Linear normal form. The linearization at the origin with respect to the coordinates (q_1, q_2, u_1, u_2) is

$$\begin{pmatrix} 0 & \frac{1}{2}(f_1+f_1) & f_1+f_2 & 0 \\ -\frac{1}{2}(f_1+f_2) & 0 & 0 & f_1+f_2 \\ -\frac{(f_1-f_2)^2}{4(f_1+f_2)} & 0 & 0 & \frac{1}{2}(f_1+f_2) \\ 0 & -\frac{(f_1-f_2)^2}{4(f_1+f_2)} & -\frac{1}{2}(f_1+f_2) & 0 \end{pmatrix}$$

and four linearly independent eigenvectors are

$$\begin{aligned} e_1 &= \begin{pmatrix} 2(f_1+f_2) & 0 & 0 & f_2-f_1 \end{pmatrix} \\ e_2 &= \begin{pmatrix} 0 & 2(f_1+f_2) & f_1-f_2 & 0 \end{pmatrix} \\ e_3 &= \begin{pmatrix} 0 & 2(f_1+f_2) & f_2-f_1 & 0 \end{pmatrix} \\ e_4 &= \begin{pmatrix} 2(f_1+f_2) & 0 & 0 & f_1-f_2 \end{pmatrix} \end{aligned}$$

with respect to which the symplectic form is canonical with multiplier $4(f_1-f_2)(f_1+f_2)$, which can be ignored. The coordinates (Q_1, P_1, Q_2, P_2) defined by

$$\begin{pmatrix} q_1 & q_2 & u_1 & u_2 \end{pmatrix} = Q_1 e_1 + P_1 e_2 + Q_2 e_3 + P_2 e_4$$

normalizes the quadratic part of H to

$$\frac{\omega_1}{2}(Q_1^2+P_1^2) - \frac{\omega_2}{2}(Q_2^2+P_2^2),$$

where

$$\omega_1 = 4(f_1-f_2)(f_1+f_2)f_1, \qquad \omega_2 = 4(f_1-f_2)(f_1+f_2)f_2.$$

The SO(2) action for vertical ν explained in Remark 4.3 is anticlockwise rotations on (Q_1, P_1) and clockwise rotations on (Q_2, P_2). The momentum of this action can be taken as

$$\boldsymbol{J} = -\frac{1}{2}(Q_1^2+P_1^2) + \frac{1}{2}(Q_2^2+P_2^2) = -\mathcal{I}_1 + \mathcal{I}_2. \tag{4.25}$$

3. Nonlinear normal form. Since the Hamiltonian H is invariant under an SO(2) symmetry, we choose to seek normalizations within the corresponding class of invariant functions. Convenient invariants are

$$\begin{aligned} w_1 &= \mathcal{I}_1 = \frac{1}{2}(Q_1^2+P_1^2), & w_2 &= \mathcal{I}_2 = \frac{1}{2}(Q_2^2+P_2^2), \\ w_3 &= \frac{1}{\sqrt{2}}(Q_1Q_2 - P_1P_2), & w_4 &= \frac{1}{\sqrt{2}}(Q_1P_2 + Q_2P_1), \end{aligned}$$

and these satisfy the relation

$$2w_1w_2 - w_3^2 - w_4^2 = 0. \tag{4.26}$$

The Poisson bracket of the w_i is closed and the matrix $\{w_i, w_j\}$ is

$$\{w_i, w_j\} = \begin{pmatrix} 0 & 0 & w_4 & -w_3 \\ 0 & 0 & w_4 & -w_3 \\ -w_4 & -w_4 & 0 & -w_1 - w_2 \\ w_3 & w_3 & w_1 + w_2 & 0 \end{pmatrix} \tag{4.27}$$

Since H is invariant it can be expanded in a Taylor series in w_1, w_2, w_3, w_4 as

$$H = H_2 + H_4 + \cdots + H_{2N} + H^\dagger$$

where

$$H_2 = \omega_1\mathcal{I}_1 - \omega_2\mathcal{I}_2 = \omega_1 w_1 - \omega_2 w_2$$

and the H_{2i} are degree i homogeneous polynomials in w_1, w_2, w_3, w_4. The normalization process is formally the same in the space of invariants as it is in the space of functions of q_1, p_1, q_2, p_2. See [9, Chapter VII.C]. The normal form is achieved at degree $k \geq 2$ successively in k. After normalizing the terms of degree smaller than or equal to k, at the beginning of step k, we start with the Hamiltonian H_{2k} from the previous step. The Hamiltonian H_{2k} is in normal form up to order k, i.e. it is the sum of homogeneous degree $i \leq k$ polynomials in w_1, w_2 and homogeneous degree $i > k$ polynomials in w_1, w_2, w_3, w_4. We now look for a homogeneous polynomial G of degree $k+1$ in w_1, w_2, w_3, w_4 such that the time 1 flow of G transforms H into a function

$$\tilde{H} = H + \{G, H\} + \frac{1}{2!}\{G, \{G, H\}\} + \cdots,$$

which is in normal form at degree $k+1$. The degree $k+1$ term of $\tilde{H}$ is $H_{2(k+1)} + \{G, H_2\}$ because

$$\deg\{G, H_{2i}\} = \deg \mathrm{D}G + \deg \mathrm{D}H_{2i} + 1 = k + (i-1) + 1 = k + i,$$

so $\deg\{G, H_{2i}\} = k+1$ only for $i = 1$. The coefficients of G are adjusted so that

$$\tilde{H}_{2k} = H_{2k} + \{G, H_2\} \tag{4.28}$$

is a function only of w_1 and w_2, after all powers of w_4 greater than 1 are eliminated using (4.26). The Poisson brackets here are computed using (4.27). Then H is replaced by $\tilde{H}$ and the computation proceeds to step $k+1$.

At step k, there always exists a G such that $\tilde{H}_{2k}$ is normalized. Indeed, $\{\cdot, H_2\}$ maps the finite dimensional vector space $\mathbb{P}_{k+1}$ of homogeneous degree $k+1$ polynomials to itself. The space $\mathbb{P}_{k+1}$ can be regarded as an inner product space by

$$\langle f, g\rangle = \int_{B_{12}\times B_{34}} fg$$

where B_{12}, B_{34} are the unit disks in the w_1, w_2 and w_3, w_4 variables. The linear map $\{\cdot, H_2\}$ is antisymmetric with respect to this inner product because

$$\begin{aligned}\langle f, \{g, H_2\}\rangle + \langle \{f, H_2\}, g\rangle &= \int_{B_{12}\times B_{34}} \{fg, H_2\} \\ &= \int_{B_{12}\times B_{34}} (\omega_1 - \omega_2)\left(-w_4\frac{\partial(fg)}{\partial w_3} + w_3\frac{\partial(fg)}{\partial w_4}\right) \\ &= -(\omega_1 - \omega_2)\int_{B_{12}} \oint_{\partial B_{34}} fg(w_3\,\mathrm{d}w_3 + w_4\,\mathrm{d}w_4) \\ &= 0\end{aligned}$$

because the vector field in the line integral is always orthogonal to its path. Consequently, the image of $\{\cdot, H_2\}$ is orthogonal to $\ker\{\cdot, H_2\}$ and G in (4.28) can be chosen so that $\tilde{H}_{2k} \in \ker\{\cdot, H_2\}$. Moreover, $f \in \mathbb{P}_{k+1}$ is in $\ker\{\cdot, H_2\}$ if and only if

$$(\omega_1 - \omega_2)\left(-w_4\frac{\partial f}{\partial w_3} + w_3\frac{\partial f}{\partial w_4}\right) = 0,$$

from which $f = \tilde{f}(w_1, w_2, w_3^2 + w_4^2)$, since $\omega_1 \neq \omega_2$. But in view of the relation (4.26), this implies that f is a function of w_1, w_2, as required. We note that only the nonresonance condition $\omega_1 \neq \omega_2$ is required and the normalization could in theory be achieved at any order.

The normal form cannot be hand-computed, but can be computed using a symbolic manipulator. The result up to and including H_8 (the notation $1 \leftrightarrow 2$ means the preceeding fragment with w_1 and w_2 exchanged) is

$$\begin{aligned}H_4 &= 8(f_1+f_2)^3\Big(A_{20}w_1^2 + \frac{1}{2}A_{11}w_1w_2 + 1 \leftrightarrow 2\Big), \\ H_6 &= -\frac{32(f_1+f_2)^5}{(f_1-f_2)^2}\Big(A_{30}w_1^4 + A_{31}w_1^2w_2 + 1 \leftrightarrow 2\Big), \\ H_8 &= -\frac{64(f_1+f_2)^7}{(f_1-f_2)^4}\Big(A_{40}w_1^4 + A_{41}w_1^3w_2 + \frac{1}{2}A_{42}w_1^2w_2^2 + 1 \leftrightarrow 2\Big),\end{aligned}$$

$$\begin{aligned}A_{20} &= (-f_1 - f_2)\mu + (f_1+f_2)f_2, \\ A_{21} &= (-4f_1 - 4f_2)\mu + 8f_1f_2, \\ A_{30} &= 2(f_1+f_2)^2\mu^2 - 2(f_1+f_2)(3f_1+f_2)\mu f_2 + 4(f_1+f_2)f_1f_2^2, \\ A_{31} &= 15(f_1+f_2)^2\mu^2 - (f_1+f_2)(5f_1^2 + 44f_1f_2 + 11f_2^2)\mu \\ &\qquad + (5f_1^2 + 38f_1f_2 + 17f_2^2)f_1f_2, \\ A_{32} &= 15(f_1+f_2)^2\mu^2 - (f_1+f_2)(11f_1^2 + 44f_1f_2 + 5f_2^2)\mu \\ &\qquad + (17f_1^2 + 38f_1f_2 + 5f_2^2)f_1f_2, \\ A_{40} &= -16(f_1+f_2)^3\mu^3 + 2(f_1^2 + 36f_1f_2 + 11f_2^2)(f_1+f_2)^2\mu^2 - 2(f_1+f_2)(2f_1^3 \\ &\qquad + 55f_1^2f_2 + 36f_1f_2^2 + 3f_2^3)\mu f_2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2(f_1+f_2)(f_1^2+26f_1f_2+5f_2^2)f_1f_2^2,\\
A_{41} &= -182(f_1+f_2)^3\mu^3 + 26(3f_1^2+31f_1f_2+8f_2^2)(f_1+f_2)^2\mu^2\\
&\quad - 2(13f_1+f_2)(f_1+f_2)(8f_1^2+49f_1f_2+21f_2^2)\mu f_2\\
&\quad + 2(65f_1^3+367f_1^2f_2+267f_1f_2^2+29f_2^3)f_1f_2^2,\\
A_{42} &= -354(f_1+f_2)^3\mu^3 + 3(95f_1^2+518f_1f_2+95f_2^2)(f_1+f_2)^2\mu^2 - 3(f_1+f_2)\\
&\quad (9f_1^4+274f_1^3f_2+850f_1^2f_2^2+274f_1f_2^3+9f_2^4)\mu + 3(9f_1^4+204f_1^3f_2\\
&\quad + 518f_1^2f_2^2+204f_1f_2^3+9f_2^4)f_1f_2.
\end{aligned}
$$

4. Twist condition The twist condition in the Arnold stability theorem is $D_{2k} = H_{2k}(\omega_2,\omega_1) \neq 0$ for some $k \geq 2$. From the normal form, the first three D_{2k} are

$$
\begin{aligned}
D_4 &= 128(f_1-f_2)^2(f_1+f_2)^5(-(f_1+f_2)(f_1^2+4f_1f_2+f_2^2)\mu\\
&\quad + (f_1^2+10f_1f_2+f_2^2)f_1f_2),\\
D_6 &= -2048(f_1+f_2)^9((2f_1^2+13f_1f_2+2f_2^2)(f_1+f_2)^2\mu^2 - (f_1+f_2)(11f_1^2\\
&\quad + 46f_1f_2+11f_2^2)\mu f_1f_2 + (9f_1^2+50f_1f_2+9f_2^2)f_1^2f_2^2)(f_1-f_2),\\
D_8 &= 16384(f_1+f_2)^{11}(-2(8f_1^4+91f_1^3f_2+177f_1^2f_2^2+91f_1f_2^3+8f_2^4)(f_1+f_2)^3\mu^3\\
&\quad + (2f_1^6+150f_1^5f_2+1113f_1^4f_2^2+1970f_1^3f_2^3+1113f_1^2f_2^4\\
&\quad + 150f_1f_2^5+2f_2^6)(f_1+f_2)^2\mu^2 - (f_1+f_2)(4f_1^6+345f_1^5f_2+2226f_1^4f_2^2\\
&\quad + 3850f_1^3f_2^3+2226f_1^2f_2^4+345f_1f_2^5+4f_2^6)\mu f_1f_2 + (2f_1^6+211f_1^5f_2\\
&\quad + 1466f_1^4f_2^2+2642f_1^3f_2^3+1466f_1^2f_2^4+211f_1f_2^5+2f_2^6)f_1^2f_2^2).
\end{aligned}
$$

We assume D_4, D_6, D_8 vanish and argue by contradiction. The expressions are homogeneous in f_1, f_2, so one can assume $f_2 = 1$. Since D_4 is linear in μ, if the coefficient of μ vanishes, then the assumption $D_4 = 0$ implies the constant term in μ also vanishes, i.e.

$$
-(f_1+1)(1+4f_1+f_1^2) = 0, \quad \text{and} \quad f_1(f_1^2+10f_1+1) = 0.
$$

But these two polynomials in f_1 cannot simultaneously vanish because their gcd is 1, as can be computed with the Euclidean algorithm. So the coefficient of μ in D_4 does not vanish. Solving D_4 for μ and substituting into D_6 and D_8 results in

$$
\begin{aligned}
0 &= 10240(f_1+1)^9f_1^3(f_1+5)(5f_1+1)(f_1-1)^3\\
0 &= -245760(f_1+1)^{11}f_1^4(21f_1^4+172f_1^3+334f_1^2+172f_1+21)(f_1-1)^4
\end{aligned}
$$

But these cannot simultaneously vanish because their gcd is $10240(f_1+1)^9f_1^3(f_1-1)^3$, which is nonzero in view of $f_1 > f_2 = 1$.

Remark 4.7. The computer generated normal form used in Theorem 4.6 requires an independent check, and one possible check follows. The reduction by SO(2) of the SO(2)-invariant Hamiltonian (4.24) gives a one parameter family of 2-dimensional

$10^4\epsilon$	$T(\epsilon)/T_0$	$2i=4$		$2i=6$		$2i=8$	
		$T_4^{\rm nf}(\epsilon)/T_0$	$r_{4,\epsilon}$	$T_6^{\rm nf}(\epsilon)/T_0$	$r_{6,\epsilon}$	$T_8^{\rm nf}(\epsilon)/T_0$	$r_{8,\epsilon}$
1	.9995410553688	.99954076	4.0	.99954105570	6.0	.9995410553684	8.2
2	.9981715477722	.99816682	4.0	.99817156906	6.0	.9981715476563	8.0
4	.9928005386538	.99272712	3.3	.99280185244	5.8	.9928005101406	7.8
8	.9728669887964	.97182458	3.5	.97294031478	5.4	.9728607222126	7.2
16	.9108473444693	.89972618		.91385268217		.9098792303872	

Table 1: *Comparison, at the same energy-momentum values, of numerically computed periods, and periods computed from the normal forms $H_{2i}^{\rm nf}$. The correct normal form is indicated because the values of r in the columns are nearly the corresponding values of $2i$.*

Hamiltonian systems parametrized by the SO(2) angular momentum. All trajectories of these reduced Hamiltonian systems near $q=0, u=0$ are periodic. The periods of these orbits depend only on the energy and the SO(2) momentum, and may be numerically computed. On the other hand, the successive normal form Hamiltonians

$$\begin{aligned} H_4^{\rm nf} &= H_2 + H_4, \\ H_6^{\rm nf} &= H_2 + H_4 + H_6, \\ H_8^{\rm nf} &= H_2 + H_4 + H_6 + H_8, \end{aligned} \tag{4.29}$$

are functions of w_1 and w_2. Using the Poisson bracket (4.27), the differential equations for w_i are

$$\frac{{\rm d}w_1}{{\rm d}t} = \frac{{\rm d}w_2}{{\rm d}t} = 0, \quad \frac{{\rm d}w_3}{{\rm d}t} = -\left(\frac{\partial H_{2i}^{\rm nf}}{\partial w_1} + \frac{\partial H_{2i}^{\rm nf}}{\partial w_2}\right) w_4, \quad \frac{{\rm d}w_4}{{\rm d}t} = \left(\frac{\partial H_{2i}^{\rm nf}}{\partial w_1} + \frac{\partial H_{2i}^{\rm nf}}{\partial w_2}\right) w_3.$$

These differential equations are the SO(2) reduction of the Hamiltonian system (4.24), and the orbits have periods

$$T_{2i}^{\rm nf} = \left| \frac{\partial H_{2i}^{\rm nf}}{\partial w_1} + \frac{\partial H_{2i}^{\rm nf}}{\partial w_2} \right|, \tag{4.30}$$

which also depend only on the energy and the SO(2) momentum. For energy and momentum obtained from initial conditions $\epsilon q^0, \epsilon p^0$, the periods of $H_{2i}^{\rm nf}$, computed directly from (4.30), can be compared with numerically computed periods of the Hamiltonian (4.24). If the normal form is correct, then the difference of these must fall as ϵ^{2i}, because the period from $H_{2i}^{\rm nf}$ is accurate to order $i-1$ in w, and therefore its difference falls as order i, which is order $2i$ in ϵ. If the normal form is not correct, then it is not likely that the anharmonic periods it predicts will agree at the proper order with the periods of its non-normalized precursor.

We have implemented this check for the data

$$\begin{aligned} &f_1 = \frac{\sqrt{5}}{2}, \quad f_2 = 1, \quad \mu = \frac{1}{2}, \quad q^0 = (0.5, 1.0), \quad p^0 = (-0.75, 0), \\ &\epsilon = 0.0001,\ 0.0002,\ 0.0004,\ 0.0008,\ 0.0016. \end{aligned}$$

The choice of f_1 is convenient because then $4(f_1 - f_2)(f_1 + f_2) = 1$ and the symplectic multiplier of Theorem 4.6 does not have to be accounted for. In Table 1 are the numerically computed periods $T(\epsilon)$, and the periods $T_{2i}^{\mathrm{nf}}(\epsilon)$ computed using the normal forms, as ratios of the period

$$T_0 = \frac{2\pi}{f_1 - f_2} = \frac{2\pi}{\frac{\sqrt{5}}{2} - 1},$$

of the linearization. Also tabulated are the values called $r_{2i,\epsilon}$, which are the base two logarithms of the successive (as ϵ is successively doubled) ratios of the differences between the periods $T_{2i}^{\mathrm{nf}}(\epsilon)$ and $T(\epsilon)$ i.e.

$$r_{2i,\epsilon} = \log_2 \left| \frac{T_{2i}^{\mathrm{nf}}(2\epsilon) - T(2\epsilon)}{T_{2i}^{\mathrm{nf}}(\epsilon) - T(\epsilon)} \right|.$$

In the column corresponding to H_{2i}^{nf}, there is agreement between $r_{2i,\epsilon}$, which is the observed order that the difference falls as, and $2i$, which is the predicted order. We conclude that there is numerical evidence that the normal forms H_{2i}^{nf} are correct, because they predict the correct anharmonic period with an error of order $2i$.

5 Numerics

In this section we simulate small dissipation directly in the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduction of the Kirchhoff model, as computed in Proposition 4.1. We verify that, after addition of small dissipation,

(a) stability is maintained for the axisymmetric relative equilibria x_e which are in the EM-region determined by (4.16);

(b) stability is destroyed for the axisymmetric relative equilibria x_e which are in the gap determined by (4.16) and (4.17).

This confirms that the stability of the system in the gap is qualitatively less robust than its stability in the EM-region, and therefore confirms that *energy-momentum confinement fails in the gap, with physically significant consequences.* The approach of simulating the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ reduced system, rather than the full Kirchhoff model (4.1), is consistent with our development in Sections 2 and 3, and is convenient for the simulation of energy dissipation which preserves the $\mathbb{R}^3 \times \mathrm{SO}(2)_R$ momentum and symmetry (see Remark 3.7).

5.1 Splitting method

The Hamiltonian (4.10) may be split as

$$H = H_0 + H_1 + H_2 + H_3$$

where

$$H_0 = H(q_1, q_2, 0, 0),$$

$$H_1 = \frac{F_p}{2}(p_1^2 + p_2^2 - 2q_2p_2q_1p_1 - q_1^2p_1^2 - q_2^2p_2^2),$$
$$H_2 = F_pF_l(q_1p_1 + q_2p_2)\sqrt{1 - q_1^2 - q_2^2},$$
$$H_3 = -\frac{F_p\nu^\theta(p_2q_1 - p_1q_2)}{1 + \sqrt{1 - q_1^2 - q_2^2}}.$$

Only H_0 depends on ν_a, and all but H_0 are SO(2) invariant under the diagonal action of SO(2) on $(q_1, q_2), (p_1, p_2)$, whereas H_0 has this invariance only for vertical ν (see Remark 4.3).

There are exact formulas for the flows of each H_i, as follows:

(a) H_0 is a function of q only and hence its flow is $p \mapsto p - t\nabla H_0$.

(b) The flow of H_1 corresponds to the motion of a particle of mass $1/F_p$ freely moving on a sphere of radius 1. Indeed, for the top half of the sphere with coordinates $x = q_1\mathbf{e}_1 + q_2\mathbf{e}_1 + \sqrt{1 - q_1^2 - q_2^2}\,\mathbf{e}_3$, the kinetic energy metric of the particle motion, and its inverse, are respectively

$$\frac{1}{F_p\Gamma^2}\begin{pmatrix} 1 - q_2^2 & q_1q_2 \\ q_1q_2 & 1 - q_1^2 \end{pmatrix}, \qquad F_p\begin{pmatrix} 1 - q_1^2 & -q_1q_2 \\ -q_1q_2 & 1 - q_2^2 \end{pmatrix},$$

the second of which is twice the matrix corresponding to the quadratic form H_1.

(c) Since H_2 is linear in p, and p_1, p_2 are separated, the differential equations for q_1, q_2 close. As is easily verified, q_1/q_2 is conserved for these equations and so q_1, q_2 evolve along radial lines. The radial equation can be separated and integrated, and the evolution of p_1, p_2 follows that from conservation of energy and momentum.

(d) As is easily verified, $q_1^2 + q_2^2$ is conserved for the flow of H_3. Using this, the differential equations corresponding to H_3 become linear and may be integrated.

For a dissipation vector field, one could use a positive multiple of the negative gradient in p_1, p_2 of the reduced Hamiltonian (4.10), i.e. the vector field

$$\begin{aligned}\left(-(1 - q_1^2)p_1 + q_1q_2p_2 - \frac{ml\nu_3^a\Gamma_3}{M_1}q_1 - \frac{\nu^\theta}{1 + \Gamma_3}q_2\right)\frac{\partial}{\partial p_1} \\ + \left(q_1q_2p_1 - (1 - q_2^2)p_2 - \frac{ml\nu_3^a\Gamma_3}{M_1}q_2 + \frac{\nu^\theta}{1 + \Gamma_3}q_1\right)\frac{\partial}{\partial p_2}.\end{aligned} \tag{5.1}$$

This does not preserve the momentum $q_1p_2 - p_1q_2$ which occurs for vertical ν (see Remark 4.3). But the direction of (5.1) relative to the $\ker \mathrm{D}H$ hyperplane (for q_1, q_2 constant) is unaltered by projection: any projection of (5.1) onto any subspace will still dissipate energy. So we project onto the line tangent to the level set of the function $q_1p_2 - p_1q_2$, i.e. onto the vector (q_1, q_2). Since the inner product of (5.1) and (q_1, q_2) is

$$\begin{aligned}(-1 + q_1^2 + q_2^2)q_1p_1 + (-1 + q_1^2 + q_2^2)q_2p_2 - \frac{ml\nu_3^a\Gamma_3}{M_1}(q_1^2 + q_2^2) \\ = -\Gamma_3^2(q_1p_1 + q_2p_2) - \frac{ml\nu_3^a\Gamma_3}{M_1}(q_1^2 + q_2^2),\end{aligned}$$

we use for the dissipative perturbation vector field ϵR, where ϵ is small and positive, and

$$\begin{aligned} R = \big(-(q_1p_1 + q_2p_2)\Gamma_3 - F_l(q_1^2 + q_2^2)\big)\, q_1 \frac{\partial}{\partial p_1} \\ + \big(-(q_1p_1 + q_2p_2)\Gamma_3 - F_l(q_1^2 + q_2^2)\big)\, q_2 \frac{\partial}{\partial p_2}. \end{aligned} \tag{5.2}$$

The differential equations corresponding to this vector field are linear and easily solved exactly.

The full system, dissipation included, is given by the vector field

$$\epsilon R + \sum_{i,s} \left(\frac{\partial H_s}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H_s}{\partial q_i} \frac{\partial}{\partial p_i} \right). \tag{5.3}$$

Letting the flows of the Hamiltonians H_s, $0 \le s \le 3$, be $F_t^{H_s}$, and the flow of the dissipation ϵR be G_t, the concatenation

$$G_{\Delta t/2} F^{H_0}_{\Delta t/2} F^{H_3}_{\Delta t/2} F^{H_2}_{\Delta t/2} F^{H_1}_{\Delta t} F^{H_2}_{\Delta t/2} F^{H_3}_{\Delta t/2} F^{H_0}_{\Delta t/2} G_{\Delta t/2}$$

is a second order one step method, which we use to simulate the system. At vertical ν, this method preserves the conserved quantity $q_1p_2 - q_2p_1$ because every one of its steps does that separately. The method is symplectic if $\epsilon = 0$.

5.2 Simulations

The simulations that we report in this section are done at parameter values

$$\begin{aligned} &I_1 = 4, \quad M_1 = 1, \quad M_3 = \frac{1}{2}, \quad m = 1, \quad l = 1, \quad g = 1, \quad S_e = 6, \\ &\nu_3^a = P_e, \quad \nu^\theta = S_e, \end{aligned} \tag{5.4}$$

i.e. units are such that $g = 1$. The initial conditions are chosen so that

$$q_2 = 0, \quad p_1 = 0, \quad p_2 = 0, \quad \nu_2^a = 0.$$

The perturbation from the equilibrium is achieved by choosing small nonzero q_1 and ν_1^a. From (4.4), energy-momentum confinement occurs for $P_e^2 < C_1$ and spectral stability occurs for $P_e^2 < C_2$, where

$$C_1 = \frac{M_1M_3}{M_1 - M_3}\, mgl, \quad C_2 = \frac{M_1M_3}{M_1 - M_3} \left(mgl + \frac{M_1}{4(I_1M_1 - m^2l^2)} S_e^2 \right).$$

For the parameter values (5.4), $C_1 = 1$ and $C_2 = 4$, so

$2 < P_e$:	spectral instability;
$1 < P_e < 2$:	gap; spectral stability;
$0 \le P_e < 1$:	EM-region; stability.

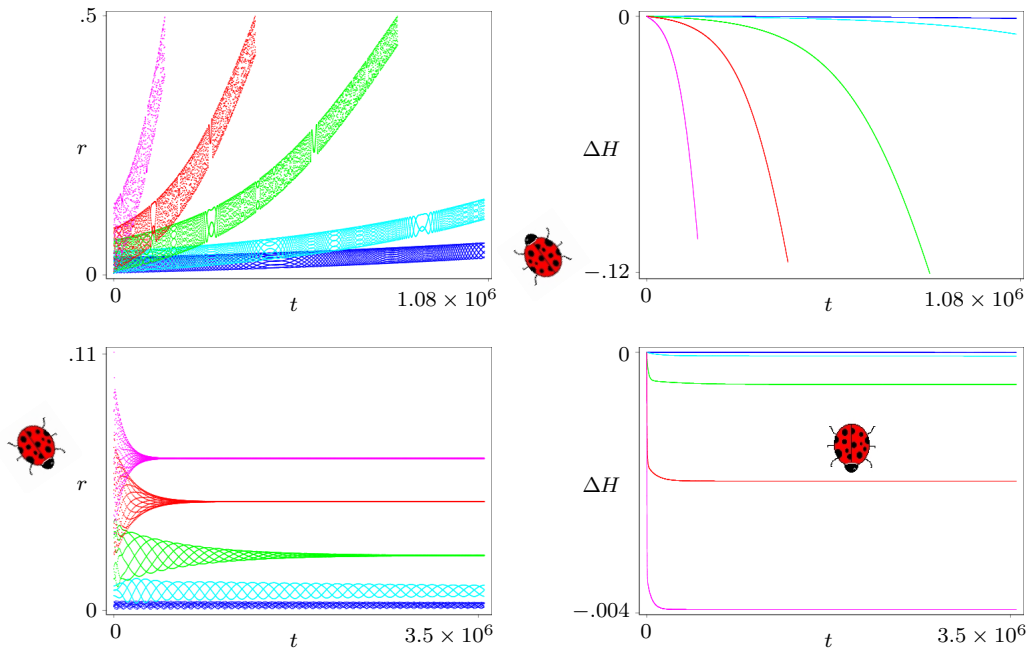

Figure 1: *Left top: instability in the gap. Left bottom: stability in the EM-region. Dissipation is added in both the top and bottom, and the energy decay is shown in the graphs at the right.*

5.2.1 Stability

On the left, Figure 1 shows a regular time sampling of the radius $r = \sqrt{q_1^2 + q_2^2}$ vs. time. The top and bottom graphs correspond respectively to $P_e = 1.5$ (which is inside the gap) and $P_e = .5$ (which is inside the EM-region). The graphs have been obtained from

$$\epsilon = .05, \quad q_1 = 0.05/r, \quad \nu_1^a = 0.01/k, \quad k = 1,\ 1.5,\ 2,\ 3,\ 4,$$
$$\epsilon = .1, \quad q_1 = 0.05/r, \quad \nu_1^a = 0.01/k, \quad k = \sqrt{2},\ (\sqrt{2})^2,\ (\sqrt{2})^3,\ (\sqrt{2})^6,\ (\sqrt{2})^{10}.$$

and for 3×10^5 and 1.2×10^6 periods, respectively. Here, "period" means the smallest period of the normal modes of the linearization at the equilibrium $q = p = 0$, i.e. $2\pi/\omega$ where ω is the largest of the imaginary parts of the eigenvalues computed from (4.19). The top and bottom of Figure 1 corresponds to the first and second lines of initial data above, respectively. The time step for the simulations was $\Delta t \approx .04453$. The simulations were stopped if r exceeded .5, corresponding to a vehicle configuration which is skewed 30 degrees to the vertical.

At the top left, instability is indicated in the gap because large deviations from the equilibrium are observed for sufficiently long time. At the bottom left, stability is indicated in the EM-region because of the decreasing deviation from the equilibrium for decreasing initial conditions, irrespective of the elapsed simulation time. At the

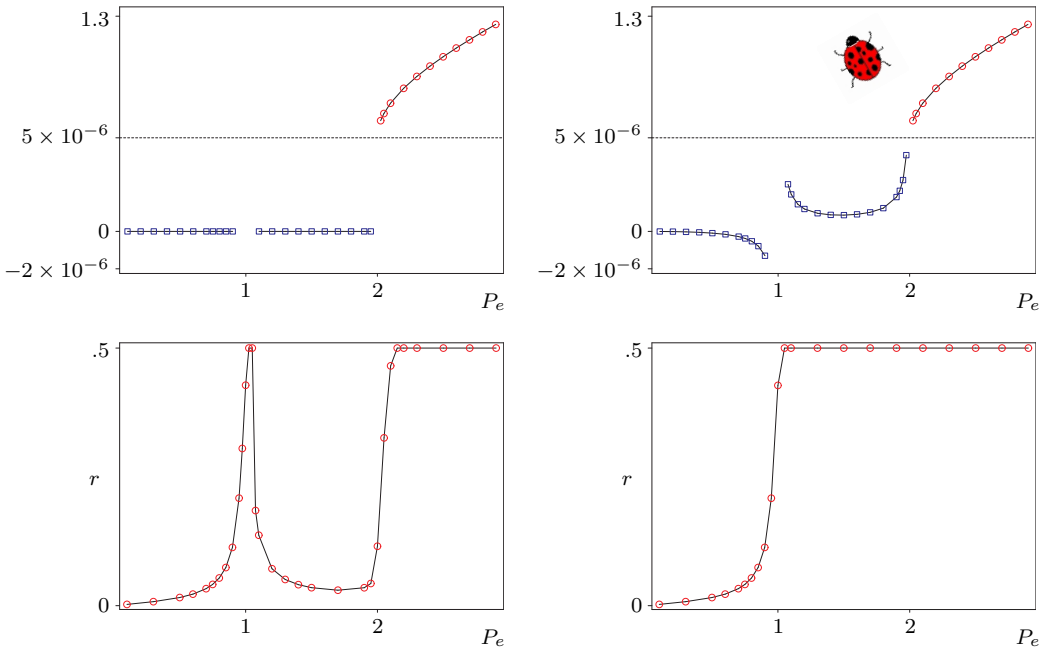

Figure 2: *Destruction of KAM stability of an axisymmetric relative equilibria in the gap between C_1 and C_2 as a result of adding small dissipation. Left: no dissipation, right: added dissipation. Top: plots the real parts of the spectrum of nearby persisting relative equilibria. The scale on the lower parts of the graphs differs from the scale of the upper parts so that the sign change at C_1 is visible. The computation was done with the program MAPLE at 18 digits of accuracy. Bottom: large r indicates instability.*

right, energy dissipation is demonstrated by plotting the energy minus the initial energy (ΔH) against time.

5.2.2 Transitions

On the left bottom, Figure 2 shows the maximum value of r over 1.2×10^6 periods, for

$$\epsilon = 0, \quad q_1 = 0.0125, \quad \nu_1^a = 0.0025.$$

Δt was adjusted to 40 time steps per period. On the right is the same except for $\epsilon = .05$. For the no-dissipation runs on the left, stability in the gap between C_1 and C_2 is indicated by the small maximum values for r. On the right, large deviations from the equilibrium occur over the entire gap between C_1 and C_2.

The peak at C_1 in the left graph indicates loss of stability at that transition, even in the purely Hamiltonian context. This is due to the presence of a zero eigenvalue in the linearization (see (4.19) when $F_q = 0$). At C_1, KAM stability as discussed in Sections 3.1 and 4.4 is not present because it requires perturbation from an elliptic equilibrium but the equilibrium is not elliptic, it has a 0 eigenvalue.

The equilibria of the $\dot{w}$-equation (5.3) at $q = p = 0$ and vertical momentum ν persist to nearby nonvertical momenta. As further evidence of dissipation induced instability, we have computed the corresponding spectrum of the linearization of these equilibria, which were found numerically by Newton's method with start at $q = p = 0$. As shown in the top of Figure 2, at zero dissipation, we observe zero real parts in the spectrum throughout both the EM-region and the gap. For nonzero dissipation, the spectrum splits and has negative real parts in the EM-region and positive real parts in the gap. The real parts in the gap are small as compared to the real parts after the Hopf bifurcation, necessitating the two vertical scales in the Figure.

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