Relative Lyapunov Centre Bifurcations

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Abstract

Relative equilibria and relative periodic orbits (RPOs) are ubiquitous in symmetric Hamiltonian systems and occur for example in celestial mechanics, molecular dynamics and rigid body motion. Relative equilibria are equilibria and RPOs are periodic orbits of the symmetry reduced system. Relative Lyapunov centre bifurcations are bifurcations of relative periodic orbits from relative equilibria corresponding to Lyapunov centre bifurcations of the symmetry reduced dynamics. In this paper we first prove a relative Lyapunov centre theorem by combining recent results on persistence of RPOs in Hamiltonian systems with a symmetric Lyapunov centre theorem of Montaldi et al. We then develop numerical methods for the detection of relative Lyapunov centre bifurcations along branches of RPOs and for their computation. We apply our methods to Lagrangian relative equilibria of the N-body problem.

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Contents

1 Introduction

Relative equilibria (REs) and relative periodic orbits (RPOs) are ubiquitous in symmetric Hamiltonian systems; they occur in many models of celestial mechanics, molecular dynamics, fluid dynamics and continuum mechanics. Relative equilibria are equilibria and RPOs are periodic orbits of the symmetry reduced system. In the original phase space RPOs represent a periodic vibrational dynamics superimposed with a drift along the symmetry group, e.g., superimposed with a rotation. In recent years much progress has been made in the bifurcation theory of Hamiltonian relative equilibria and RPOs, see, e.g., [3, 10, 14, 17, 22, 23, 25, 26, 27, 28, 30, 31]. However, a general theory of generic bifurcations of REs and RPOs so far only exists for dissipative systems, see e.g., [11, 15, 16, 33]. The additional structure of symmetric Hamiltonian systems changes the generic behaviour dramatically compared to general systems. As a result of this, a general bifurcation theory of Hamiltonian relative equilibria and RPOs, and numerical methods for the detection and computation of these bifurcations, are still to be developed. In this paper we make some progress towards this goal.

The paper divides into three parts. In the first part we prove a general theorem on relative Lyapunov bifurcations. These are bifurcations of RPOs from relative equilibria in symmetric Hamiltonian systems and correspond to bifurcations of periodic orbits from equilibria for the symmetry-reduced dynamics. In the second part we develop numerical methods for the detection and computation of relative Lyapunov bifurcations that occur during numerical continuation of RPOs. In the third and final part of the paper we apply both our theoretical and numerical results, to rotating choreographies of the N-body problem.

Let us start by recalling the "plain" Lyapunov centre theorem, see e.g. [20]. Consider a Hamiltonian system

$$\dot{w} = \mathbb{J}\nabla H(w), \tag{1.1}$$

where $w \in \mathbb{R}^n$, $n = 2d$, $\mathbb{J} \in \mathrm{Mat}(n)$ is skew symmetric and invertible, $H : \mathcal{D} \to \mathbb{R}$ is smooth, $\mathcal{D} \subseteq \mathcal{W} := \mathbb{R}^n$ is open, and $\nabla H := (\mathrm{D}H)^T$ is a column vector. Assume that $w = 0$ is an equilibrium of (1.1), with energy $H(0) = 0$, and let $A = \mathbb{J}\mathrm{D}^2 H(0)$. Assume that A has a simple, purely imaginary eigenvalue $\mathrm{i}\bar{\omega}$ and no eigenvalues $\mathrm{i}k\bar{\omega}$, $k \in \mathbb{Z}\setminus\{\pm 1\}$. Then the *Lyapunov centre theorem* states the following:

Theorem 1.1 *Under the above assumptions a smooth family of periodic orbits $\mathcal{P}(s)$ through $w(s)$, $s > 0$, $s \approx 0$, bifurcates from the equilibrium $w(0) = 0$, with energy $E(s) = O(\sqrt{s})$ and period $T(s)$ such that $T(0) = 2\pi/\bar{\omega}$.*

The bifurcating periodic orbits are also called (nonlinear) normal modes. In the case where a compact symmetry group K acts symplectically on $\mathcal{W}$ (see Section 2.1 for a definition) and H is K-invariant, the eigenvalue $\pm\mathrm{i}\bar{\omega}$ might not be simple, owing to symmetry [11]. Montaldi et al [23] have developed a general topological approach for periodic orbits bifurcating from stable K-symmetric equilibria taking into account their symmetries. As a special case, they also prove a symmetric Lyapunov centre theorem based purely on the implicit function theorem — and hence amenable to numerical continuation methods.

In the case of a symplectic continuous symmetry group Γ, the Hamiltonian system conserves, by Noether's theorem, the momentum map of Γ. In general Γ is noncommutative, so both REs and RPOs correspond to a group orbit of momentum values. In the case of relative Lyapunov centre bifurcation this leads to the problem of determining the dimension of the families of RPOs bifurcating from a relative equilibrium to nearby momentum-level sets. To tackle this problem we express the Hamiltonian system in suitable symmetry-adapted coordinates near Hamiltonian relative equilibria, using the bundle equations from [28]. Then we use the persistence results from [27, 30, 31] for transversal relative equilibria and RPOs with regular velocity-momentum pair and regular drift momentum pair, respectively (see Section 2 below for definitions of these terms). In particular, the bifurcating RPOs we obtain have a regular drift-momentum pair. We obtain a relative Lyapunov centre theorem (Theorem 3.2) by applying the symmetric Lyapunov centre theorem of Montaldi et al [23] to a subsystem of the bundle equations of [28] and combining this with the persistence results mentioned above. Theorem 3.2 provides a general result on the existence of smooth branches of RPOs near REs by constructive methods, that are amenable to numerical continuation. We restrict attention to compact symmetry groups Γ.

Let us mention results in the literature which are related to Theorem 3.2: Ginzburg and Lerman [10] prove the existence of RPOs near positive definite relative equilibria in the momentum-level set of the relative equilibrium by means of a relative Moser-Weinstein theorem, extending results of Lerman and Tokieda [17]. Ortega [25] gives topological estimates on the number of RPOs near a symmetric equilibrium; he also obtains results on the existence of RPOs near a relative equilibrium. The results of [10, 17, 25] are based on topological methods and therefore do not guarantee the existence of smooth branches of periodic orbits which are obtained in our setting. Other related work includes results on bifurcations of relative equilibria from symmetric equilibria with continuous isotropy, see, e.g., [3, 14, 25] and results on Hamiltonian Hopf bifurcation from symmetric equilibria by Chossat et al [8]. In this paper we restrict attention to finite rather than continuous isotropy subgroups and do not deal with Hamiltonian Hopf bifurcation, i.e., collisions of two pairs of complex eigenvalues on the imaginary axis. Instead, we restrict attention to the extension of the Lyapunov centre theorem 1.1 to Hamiltonian systems with continuous symmetries.

In the second part of the paper, we develop, based on the Theorem 3.2, general numerical methods for the detection and computation of relative Lyapunov centre bifurcations along branches of relative periodic orbits in the case of compact group actions. Our methods build on previous work by Galan et al [9, 24] who have developed numerical methods for the continuation of normal periodic orbits of symmetric Hamiltonian systems in external parameters, and on our own previous work [35], where we have extended the methods of Galan et al to the continuation of RPOs in energy and momentum.

In the third part of the paper we apply our results to Lagrangian relative equilibria of the N-body problem with identical masses. In [7] Chenciner and Féjoz analyze bifurcations of RPOs from Lagrangian relative equilibria of the N-body problem by means of the Moser-Weinstein theorem. We show that $(N-2)$ different smoothly parametrized non-planar families of rotating choreographies (in the sense specified in Definition 5.4 below) bifurcate from the Lagrangian relative equilibrium of the N-body problem, extending the results of [7] by proving the existence of smooth branches under suitable non-resonance conditions.

Finally we restrict attention to the gravitational three-body problem. Chenciner et al. [5] have proved that three families of rotating choreographies bifurcate from the famous Figure Eight solution of Chenciner and Montgomery [4], these being two non-planar families and one planar family. Two of these families were already known to exist — the planar family and one of the non-planar families which connects to the Lagrangian relative equilibrium (c.f. the discussion in [5]). In [6] Chenciner and Féjoz prove that exactly one non-planar family of RPOs, namely the aforementioned family of rotating Figures of Eight, bifurcates from the Lagrangian relative equilibrium. We have previously applied our numerical continuation methods to this problem too and have shown that the two non-planar families of rotating Figures of Eight are connected via a symmetry breaking bifurcation [35]. In this paper we demonstrate how to detect numerically the relative Lyapunov centre bifurcation that occurs along the non-planar families of rotating Figures of Eight at the Lagrangian relative equilibrium.

The rest of the paper is structured as follows. In Section 2 we introduce symmetric Hamiltonian systems, Hamiltonian relative equilibria and RPOs. In Section 3 we present Theorem 3.2 on relative Lyapunov bifurcations. In Section 4 we develop numerical methods for relative Lyapunov centre bifurcations. In Section 5 we study relative Lyapunov centre bifurcations in the N-body problem.

2 Relative equilibria and RPOs in Hamiltonian systems

In this section we introduce symmetric Hamiltonian systems, relative equilibria and RPOs and review the "bundle equations" near relative equilibria from [28] which we need for the proof of Theorem 3.2 on relative Lyapunov centre bifurcations.

2.1 Symmetric Hamiltonian systems

We consider a Hamiltonian system

$$\dot{x} = f_H(x) = \mathbb{J}\nabla H(x) \tag{2.1}$$

with Hamiltonian (energy) $H(x)$ on a finite-dimensional symplectic vector space $\mathcal{X} = \mathbb{R}^{2d}$ with symplectic structure matrix $\mathbb{J}$ (i.e., $\mathbb{J}$ is skew-symmetric and invertible). Let

$$\Omega(v, w) = \langle \mathbb{J}^{-1}v, w\rangle \tag{2.2}$$

be the symplectic form generated by $\mathbb{J}$. We denote the flow of (2.1) by $\Phi^t(x_0)$, i.e., $x(t) = \Phi^t(x_0)$ is a solution of (2.1) with initial value $x(0) = x_0$. Then the energy $H(x)$ is a conserved quantity of (2.1): $H(\Phi^t(x_0)) = H(x_0)$ for all x_0, t. We assume that a finite-dimensional compact Lie group Γ acts on $\mathcal{X}$ faithfully, linearly and symplectically (i.e., Ω is Γ-invariant) and that the Hamiltonian H is Γ-invariant. This implies that (2.1) is Γ-equivariant, i.e., f and γ commute for all $\gamma \in \Gamma$. We call the elements of Γ the *symmetries* of (2.1). Let $\mathbf{g} = \mathcal{T}_{\mathrm{id}}\Gamma$ denote the Lie algebra of Γ. Since Γ is compact, there is a Γ-invariant inner product which we equip $\mathcal{X}$ with. By Noether's theorem locally there is a conserved quantity $\mathbf{J}_\xi$ of (2.1) for each $\xi \in \mathbf{g}$ which is linear in ξ, so that $\mathbf{J}$, called the momentum map, maps to the dual $\mathbf{g}^*$ of the Lie algebra $\mathbf{g}$ of Γ [1, 19]. Let Ad_γ, $\gamma \in \Gamma$, and ad_ξ, $\xi \in \mathbf{g}$, denote the adjoint (infinitesimal adjoint) actions of Γ and $\mathbf{g}$ on $\mathbf{g}$: $\mathrm{Ad}_\gamma\xi = \gamma\xi\gamma^{-1}$, $\eta \in \mathbf{g}$, $\gamma \in \Gamma$, $\mathrm{ad}_\xi\eta = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{\exp(t\eta}\xi)|_{t=0} = [\xi, \eta]$, and consider the coadjoint action of Γ on $\mathbf{g}^*$ given by $\gamma\mu = (\mathrm{Ad}^*_\gamma)^{-1}\mu$, $\gamma \in \Gamma$, where $(\mathrm{Ad}^*_\gamma\mu)(\xi) = \mu(\mathrm{Ad}_\gamma\xi)$ for $\mu \in \mathbf{g}^*$, $\xi \in \mathbf{g}$, $\gamma \in \Gamma$. We assume throughout the paper that $\mathbf{J}$ is defined on the whole of $\mathcal{X}$ and that it is Γ-equivariant with respect to the Γ-action on $\mathcal{X}$ and the co-adjoint action on $\mathbf{g}^*$. Moreover we choose an Ad-invariant inner product on $\mathbf{g}$ such that the adjoint action on $\mathbf{g}$ is by orthogonal matrices, $\mathbf{g}$ can be identified with $\mathbf{g}^*$ and the adjoint and co-adjoint actions are identical.

For an action of a group Γ on a space $\mathcal{X}$ we define the *isotropy subgroup* of $x \in \mathcal{X}$ as $\Gamma_x = \{\gamma \in \Gamma,\ \ \gamma x = x\}$, see [11], and denote its Lie algebra by $\mathbf{g}_x$. For any subgroup K or element γ of Γ we define the *fixed point space* of K and γ as $\mathrm{Fix}_{\mathcal{X}}(K) = \{x \in \mathcal{X},\ \ \gamma x = x\ \forall\gamma \in K\}$ and $\mathrm{Fix}_{\mathcal{X}}(\gamma) = \{x \in \mathcal{X},\ \ \gamma x = x\}$, respectively. We denote by $N(K) = \{\gamma \in \Gamma, \gamma K\gamma^{-1} = K\}$ the *normalizer* of any subgroup K of Γ. For any $\alpha \in \Gamma$ we define $Z(\alpha) = \{\gamma \in \Gamma,\ \ \gamma\alpha = \alpha\gamma\}$

to be the *centralizer* of α. For any group Γ define Γ^{id} to be the connected component of Γ containing the identity. Note that in this Section and in Section 3 we could as well assume that $\mathcal{X}$ is a symplectic manifold. But in Sections 4 and 5 we need $\mathcal{X}$ to be a symplectic vector space.

Example 2.1 In the case of rotational symmetries $\Gamma = \mathrm{SO}(3)$ we have $\mathbf{g}^* = \mathrm{so}(3)^* \equiv \mathbb{R}^3$ and $\mathbf{J} : \mathcal{X} \to \mathbb{R}^3$ is the angular momentum; see Section 5 below for an example from celestial mechanics. In this case $\mathbf{g} = \mathrm{so}(3) \simeq \mathbb{R}^3$, where we identify $\omega \in \mathbb{R}^3$ with an infinitesimal rotation so(3) of frequency $|\omega|$ around the vector ω. Then the adjoint and co-adjoint actions are just the usual multiplication by matrices in SO(3). The Lie bracket becomes $[\xi, \eta] = \xi \times \eta$, where $\xi, \eta \in \mathbb{R}^3 \simeq \mathrm{so}(3)$, see, e.g., [1, 19].

2.2 Relative equilibria

A point $\bar{x} \in \mathcal{X}$ lies on a *relative equilibrium* $\Gamma\bar{x}$ if there is some $\bar{\xi} \in \mathbf{g}$ such that $\bar{\xi}\bar{x} = f_H(\bar{x})$, i.e., the relative equilibrium through $\bar{x}$ is an equilibrium of the Hamiltonian system (2.1) in a frame moving with velocity $\bar{\xi}$. We call $\bar{\xi}$ the drift velocity of the relative equilibrium at $\bar{x}$ and denote by $\bar{K} = \Gamma_{\bar{x}} = \{\gamma \in \Gamma, \gamma\bar{x} = \bar{x}\}$ the isotropy subgroup of the relative equilibrium. In this paper we assume that $\bar{K}$ is finite.

Momentum conservation implies that the drift velocity $\bar{\xi}$ and momentum $\bar{\mu} = \mathbf{J}(\bar{x})$ of a relative equilibrium satisfy $\mathrm{ad}^*_{\bar{\xi}}\bar{\mu} = 0$ [27]. As in [31] we call pairs $(\xi, \mu) \in \mathbf{g} \oplus \mathbf{g}^*$ satisfying $\mathrm{ad}^*_{\xi}\bar{\mu} = 0$ *velocity-momentum pairs* and denote the space of velocity-momentum pairs by $(\mathbf{g} \oplus \mathbf{g}^*)^c$. We define an action of Γ on the space of velocity-momentum pairs as $\gamma(\xi, \mu) = (\mathrm{Ad}_\gamma\xi, (\mathrm{Ad}^*_\gamma)^{-1}\mu)$, for $\gamma \in \Gamma$, $(\xi, \mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^c$. For later purposes we define $r_{(\xi,\mu)} = \dim \mathbf{g}_{(\xi,\mu)}$ for $(\xi, \mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^c$. Moreover we define $\Gamma_\xi = \Gamma_{(\xi,0)}$ for $\xi \in \mathbf{g}$ and $\Gamma_\mu = \Gamma_{(0,\mu)}$ for $\mu \in \mathbf{g}^*$. As in [27, 31] we call a velocity-momentum pair $(\xi, \mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^c$ *regular* if $\dim \mathbf{g}_{(\xi,\mu)}$ is locally constant in the space of velocity-momentum pairs. We call $\mu \in \mathbf{g}^*$ regular if $\dim \mathbf{g}_\mu$ is locally constant in $\mathbf{g}^*$. Regular velocity-momentum pairs and regular momenta are generic in their respective spaces.

Example 2.2 In the case $\Gamma = \mathrm{SO}(3)$ a velocity-momentum pair (ξ, μ) satisfies $\xi \times \mu = 0$ (see Example 2.1) and so $\xi || \mu$. Then (ξ, μ) is regular if $(\xi, \mu) \neq 0$. In this case $\mathbf{g}_{(\xi,\mu)} = \mathrm{span}(\xi, \mu) \simeq \mathbb{R}$.

Denote by $\mathcal{N}$ a normal space transverse to $\Gamma\bar{x}$ at $\bar{x}$, i.e., $\mathcal{X} = \mathcal{T}_{\bar{x}}\Gamma\bar{x} \oplus \mathcal{N}$. Then $\mathcal{N}$ is a model for the space of group orbits $\mathcal{X}/\Gamma$ near $\bar{x}$. As in [28] let us decompose $\mathcal{X} = \mathcal{T} \oplus \mathcal{N}$, $\mathcal{T} = \mathbf{g}\bar{x} = \mathcal{T}_0 \oplus \mathcal{T}_1$ and define $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$, where

$$\mathcal{T}_0 = \mathcal{T} \cap \mathcal{T}^\Omega, \quad \mathcal{T}_1 = \mathcal{T} \cap \mathcal{T}_0^\perp, \quad \mathcal{N}_0 = (\mathcal{T} + \mathcal{T}^\Omega)^\perp, \quad \mathcal{N}_1 = \mathcal{T}^\Omega \cap \mathcal{T}_0^\perp \tag{2.3}$$

for some $\bar{K}$-invariant inner product and

$$\mathcal{T}^\Omega = \{x \in \mathcal{X}, \quad \Omega(x, v) = 0 \quad \forall v \in \mathcal{T}\} = \ker \mathrm{D}\mathbf{J}(\bar{x}).$$

Here

$$\mathbb{J}\mathcal{T}_0 = \mathcal{N}_0, \quad \mathbb{J}\mathcal{N}_0 = \mathcal{T}_0, \quad \mathbb{J}\mathcal{T}_1 = \mathcal{T}_1, \quad \mathbb{J}\mathcal{N}_1 = \mathcal{N}_1. \tag{2.4}$$

The space $\mathcal{N}_1$ with symplectic structure matrix $\mathbb{J}_{\mathcal{N}_1}$ is called *symplectic normal space*, and all spaces $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{T}_0$ and $\mathcal{T}_1$ are $\bar{K}$-invariant. Let $\mathbf{n}_{\bar{\mu}}$ be a $\Gamma_{\bar{\mu}}$-invariant complement to $\mathbf{g}_{\bar{\mu}}$ in $\mathbf{g}^*$. Then the annihilator $\mathrm{ann}(\mathbf{n}_{\bar{\mu}})$ of $\mathbf{n}_{\bar{\mu}}$ in $\mathbf{g}$ is a $\Gamma_{\bar{\mu}}$-invariant section transverse to the momentum group orbit $\Gamma\bar{\mu}$ at $\bar{\mu}$ in $\mathbf{g}^*$ and

$$\mathcal{N}_0 \simeq \mathbf{g}^*_{\bar{\mu}} \simeq \mathrm{ann}(\mathbf{n}_{\bar{\mu}}), \quad \mathcal{T}_0 \simeq \mathbf{g}_{\bar{\mu}}, \quad \mathcal{T}_1 \simeq \mathbf{n}_{\bar{\mu}}. \tag{2.5}$$

Furthermore, there are coordinates $x \simeq (\gamma, v)$, $\gamma \in \Gamma$, $v \in \mathcal{N}$, in a Γ-invariant neighbourhood $\mathcal{U}$ of $\Gamma\bar{x}$ such that $\mathcal{U}$ is symplectomorphic to

$$\mathcal{U} \simeq \Gamma \times_{\bar{K}} \mathcal{N}, \tag{2.6}$$

where $\bar{x} \simeq (\mathrm{id}, 0)$ [12]. Here we identify $(\gamma, v) \simeq (\gamma k^{-1}, kv)$ for $k \in \bar{K}$. We decompose $v \in \mathcal{N}$ as $v = (\nu, w)$, $\nu \in \mathcal{N}_0$, $w \in \mathcal{N}_1$. Then the dynamics in the coordinates (γ, ν, w) takes the form [28]:

$$\dot{\gamma} = f_\Gamma(\nu, w) = \gamma \mathrm{D}_\nu h(\nu, w), \quad \dot{\nu} = f_{\mathcal{N}_0}(\nu, w) = \mathrm{ad}^*_{\mathrm{D}_\nu h(\nu,w)}\nu, \quad \dot{w} = f_{\mathcal{N}_1}(\nu, w) = \mathbb{J}_{\mathcal{N}_1}\mathrm{D}_w h(\nu, w). \tag{2.7}$$

The original relative equilibrium corresponds to the equilibrium $(\nu, w) = 0$ of the (ν, w)-subsystem $\dot{v} = f_{\mathcal{N}}(v)$ of (2.7). The Hamiltonian $h(\nu, w)$ of (2.7) is $\bar{K}$-invariant where $\bar{K}$ acts as $\gamma\nu = (\mathrm{Ad}^*_\gamma)^{-1}\nu$, $\gamma \in \bar{K}$, $\nu \in \mathcal{N}_0$. The momentum map in these coordinates takes the form $\mathbf{j}(\gamma, \nu, w) = \gamma(\bar{\mu} + \nu)$.

As in [31] we call a relative equilibrium $\Gamma\bar{x}$ *nondegenerate* if $\mathrm{D}_w f_{\mathcal{N}_1}(0, 0)$ is invertible. For a velocity-momentum pair (ξ, μ) let $r = r_{(\xi,\mu)} = \dim \mathbf{g}_{(\xi,\mu)}$. A nondegenerate relative equilibrium $\Gamma\bar{x}$ with regular velocity-momentum pair $(\bar{\xi}, \bar{\mu})$ can be continued to a smooth $r_{(\bar{\xi},\bar{\mu})}$-dimensional family $\Gamma x(\chi)$ of REs, with drift velocity $\xi(\chi) \in \mathbf{g}_{(\bar{\xi},\bar{\mu})}$ and with momentum $\mathbf{J}(x(\chi)) = \bar{\mu} + \chi$, where $\chi \in \ker \mathrm{ad}^*_{\bar{\xi}}|_{\mathbf{g}^*_{\bar{\mu}}} \simeq \mathbf{g}^*_{(\bar{\xi},\bar{\mu})} \simeq \mathbb{R}^r$, see [27, 31]. For later purposes, we define a relative equilibrium $\Gamma\bar{x}$ to be *L-nondegenerate* for any subgroup L of $\Gamma_{\bar{x}}$ if $\mathrm{D}_w f_{\mathcal{N}_1}(0, 0)|_{\mathrm{Fix}_{\mathcal{N}_1}(L)}$ is nondegenerate [31].

2.3 Relative periodic orbits

A point $\bar{x} \in \mathcal{X}$ lies on a *relative periodic orbit* (RPO) if there exists $t > 0$ such that $\Phi^t(\bar{x}) \in \Gamma\bar{x}$. The infimum $\bar{\tau}$ of such t is called the *relative period* of the RPO and the element $\bar{\sigma} \in \Gamma$ such that $\bar{\sigma}\Phi^{\bar{\tau}}(\bar{x}) = \bar{x}$ is called *drift symmetry* of the RPO. The relative periodic orbit $\bar{\mathcal{P}}$ itself is given by $\bar{\mathcal{P}} = \{\gamma\Phi^\theta(\bar{x}),\ \gamma \in \Gamma, \theta \in \mathbb{R}\}$. We assume that $\bar{\tau} > 0$ so that $\bar{\mathcal{P}}$ is a proper RPO (i.e., not a relative equilibrium). Let $K := \Gamma_{\bar{x}}$ be the isotropy subgroup (spatial symmetry group) of the RPO through $\bar{x}$. In what follows in this section we assume that K is trivial; if not we restrict (2.1) to $\mathrm{Fix}_{\mathcal{X}}(K)$ and redefine Γ as $N(K)/K$, c.f. Lemma 3.4 below.

The RPO through $\bar{x}$ becomes a periodic orbit with period $\bar{T} = \ell\bar{\tau}$ in a comoving frame $\bar{\xi} \in \mathbf{g}$ [30, 33, 35] for some $\ell \in \mathbb{N}$. This is due to the fact that there are $\ell \in \mathbb{N}$, $\bar{\xi} \in \mathbf{g}$, $\alpha \in \Gamma$ so that we can decompose

$$\bar{\sigma} = \alpha \exp(-\bar{\tau}\bar{\xi}), \quad \alpha^\ell = \mathrm{id}, \quad \mathrm{Ad}_\alpha \bar{\xi} = \bar{\xi}, \quad \mathbf{g}_{\bar{\sigma}} = \mathbf{g}(\alpha) \cap \mathbf{g}_{\bar{\xi}}. \tag{2.8}$$

We call $\bar{\xi}$ an *average drift velocity* of the RPO.

Similarly as for relative equilibria momentum conservation implies that the drift symmetry $\bar{\sigma}$ and momentum $\bar{\mu} = \mathbf{J}(\bar{x})$ of an RPO through $\bar{x}$ satisfy $\bar{\sigma}\bar{\mu} = \bar{\mu}$ [30]. This implies that the average drift velocity $\bar{\xi}$ and the drift symmetry α in the comoving frame in (2.8) satisfy

$$\bar{\xi} \in \mathbf{g}_{(\bar{\sigma},\bar{\mu})}, \quad \bar{\alpha} \in \Gamma_{\bar{\mu}}, \tag{2.9}$$

c.f. [30, 35]. Analogously to relative equilibria, we call pairs $(\sigma, \mu) \in \Gamma \times \mathbf{g}^*$ satisfying $\bar{\sigma}\bar{\mu} = \bar{\mu}$ *drift-momentum pairs* and denote the space of drift-momentum pairs by $(\Gamma \times \mathbf{g}^*)^c$. We define an action of Γ on pairs (σ, μ) as $\gamma(\sigma, \mu) = (\gamma\sigma\gamma^{-1}, (\mathrm{Ad}^*_\gamma)^{-1}\mu)$, where $\gamma \in \Gamma$. For later purposes we define $\Gamma_\sigma = \Gamma_{(\sigma,0)}$ for any $\sigma \in \Gamma$ and set $r_{(\sigma,\mu)} = \dim \mathbf{g}_{(\sigma,\mu)}$ where $\mathbf{g}_{(\sigma,\mu)}$ is the isotropy subalgebra of $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ with respect to this action. As in [30, 35] we call a drift-momentum pair $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ *regular* if $r_{(\sigma,\mu)}$ is locally constant in the space of drift-momentum pairs. Regular drift-momentum pairs (σ, μ) are generic in $(\Gamma \times \mathbf{g}^*)^c$.

Similarly as for REs (see Section 2.2), an RPO through $\bar{x}$ is called nondegenerate if the eigenvalue 1 of its linearization $\mathrm{D}\bar{\sigma}\Phi^{\bar{\tau}}(\bar{x})$ has minimal multiplicity as enforced by symmetry and conserved quantities. Then, analogously to REs, a nondegenerate RPO through $\bar{x}$ with regular-drift momentum pair $(\bar{\sigma}, \bar{\mu})$ can be continued to an $(r_{(\bar{\sigma},\bar{\mu})} + 1)$-dimensional families of RPOs $\mathcal{P}(E, \chi)$ through $x(E, \chi) \approx \bar{x}$, parametrized by energy $H(x(E, \chi)) = E$ and momentum

$\mathbf{J}(x(E,\chi)) = \bar{\mu} + \chi$, $\chi \in \mathbf{g}^*_{(\bar{\sigma},\bar{\mu})} \simeq \mathbb{R}^r$, where $r = r_{(\bar{\sigma},\bar{\mu})} = \dim \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$, see [30]. This continuation result is also valid if the non-degeneracy condition fails, but the RPO is still *transversal*, a condition which allows for folds in (E,χ). In this case the bifurcating branch $\mathcal{P}(s)$ of RPOs is parametrized by $s \in \mathbb{R}^{r+1}$, see [35] for details.

3 Relative Lyapunov-centre bifurcations

In this section we present an extension of the Lyapunov centre theorem (Theorem 1.1) to nondegenerate relative equilibria with regular velocity-momentum pair as defined in Section 2.2. We start by reviewing the symmetric Lyapunov centre theorem of Montaldi et al [23].

3.1 Lyapunov centre bifurcations from symmetric equilibria

Let us consider a Hamiltonian system

$$\dot{w} = \mathbb{J}\nabla h(w) \tag{3.1}$$

on a symplectic space $\mathcal{W}$ with the symplectic action of a finite group Γ, and assume that $h(w)$ is Γ-invariant and that $w = 0$ is a Γ-invariant equilibrium of (3.1), i.e., $\bar{K} = \Gamma_{\bar{x}} = \Gamma$. Let $A := \mathbb{J}\mathrm{D}^2_w h(0)$ and let $\pm \mathrm{i}\omega$, $\omega \neq 0$, be semi-simple eigenvalues of A. As in [23] we use the following notation:

a) $\mathcal{W}^\omega$ is the real eigenspace of A to all eigenvalues in $\mathrm{i}\omega\mathbb{Z} \setminus \{0\}$ (not the generalized real eigenspace) and $A^\omega = A|_{\mathcal{W}^\omega}$.

b) $\mathcal{W}_{j\omega}$ is the real eigenspace of A to the eigenvalues $\mathrm{i}j\omega$, for $j \in \mathbb{Z} \setminus \{0\}$ and is $\{0\}$ if $\mathrm{i}j\omega$ is not an eigenvalue of A.

Then $\mathcal{W}^\omega$ is Γ-invariant, [11], and we can write $w \in \mathcal{W}^\omega$ as $w = \sum_{j=1}^\infty w_j$ where $w_j \in \mathcal{W}_{j\omega}$. Let $\Sigma := \bar{K} \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then we can define a Σ-action on $\mathcal{W}^\omega$ as

$$((\gamma,\theta)w)_j := \gamma \exp\left(\frac{\theta 2\pi}{\omega} A^\omega\right) w_j, \quad j \in \mathbb{Z} \setminus \{0\}. \tag{3.2}$$

Let Λ be an isotropy subgroup of the action (3.2) of Σ and let

$$L = \{\gamma \in \Gamma, \ \ (\gamma,\theta) \in \Lambda \ \text{ for some } \theta\}. \tag{3.3}$$

Then there is a map $\Theta : L \to \mathbb{R}/\mathbb{Z}$ such that any $(\gamma,\theta) \in \Lambda$ satisfies $\theta = \Theta(\gamma)$ [11]. Such subgroups of Σ are called twisted subgroups. Let

$$K = \{\gamma \in \bar{K}, \ \ (\gamma, 0) \in \Lambda\} = \ker \Theta. \tag{3.4}$$

Then K is normal in L and, since Γ is finite, $L/K \simeq \mathbb{Z}_\ell(\alpha)$ for some $\ell \in \mathbb{N}$, where $(\alpha, \frac{1}{\ell}) \in \Lambda$, see [11, 23]. We define the operation of $\Sigma = \bar{K} \times \mathbb{S}^1$ on a T-periodic solution $w(\cdot)$ as

$$((\gamma,\theta)w)(t) = \gamma w(t + \theta T) \quad \text{for} \quad (\gamma,\theta) \in \bar{K} \times \mathbb{S}^1.$$

The *spatio-temporal symmetry group* $\Lambda \subset \bar{K} \times \mathbb{S}^1$ of $w(\cdot)$ is the isotropy subgroup of $w(\cdot)$ with respect to this group action, and K from (3.4) is its *spatial symmetry group.* Let $L/K \simeq \mathbb{Z}_\ell(\alpha)$, where $\Theta(\alpha) = \frac{1}{\ell}$; then we call α a *drift symmetry* of the symmetric periodic orbit $w(\cdot)$.

Montaldi et al prove a symmetric Moser-Weinstein theorem in [23, Theorem 1.1] which contains a symmetric Lyapunov centre theorem as a special case, cf. [23, Remark 1.2b)]. Their result easily extends to the following:

Theorem 3.1 *Let $w = 0$ be an equilibrium of the Γ-equivariant Hamitonian system (3.1) and let $\pm \mathrm{i}\bar{\omega}$, $\bar{\omega} > 0$, be semi-simple eigenvalues of A. Let $\Lambda \subseteq \Gamma \times \mathbb{S}^1$ be a symmetry group such that $L/K \simeq \mathbb{Z}_\ell$ for some $\ell \in \mathbb{N}$ with K as in (3.4). Assume*

$$\mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda) \subseteq \mathcal{W}_{\bar{\omega}} \tag{3.5}$$

and

$$\dim \mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda) = 2 \tag{3.6}$$

for the action (3.2). Define L as in (3.3). Assume 0 is an L-nondegenerate equilibrium. Then there is a unique branch $w(t;\epsilon)$, $\epsilon \geq 0$, of periodic solutions with amplitude $O(\epsilon)$ bifurcating from $w = 0$ with $\mathrm{D}_\epsilon w(t;0) \in \mathrm{Fix}_{\mathcal{W}_{\bar{\omega}}}(\Lambda)$ with energy $E(\epsilon)$ such that $E(0) = \bar{E}$, where $\bar{E} = h(0)$, $E'(0) = 0$, $E''(0) \neq 0$, with minimal period $T(\epsilon)$, such that $T(0) = 2\pi/\bar{\omega}$, $T'(0) = 0$, and with Λ as spatio-temporal symmetry group.

Proof. Replace $\mathcal{W}$ by $\mathrm{Fix}_{\mathcal{W}}(K)$ and Γ by $N(K)/K$, see Lemma 3.4 a) below, and denote the flow of (3.1), as before, by $\Phi^t(\cdot)$. By assumption there is an eigenvector $\bar{w}$ of A to the eigenvalue $\mathrm{i}\bar{\omega}$ with $\mathrm{Re}(\bar{w}), \mathrm{Im}(\bar{w}) \in \mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda)$. Choose $\bar{w}$ such that $\|\mathrm{Re}(\bar{w})\|_2 = 1$. Let P be a projection onto $\mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda)$ which commutes with A. Write $w = \delta_1 \mathrm{Re}(\bar{w}) + \delta_2 \mathrm{Im}(\bar{w}) + u$ where $u \in (\mathrm{id} - \mathrm{P})\mathcal{W}$, $\delta_1, \delta_2 \in \mathbb{R}$. Then $\mathcal{S} = \{\delta_2 = 0, \delta_1 > 0\} \times (\mathrm{id} - \mathrm{P})\mathcal{W}$ is a section transverse to the flow for $w \neq 0$, $w \approx 0$. Define the Poincaré-map $\Pi : \mathcal{S} \to \mathcal{S}$ as $\Pi(w) = \alpha \Phi^{\tau(w)}(w)$ where $\tau(w)$ is such that $\Phi^{\tau(w)}(w) \in \alpha^{-1}\mathcal{S}$ and $\tau(0) = 2\pi/(\bar{\omega}\ell)$. Here $(\alpha, 1/\ell) \in \Lambda$ as before. Due to (3.6), the L-nondegeneracy of $w = 0$ and due to the fact that $\mathrm{i}\bar{\omega}$ is a semi-simple eigenvalue the matrix $(\mathrm{id} - \mathrm{P})(\mathrm{D}\Pi(0) - \mathrm{id})$ is invertible. Denote by $u(\delta_1)$ the solution of $(\mathrm{id} - \mathrm{P})\Pi(u + \delta_1 \mathrm{Re}(\bar{w})) = u$ and let $\tau(\delta_1) = \tau(u(\delta_1) + \delta_1 \mathrm{Re}(\bar{w}))$. Define

$$\epsilon(\delta_1) = \langle \mathrm{Re}\,\bar{w}, \int_0^1 \mathrm{e}^{(1-s)\frac{2\pi}{\bar{\omega}}A} \Phi^{s\ell\tau(\delta_1)}(u(\delta_1) + \delta_1 \mathrm{Re}(\bar{w}))\mathrm{d}s\rangle = \delta_1 + O(\delta_1^2)$$

and $w(\epsilon) = u(\delta_1(\epsilon)) + \delta_1(\epsilon)\mathrm{Re}(\bar{w})$, $\epsilon \geq 0$. Then $w(0) = 0$, $\mathrm{D}_\epsilon w(0) = \mathrm{Re}(\bar{w})$ and $w(\epsilon)$ has energy $E(\epsilon)$ satisfying $E(0) = \bar{E}$ and $\mathrm{D}_\epsilon E(0) = H'(0)w'(0) = 0$. Moreover $\mathrm{D}_\epsilon^2 E(0) = \langle \mathrm{D}_w^2 h(0)\mathrm{Re}(\bar{w}), \mathrm{Re}(\bar{w})\rangle = -\bar{\omega}\langle \mathbb{J}^{-1}\mathrm{Im}(\bar{w}), \mathrm{Re}(\bar{w})\rangle \neq 0$. So energy changes when we move along the line spanned by $\mathrm{Re}(\bar{w})$ near $w = 0$. Therefore energy conservation implies that $w(\epsilon)$ is a fixed point of Π and hence we obtain a family of periodic solutions through $w(\epsilon)$, $\epsilon \geq 0$, of (3.1) with spatio-temporal symmetry Λ and period $T(\epsilon) = \ell\tau(\delta_1(\epsilon))$. We have

$$w(-\epsilon) = \Phi^{T(\epsilon)/2}(w(\epsilon)). \tag{3.7}$$

Therefore $T(\epsilon)$ is even in ϵ and so $\mathrm{D}_\epsilon T(0) = 0$. ∎

3.2 A relative Lyapunov centre theorem

In this section we prove the following extension of Theorem 3.1:

Theorem 3.2 *Let $\bar{x}$ lie on a relative equilibrium $\Gamma\bar{x}$ with drift velocity $\bar{\xi}$, momentum $\bar{\mu}$ and discrete isotropy subgroup $\bar{K}$. Assume that $\mathbb{J}_{\mathcal{N}_1}\mathrm{D}_w^2 h(0)$ has a pair of semi-simple eigenvalues $\pm\mathrm{i}\bar{\omega}$, $\bar{\omega} > 0$. Let $\Lambda \subseteq \Sigma := \bar{K} \times \mathbb{S}^1$ be such that (3.5) and (3.6) hold for the action (3.2) with $\mathcal{W} = \mathcal{N}_1$ and A replaced by $\mathbb{J}_{\mathcal{N}_1}\mathrm{D}_w^2 h(0)$. Define K as in (3.4), L as in (3.3), and, as before, let $\ell \in \mathbb{N}$ be such that $L/K \simeq \mathbb{Z}_\ell$ and let $\alpha \in L$ be such that $(\alpha, \frac{1}{\ell}) \in \Lambda$. Assume that $\Gamma\bar{x}$ is an L-nondegenerate relative equilibrium and that $(\bar{\xi}, \bar{\mu})$ is a regular velocity-momentum pair for the group $N(L)/L$. Denote the dimension of its isotropy subalgebra for the group $N(L)/L$ by r. Then:*

a) there is an r-dimensional family of L-nondegenerate relative equilibria $\Gamma x(\chi)$ with momentum $\mathbf{J}(x(\chi)) = \bar{\mu} + \chi$, $\chi \in \mathrm{Fix}_{\mathbf{g}^*_{(\bar{\xi},\bar{\mu})}}(L) \simeq \mathbb{R}^r$, and drift velocity $\xi(\chi) \in \mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L)$ at $x(\chi) \in \mathrm{Fix}_{\mathcal{X}}(L)$ such that $x(0) = \bar{x}$, and all relative equilibria near $\bar{x}$ inside $\mathrm{Fix}_{\mathcal{X}}(L)$ belong to this family.

b) Let $\bar{\sigma} = \alpha \exp(-\bar{\tau}\bar{\xi})$ with $\bar{\tau} = \frac{2\pi}{\bar{\omega}\ell}$. Assume that

$$\dim \mathrm{Fix}_{\mathbf{g}_{(\bar{\sigma},\bar{\mu})}}(K) = r. \tag{3.8}$$

Then there is an $(r+1)$-dimensional family of RPOs $\mathcal{P}(\epsilon,\chi)$, $\epsilon > 0$, $\chi \in \mathbb{R}^r$, $(\epsilon,\chi) \approx (0,0)$, and there are smooth functions $x(\epsilon,\chi) \in \mathrm{Fix}_{\mathcal{X}}(K)$, $\tau(\epsilon,\chi) > 0$, $\xi(\epsilon,\chi) \in \mathbf{g}$ with $x(0,\chi) = x(\chi)$, $\tau(0,0) = \bar{\tau}$, $\xi(0,\chi) = \xi(\chi)$ such that $x(\epsilon,\chi)$ lies on $\mathcal{P}(\epsilon,\chi)$, $\mathcal{P}(\epsilon,\chi)$ has relative period $\tau(\epsilon,\chi)$, and has

$$\begin{array}{l} \textit{momentum} \ \ \mathbf{J}(x(\epsilon,\chi)) = \bar{\mu} + \chi, \quad \chi \in \mathrm{Fix}_{\mathbf{g}^*_{(\bar{\sigma},\bar{\mu})}}(L) \simeq \mathbb{R}^r, \\ \textit{drift symmetry} \ \ \sigma(\epsilon,\chi) = \alpha \exp(-\tau(\epsilon,\chi)\xi(\epsilon,\chi)) \in N(K), \\ \textit{average drift velocity} \ \ \xi(\epsilon,\chi) \in \mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L) \end{array}$$

and energy $E(\epsilon,\chi)$ at $x(\epsilon,\chi)$ such that $\partial_\epsilon E(\epsilon,\chi)|_{(\epsilon,\chi)=0} = 0$, $\partial^2_\epsilon E(\epsilon,\chi)|_{(\epsilon,\chi)=0} \neq 0$. Moreover $\mathrm{D}_\epsilon \tau(0,\chi) = 0$, $\mathrm{D}_\epsilon \xi(0,\chi) = 0$.

For the proof we need the following lemmata:

Lemma 3.3

a) [27, 31] A velocity-momentum pair $(\xi,\mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^c$ is regular if and only if $\mathbf{g}_{(\xi,\mu)}$ is the Lie algebra of a maximal torus. In particular for a regular velocity-momentum pair (ξ,μ) the isotropy subalgebra $\mathbf{g}_{(\xi,\mu)}$ is abelian.

b) [30, 35] A drift momentum pair (σ,μ), where $\sigma = \alpha\exp(-\xi)$, $\mathrm{Ad}_\alpha \xi = \xi$, is regular if and only if (ξ,μ) is a regular velocity-momentum pair for the group $Z(\alpha)$. In particular, by a), $\mathbf{g}_{(\sigma,\mu)}$ is abelian if (σ,μ) is regular.

Lemma 3.4 *Let K be a finite subgroup of Γ. Then:*

a) $\mathrm{Fix}_{\mathcal{X}}(K)$ is invariant under the symmetry group $N(K)/K$.

b) The Lie algebra of $N(K)/K$ is $\mathrm{Fix}_{\mathbf{g}}(K)$.

c) The velocity-momentum pair $(\bar{\xi},\bar{\mu})$ of a relative equilibrium through $\bar{x}$ with isotropy subgroup $\bar{K}$ satisfies

$$\bar{\xi} \in \mathrm{Fix}_{\mathbf{g}}(\bar{K}), \quad \bar{\mu} \in \mathrm{Fix}_{\mathbf{g}^*}(\bar{K}).$$

Part a) of Lemma 3.4 is well known [11], for part b) see [27, 31], part c) follows immediately from a) and b).

Proof of Theorem 3.2. By assumption $(\bar{\xi},\bar{\mu})$ is regular velocity-momentum pair for the group $N(L)/L$, therefore its isotropy subalgebra, which is $\mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L)$ by Lemma 3.4 c), is abelian by Lemma 3.3 a), and, by assumption, has dimension r. Since $\bar{\xi} \in \mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L)$, $\alpha \in L$ and $\bar{\sigma} = \alpha\exp(-\bar{\tau}\bar{\xi})$, we have $\mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L) \subseteq \mathrm{Fix}_{\mathbf{g}_{(\bar{\sigma},\bar{\mu})}}(K)$. Hence condition (3.8) ensures that both these spaces have dimension r and so are identical:

$$\mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L) = \mathrm{Fix}_{\mathbf{g}_{(\bar{\sigma},\bar{\mu})}}(K). \tag{3.9}$$

We now replace $\mathcal{X}$ by $\mathrm{Fix}_{\mathcal{X}}(K)$, Γ by $N(K)/K$, and $\mathbf{g}$ by $\mathrm{Fix}_{\mathbf{g}}(K)$, the Lie algebra of $N(K)/K$ (see Lemma 3.4). Then K becomes trivial and $L = \mathbb{Z}_\ell(\alpha)$. Therefore (3.9) becomes

$$\mathbf{g}_{(\bar{\xi},\bar{\mu})} \cap \mathbf{g}_\alpha = \mathbf{g}_{(\bar{\sigma},\bar{\mu})}, \tag{3.10}$$

which is abelian. Next, instead of equipping $\mathcal{U}$ with its full symmetry group Γ, we consider it with the smaller symmetry group $\tilde{\Gamma} = \Gamma_{\bar{\sigma}}$, to exploit the fact that the momentum isotropy subalgebra $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ of μ with respect to $\tilde{\Gamma}$ is abelian, see below. We denote the slice at $\bar{x}$ for the $\tilde{\Gamma}$ action by $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 \oplus \widetilde{\mathcal{N}}_1$, c.f. (2.3). Here $\widetilde{\mathcal{N}}_0 \simeq \mathbf{g}^*_{(\bar{\sigma},\bar{\mu})}$, and, since $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ is abelian, we have $f_{\widetilde{\mathcal{N}}_0} \equiv 0$ by (2.7), with $\mathcal{N}_0$ replaced by $\widetilde{\mathcal{N}}_0$ and Γ replaced by $\tilde{\Gamma}$. So $\tilde{\nu} \in \widetilde{\mathcal{N}}_0$ is a parameter for the dynamics of the vectorfield $f_{\widetilde{\mathcal{N}}_1}$ from (2.7), with $\mathcal{N}_1$ replaced by $\widetilde{\mathcal{N}}_1$. Due to (3.10) all $\tilde{\nu} \in \widetilde{\mathcal{N}}_0$ are fixed by α. Hence the vector-field $\tilde{w} \to f_{\widetilde{\mathcal{N}}_1}(\tilde{\nu}, \tilde{w})$ is $\mathbb{Z}_\ell(\alpha)$ equivariant for all $\tilde{\nu}$ with $\mathbb{Z}_\ell(\alpha)$-invariant Hamiltonian $\tilde{h}(\tilde{\nu}, \cdot)$.

The assumption of L-nondegeneracy implies that we can solve $f_{\mathcal{N}_1}|_{\mathrm{Fix}_{\mathcal{N}_1}(L)} = 0$ for $w(\nu)$ to get equilibria of the vector-field $f_{\mathcal{N}_1}$ with isotropy subgroup L. Similarly we also obtain equilibria $\tilde{w}(\tilde{\nu})$ of the vector-field $f_{\widetilde{\mathcal{N}}_1}|_{\mathrm{Fix}_{\widetilde{\mathcal{N}}_1}(L)} = 0$. To see this, note that $M = \mathrm{D}(f_H(\bar{x}) - \bar{\xi})$ in bundle coordinates is given by

$$M = \begin{pmatrix} -\mathrm{ad}_{\bar{\xi}} & \mathrm{D}^2_\nu h(0) & \mathrm{D}^2_{\nu w} h(0) \\ 0 & \mathrm{ad}^*_{\bar{\xi}}|_{\mathbf{g}^*_{\bar{\mu}}} & 0 \\ 0 & \mathbb{J}_{\mathcal{N}_1} \mathrm{D}^2_{\nu,w} h(0) & \mathbb{J}_{\mathcal{N}_1} \mathrm{D}^2_w h(0) \end{pmatrix} \tag{3.11}$$

see (2.7) and [28]. On $\mathrm{Fix}_{\mathcal{X}}(L)$ any kernel vectors of $\mathrm{ad}_{\bar{\xi}}$ and $\mathrm{ad}^*_{\bar{\xi}}$ lie in $\widetilde{\mathcal{T}}_0$ and $\widetilde{\mathcal{N}}_0$ respectively, since $\mathrm{Fix}_{\mathcal{T}}(L) = \mathbf{g}_\alpha$ when K is trivial. Therefore $\tilde{w} = 0$ is an L-nondegenerate equilibrium of $f_{\widetilde{\mathcal{N}}_1}$. This gives the family of L-nondegenerate relative equilibria from part a).

To prove part b), the idea is to apply the symmetric Lyapunov centre theorem (Theorem 3.1) to the Hamiltonian system

$$\dot{\tilde{w}} = f_{\widetilde{\mathcal{N}}_1}(\tilde{\nu}, \tilde{w}) = \mathbb{J}_{\widetilde{\mathcal{N}}_1} \mathrm{D}_{\tilde{w}} \tilde{h}(\tilde{\nu}, \tilde{w}) \tag{3.12}$$

near each equilibrium $\tilde{w}(\tilde{\nu})$ with $\tilde{\nu}$ close to 0. For this we need to check that conditions (3.5) and (3.6) also hold if we replace $\mathcal{W} = \mathcal{N}_1$ by $\widetilde{\mathcal{W}} = \widetilde{\mathcal{N}}_1$; so we have to show that

$$\mathrm{Fix}_{\widetilde{\mathcal{W}}^{\tilde{\omega}}}(\Lambda) \subseteq \widetilde{\mathcal{W}}_{\tilde{\omega}}, \quad \dim \mathrm{Fix}_{\widetilde{\mathcal{W}}^{\tilde{\omega}}}(\Lambda) = 2. \tag{3.13}$$

Let $\tilde{w} \in \mathrm{Fix}_{\widetilde{\mathcal{W}}^{\tilde{\omega}}}(\Lambda)$ and decompose

$$\tilde{w} = \mathrm{P}_{\mathcal{T}_1}\tilde{w} + \mathrm{P}_{\mathcal{T}_0}\tilde{w} + \mathrm{P}_{\mathcal{N}_0}\tilde{w} + \mathrm{P}_{\mathcal{N}_1}\tilde{w}. \tag{3.14}$$

Here $\mathrm{P}_{\mathcal{T}_1}$ denotes a projection onto $\mathcal{T}_1$ with kernel $\mathcal{T}_0 \oplus \mathcal{N}_0 \oplus \mathcal{N}_1$ etc.. Note that $\widetilde{\mathcal{T}} = \mathbf{g}_{\bar{\sigma}}\bar{x}$, $\widetilde{\mathcal{T}}_0 = \mathbf{g}_{(\bar{\sigma},\bar{\mu})}\bar{x}$ and that $\widetilde{\mathcal{T}}_1 \simeq \mathbf{n}_{\bar{\mu}} \cap \mathbf{g}_{\bar{\sigma}}$. Then

$$\widetilde{\mathcal{N}}_1 \simeq \mathbf{l}_{\bar{\sigma}} \oplus (\mathbf{m}_{(\bar{\sigma},\bar{\mu})} \oplus \mathbf{m}^*_{(\bar{\sigma},\bar{\mu})}) \oplus \mathcal{N}_1.$$

Here $\mathbf{l}_{\bar{\sigma}}$ is a $\Gamma_{(\bar{\sigma},\bar{\mu})}$ invariant complement to $\mathbf{n}_{\bar{\mu}} \cap \mathbf{g}_{\bar{\sigma}}$ in $\mathbf{n}_{\bar{\mu}}$ such that $\mathbf{l}_{\bar{\sigma}} \simeq \mathcal{T}_1 \cap \widetilde{\mathcal{N}}_1$. Moreover $\mathbf{m}_{(\bar{\sigma},\bar{\mu})}$ is a $\Gamma_{(\bar{\sigma},\bar{\mu})}$ invariant complement to $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ in $\mathbf{g}_{\bar{\mu}}$ so that $\mathbf{m}_{(\bar{\sigma},\bar{\mu})} \simeq \mathcal{T}_0 \cap \widetilde{\mathcal{N}}_1$. Similarly, $\mathbf{m}^*_{(\bar{\sigma},\bar{\mu})}$ is an $\Gamma_{(\bar{\sigma},\bar{\mu})}$ invariant complement to $\mathbf{g}^*_{(\bar{\sigma},\bar{\mu})}$ in $\mathbf{g}^*_{\bar{\mu}}$ such that $\mathbf{m}^*_{(\bar{\sigma},\bar{\mu})} \simeq \mathcal{N}_0 \cap \widetilde{\mathcal{N}}_1$. Therefore for $\tilde{w} \in \widetilde{\mathcal{N}}_1$ we have

$$\mathrm{P}_{\mathcal{T}_j}\tilde{w} \in \mathcal{T}_j \cap \widetilde{\mathcal{N}}_1 \quad \text{and} \quad \mathrm{P}_{\mathcal{N}_j}\tilde{w} \in \mathcal{N}_j \cap \widetilde{\mathcal{N}}_1, \quad j = 1, 2. \tag{3.15}$$

Next notice that (3.11) and $\tilde{w} \in \mathrm{Fix}_{\widetilde{\mathcal{W}}^{\bar{\omega}}}(\Lambda)$ imply that

$$\mathrm{P}_{\mathcal{N}_0}\tilde{w} = (\mathrm{Ad}^*_\alpha)^{-1}\mathrm{Ad}^*_{\exp(\bar{\tau}\bar{\xi})}\mathrm{P}_{\mathcal{N}_0}\tilde{w}.$$

Due to the definition of $\bar{\sigma}$ this becomes

$$\mathrm{P}_{\mathcal{N}_0}\tilde{w} = (\mathrm{Ad}^*_{\bar{\sigma}})^{-1}\mathrm{P}_{\mathcal{N}_0}\tilde{w}$$

or $\mathrm{P}_{\mathcal{N}_0}\tilde{w} \in \mathbf{g}^*_{(\bar{\sigma},\bar{\mu})} \simeq \widetilde{\mathcal{N}}_0$. Since by (3.15) also $\mathrm{P}_{\mathcal{N}_0}\tilde{w} \in \widetilde{\mathcal{N}}_1$ this implies that $\mathrm{P}_{\mathcal{N}_0}\tilde{w} = 0$. Due to (3.11) we know that $\mathrm{P}_{\mathcal{N}_1}\tilde{w} \in \mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda)$ and therefore, by (3.5) and (3.6),

$$\mathrm{P}_{\mathcal{N}_1}\mathrm{Fix}_{\widetilde{\mathcal{W}}^{\bar{\omega}}}(\Lambda) \subseteq \mathrm{Fix}_{\mathcal{W}_{\bar{\omega}}}(\Lambda)) \simeq \mathbb{R}^2.$$

If $\mathrm{P}_{\mathcal{N}_1}$ is injective on $\mathrm{Fix}_{\widetilde{\mathcal{W}}^{\bar{\omega}}}(\Lambda)$ then this proves (3.13). Suppose not and let $\tilde{w} = \eta \in \mathcal{T} \cap \mathrm{Fix}_{\widetilde{\mathcal{W}}^{\bar{\omega}}}(\Lambda)$. Similarly as above we see that then $\eta = \mathrm{Ad}_{\bar{\sigma}}\eta$ and so $\eta \in \mathbf{g}_{\bar{\sigma}} \simeq \widetilde{\mathcal{T}}$. Since $\widetilde{\mathcal{T}} \cap \widetilde{\mathcal{N}}_1 = \{0\}$ we see that $\eta = 0$. Hence $\mathrm{P}_{\mathcal{N}_1}$ is injective on $\mathrm{Fix}_{\widetilde{\mathcal{W}}^{\bar{\omega}}}(\Lambda)$ and so (3.13) holds and, thus, (3.5) and (3.6) hold with $\mathcal{W}$ replaced by $\widetilde{\mathcal{W}} = \widetilde{\mathcal{N}}_1$.

By continuous dependence on the parameter $\tilde{\nu}$ these conditions also hold at the equilibria $\tilde{w}(\tilde{\nu})$, $\tilde{\nu} \approx 0$, of (3.12) for the eigenvalue $\mathrm{i}\omega(\tilde{\nu})$ of $\mathrm{D}f_{\widetilde{\mathcal{N}}_1}(\tilde{w}(\tilde{\nu}),\tilde{\nu})$ where $\omega(0) = \bar{\omega}$. Here we use that by (3.13) the isotypic component of $\widetilde{\mathcal{W}}_{\bar{\omega}}$ where α acts as $\mathrm{e}^{-2\pi\mathrm{i}/\ell}$ is one-dimensional, so that $\omega(\tilde{\nu})$ depends smoothly on $\tilde{\nu}$. Theorem 3.1 then gives a smooth function $\tilde{w}(\epsilon,\tilde{\nu})$, $\epsilon \geq 0$ of points on periodic orbits of (3.12) with period $T(\epsilon,\tilde{\nu}) = \ell\tau(\epsilon,\tilde{\nu})$, momentum $\mu(\tilde{\nu}) = \bar{\mu} + \tilde{\nu}$ and energy $E(\epsilon,\tilde{\nu})$ such that $\tilde{w}(0,\tilde{\nu}) = \tilde{w}(\tilde{\nu})$. These correspond to RPOs through $x(\epsilon,\chi) \simeq (\mathrm{id},\tilde{\nu},\tilde{w}(\epsilon,\tilde{\nu})) \in \tilde{\Gamma} \times \widetilde{\mathcal{N}}$ of the original system (2.1) with $\chi = \tilde{\nu} \in \mathbf{g}^*_{(\bar{\sigma},\bar{\mu})}$, $\epsilon \geq 0$. Moreover, (3.7) holds, i.e.,

$$\tilde{w}(-\epsilon,\chi) = \tilde{\Phi}^{T(\epsilon,\chi)/2}(\tilde{w}(\epsilon,\chi)), \tag{3.16}$$

where $\tilde{\Phi}$ is the flow on $\widetilde{\mathcal{N}}_1$. This shows as in the proof of Theorem 3.1 that $\mathrm{D}_\epsilon T(0,\chi) = 0$. For later purposes, let, as in the proof of Theorem 3.1, $\bar{w}$ be an eigenvector of $\tilde{A} = \mathrm{D}f_{\widetilde{\mathcal{N}}_1}(0)$ to the eigenvalue $\mathrm{i}\bar{\omega}$ such that

$$\mathrm{span}(\mathrm{Re}\,\bar{w},\mathrm{Im}\,\bar{w}) = \mathrm{Fix}_{\widetilde{\mathcal{W}}_{\bar{\omega}}}(\Lambda), \quad \mathrm{D}_\epsilon\tilde{w}(0,0) = \mathrm{D}_\epsilon x(0,0) = \mathrm{Re}\,\bar{w}. \tag{3.17}$$

By construction $\Phi^{T(\epsilon,\chi)}(x(\epsilon,\chi)) \in \Gamma_{(\bar{\sigma},\bar{\mu})}x(\epsilon,\chi)$ and so there is $\sigma(\epsilon,\chi) \in \Gamma_{(\bar{\sigma},\bar{\mu})}$, $\sigma(0,0) = \bar{\sigma}$ with $\sigma(\epsilon,\chi)\Phi^{T(\epsilon,\chi)}(x(\epsilon,\chi)) = x(\epsilon,\chi)$. Since $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ is abelian and $\sigma(\epsilon,\chi)\bar{\sigma}^{-1} \in \Gamma^{\mathrm{id}}_{(\bar{\sigma},\bar{\mu})}$ also $\sigma(\epsilon,\chi)\alpha^{-1} \in \Gamma^{\mathrm{id}}_{(\bar{\sigma},\bar{\mu})}$ and so there is $\xi(\epsilon,\chi) \in \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ such that $\sigma(\epsilon,\chi) = \alpha\exp(-\tau(\epsilon,\chi)\xi(\epsilon,\chi))$ By construction there is $\gamma(\epsilon,\chi) \in \Gamma^{\mathrm{id}}_{(\bar{\sigma},\bar{\mu})}$ such that $x(-\epsilon,\chi) \simeq (\mathrm{id},\chi,\tilde{w}(-\epsilon,\chi))$ satisfies $x(-\epsilon,\chi) = \gamma(\epsilon,\chi)\Phi^{T(\epsilon,\chi)/2}(x(\epsilon,\chi))$. Since $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ is abelian and commutes with α we therefore have

$$\begin{aligned} x(-\epsilon,\chi) &= \gamma(\epsilon,\chi)\Phi^{T(\epsilon,\chi)/2}(x(\epsilon,\chi)) = \gamma(\epsilon,\chi)\sigma(\epsilon,\chi)\Phi^{T(\epsilon,\chi)}(\Phi^{T(\epsilon,\chi)/2}(x(\epsilon,\chi))) \\ &= \sigma(\epsilon,\chi)\Phi^{T(\epsilon,\chi)}(x(-\epsilon,\chi)). \end{aligned}$$

Hence $\sigma(\epsilon,\chi) = \sigma(-\epsilon,\chi)$ and so $\xi(\epsilon,\chi)$ is even in ϵ and $\mathrm{D}_\epsilon\xi(0,\chi) = 0$. ■

Remark 3.5 We call the velocity-momentum pair $(\bar{\xi},\bar{\mu})$ of a relative equilibrium through $\bar{x}$ K-regular for any $K \subseteq \Gamma_{\bar{x}}$ if it is regular for the symmetry group $N(K)/K$. Similarly we define the notion of a K-regular drift-momentum pair of an RPO. Note that the family $\mathcal{P}(\epsilon,\chi)$ of RPOs from Theorem 3.2 has a K-regular drift-momentum pair $(\bar{\sigma},\bar{\mu})$ at $(\epsilon,\chi) = 0$. This follows from Lemma 3.3 b) and the L-regularity of $(\bar{\xi},\bar{\mu})$, after replacing Γ with $N(K)/K$. Here we use that $L/K = \mathbb{Z}_\ell(\alpha)$.

Remark 3.6 Condition (3.8) is a non-resonance condition which is generically satisfied; it states that when restricted to $\mathrm{Fix}_{\mathbf{g}_{\bar\mu}}(K)$ the matrix $\mathrm{Ad}_{\bar\sigma} = \mathrm{Ad}_\alpha \mathrm{e}^{-\bar\tau \mathrm{ad}_{\bar\xi}}$ has the fixed point space $\ker(\mathrm{ad}_{\bar\xi}) \cap \mathrm{Fix}(\mathrm{Ad}_\alpha)$. Here we used again that $L/K \simeq \mathbb{Z}_\ell(\alpha)$, c.f. (3.10). As

$$(\mathrm{Ad}_\alpha)^\ell|_{\mathrm{Fix}_{\mathbf{g}_{\bar\mu}}(K)} = \mathrm{id},$$

the eigenvalues of Ad_α as a map from $\mathrm{Fix}_{\mathbf{g}_{\bar\mu}}(K)$ to $\mathrm{Fix}_{\mathbf{g}_{\bar\mu}}(K)$ are powers of $\mathrm{e}^{\pm 2\pi i/\ell}$. So in particular, if $\mathrm{ad}_{\bar\xi}|_{\mathrm{Fix}_{\mathbf{g}_{\bar\mu}}(K)}$ does not have any eigenvalues in $\mathrm{i}\bar\omega\mathbb{Z}$ then (3.8) holds.

When condition (3.8) is violated then *resonance drift* of bifurcating RPOs is possible, i.e., drift with average velocities in directions not contained in $\mathbf{g}_{(\bar\sigma,\bar\mu)}$, see [32].

Example 3.7 Consider a nondegenerate relative equilibrium $\Gamma\bar x$ with velocity momentum pair $(\bar\xi, 0) \neq 0$ in a system with SO(3)-symmetry (for an example of a relative equilibrium of coupled rigid bodies, which has non-zero angular velocity $\bar\xi \neq 0$, but zero angular momentum, see [26]). It persists to a one-dimensional family with momentum parallel to $\bar\xi$ and $\bar\mu$, see Section 2.2 and in particular Example 2.2. Assume that its isotropy subgroup $\bar K$ is trivial and that its linearization in the corotating frame in its momentum level set (i.e., $\mathrm{D}f_{\mathcal{N}_1}(0,0)$) has a simple imaginary eigenvalue $\pm\mathrm{i}\bar\omega$, $\bar\omega > 0$, and no resonant eigenvalues. Then conditions (3.5) and (3.6) hold and $\alpha = \mathrm{id}$. In this case the linearization $\mathrm{D}f_{\mathcal{N}}(0)$ of the equilibrium of the $\dot v$ equation (where $v = (\nu, w)$, see Section 2.2) has also eigenvalues $\pm\mathrm{i}\bar\omega^{\mathrm{rot}}$ where $\bar\omega^{\mathrm{rot}}$ is the rotation frequency of the relative equilibrium. This is due to (3.11) and the fact that $\mathrm{ad}_{\bar\xi}$ has eigenvalues $\pm\mathrm{i}\bar\omega^{\mathrm{rot}}$. The non-resonance condition (3.8) is therefore satisfied if $\bar\omega^{\mathrm{rot}} \notin \bar\omega\mathbb{Z}$ where $\bar\omega^{\mathrm{rot}} = |\bar\xi|$ is the rotation frequency of the relative equilibrium. In this case Theorem 3.2 gives a two-parameter family of RPOs with angular momentum and average drift velocity pointing in the direction of $\bar\xi$.

Next consider the case where the relative equilibrium at $\bar x$ has isotropy subgroup $\bar K = \mathbb{Z}_\ell(\alpha)$ for some $\alpha \in \mathrm{SO}(3)$ of finite order $\ell > 1$. Assume, as before that $\mathrm{D}f_{\mathcal{N}_1}(0,0)$ has a simple imaginary eigenvalue $\pm\mathrm{i}\bar\omega$, $\bar\omega > 0$, and that the action of α on the corresponding real eigenspace $\mathcal{W}_{\bar\omega}$ is faithful. Then, as above, conditions (3.5) and (3.6) hold for $L = \bar K$, $K = \{\mathrm{id}\}$. Let us assume that $\bar\xi || e_1$. Then the condition $\mathrm{Ad}_\alpha\bar\xi = \bar\xi$ implies $\alpha \in \mathrm{SO}_1(2)$. Here $\mathrm{SO}_j(2)$, $j = 1, 2, 3$, is the group of rotations $R_j(\phi)$, $\phi \in [0, 2\pi]$ around the e_j-axis, $j = 1, 2, 3$, with Lie algebra $\mathrm{so}_j(2)$. In this case, since α has order ℓ, the action of α on $\mathrm{so}_2(2) \oplus \mathrm{so}_3(2)$ has eigenvalues $\mathrm{e}^{\pm 2\pi\mathrm{i}m/\ell}$ for some $m \in \mathbb{Z}$. Hence, condition (3.8) is satisfied even if $\mathrm{ad}_{\bar\xi}|_{\mathbf{g}_{\bar\mu}}$ has an eigenvalue $\mathrm{i}k\bar\omega$, $k \in \mathbb{Z}$ (i.e. $\bar\omega^{\mathrm{rot}} = k\bar\omega$), provided that $|m \pm k|$ is not a multiple of ℓ. In this case Theorem 3.2 applies as before.

Remark 3.8 Assume as before that K is trivial (by restricting to $\mathrm{Fix}(K)$). Then the non-resonance condition (3.8) is only needed on $\mathbf{g}_{\bar\mu}$, not on $\mathbf{g}$, so if, for instance, in Example 3.7 the relative equilibrium has momentum $\bar\mu \neq 0$ and trivial isotropy subgroup $\bar K$, and $\mathrm{ad}_{\bar\xi}$ has eigenvalues in $\mathrm{i}\bar\omega\mathbb{Z}$ with eigenvectors in $\mathbf{g}_{\bar\sigma} \setminus \mathbf{g}_{\bar\mu}$ then Theorem 3.2 still applies. The reason for this is that we apply the symmetric Lyapunov centre theorem on $\widetilde{\mathcal{N}}$ in the proof of Theorem 3.2, so resonances with the $\widetilde{\mathcal{T}}_1$ block of M do not matter. Such a resonance occurs at the Lagrangian relative equilibrium, see Section 5, in particular Remark 5.9.

Example 3.9 Let $N > 1$ be odd. Consider a $\mathbb{D}_N$ invariant relative equilibrium $\bar x$ of a Hamiltonian system with O(3)-symmetry: let $\bar K = \mathbb{D}_N(\kappa_1, R_3(2\pi/N))$ be its isotropy subgroup. Here, as in Example 3.7, $R_j(\phi) \in \mathrm{O}(3)$ is a rotation by ϕ around e_j, $j = 1, 2, 3$, and $\kappa_j \in \mathrm{O}(3)$ is the reflection which satisfies $\kappa_j e_j = -e_j$, $\kappa_j e_i = e_i$ for $i \neq j$. Moreover $\mathbb{D}_N(\kappa_1, R_3(2\pi/N))$ denotes the group generated by κ_1 and $R_3(2\pi/N)$. Notice that $\mathrm{Fix}_{\mathbf{g}}(\bar K) = \{0\}$ and so, by Lemma 3.4, the relative equilibrium is an equilibrium. Assume that the relative equilibrium is $\bar K$-nondegenerate, that its linearization $M = \mathrm{D}f(\bar x)$ has purely imaginary eigenvalues $\pm\mathrm{i}\bar\omega$, $\bar\omega \neq 0$, and that $\bar K$ acts faithfully on $\mathcal{W}_{\bar\omega}$. Then these eigenvalues have to be double (see [11]). Assume further that these eigenvalues have multiplicity two and that $ki\bar\omega$, $k \in \mathbb{Z}\setminus\{1\}$, is not an eigenvalue of $A = \mathrm{D}f_{\mathcal{N}_1}(0)$.

Let $L = \mathbb{Z}_N(\alpha)$ where $\alpha = R_3(-2\pi/N)$ and let K be trivial. Then conditions (3.5) and (3.6) of Theorem 3.2 are satisfied. Moreover (3.8) holds with $r = 1$, and $\mathrm{Fix}_{\mathbf{g}_{(\bar\xi,\bar\mu)}}(L) = \mathrm{so}_3(2)$. So by part a) of Theorem 3.2 the equilibrium persists as a one parameter family of $\mathbb{Z}_N(\alpha)$-invariant relative equilibria rotating about the e_3-axis; moreover, by Theorem 3.2, part b) a two-parameter family of RPOs bifurcates from $\bar x$ which rotates around e_3, has trivial isotropy subgroup K and spatio-temporal symmetry $\mathbb{Z}_N(\alpha)$ in its corotating frame.

Similarly if we set $L = K = \mathbb{Z}_2(\kappa_1)$ then $\mathrm{Fix}_{\mathbf{g}_{(\bar\xi,\bar\mu)}}(L) = \mathrm{so}_1(2)$, so the equilibrium persists as a one parameter family of $\mathbb{Z}_2(\kappa_1)$-invariant relative equilibria rotating about the e_1-axis, and, since conditions (3.5) and (3.6) hold, a two-parameter family of RPOs bifurcates from $\bar x$ which rotates around e_1 and has isotropy subgroup $K = \mathbb{Z}_2(\kappa_1)$ in its corotating frame. Finally a two-parameter family of RPOs bifurcates from $\bar x$ which rotates around e_1 and has trivial isotropy subgroup $K = \{\mathrm{id}\}$ and spatio-temporal symmetry group $L = \mathbb{Z}_2(\kappa_1)$ in its co-rotating frame.

4 Numerics of relative Lyapunov centre bifurcation

In this section we first review numerical methods for the continuation of RPOs from [35]. Then we design numerical methods for the detection and computation of relative Lyapunov centre bifurcations along branches of RPOs.

4.1 Numerical continuation of transversal RPOs

In this section we briefly recall the numerical methods presented in [35] for the continuation of transversal RPOs of compact symmetry group actions with regular drift-momentum pair.

Let $\bar x$ lie on a transversal RPO with trivial isotropy subgroup and regular drift momentum pair $(\bar\sigma, \bar\mu)$, and decompose $\bar\sigma = \alpha \exp(-\bar\tau\bar\xi)$ as in (2.8), (2.9). Let $r = r_{(\bar\sigma,\bar\mu)} = \dim \mathbf{g}_{(\bar\sigma,\bar\mu)}$. In the following, let $e_1^\xi, \ldots, e_r^\xi$ denote a basis of $\mathbf{g}_{(\bar\sigma,\bar\mu)}$ and let $e_1^\xi, \ldots, e_q^\xi$, $q \geq r$, denote a basis of $\mathbf{g}_{\bar\sigma}$, the Lie algebra of $\Gamma_{\bar\sigma} = Z(\bar\sigma)$. Finally, let $e_1^\xi, \ldots, e_g^\xi$, $g = \dim\Gamma$, denote a basis of $\mathbf{g}$. For $\xi \in \mathbf{g}_{\bar\sigma}$ let $\xi = \sum_{i=1}^q \xi_i e_i^\xi$ and identify $\xi \simeq (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$. Let $\mathbf{J}_i = \mathbf{J}_{e_i^\xi}$, $i = 1, \ldots, q$. Typically $q = r$, see [35]. Define

$$\dot x = f(x, \lambda_E, \lambda_\mu, \xi) := f_H(x) + \lambda_E \nabla H(x) + \sum_{i=1}^r \lambda_{\mu,i} \nabla \mathbf{J}_i(x) - \sum_{i=1}^q \xi_i e_i^\xi. \tag{4.1}$$

Denote by $\Phi^t(x; \xi, \lambda_E, \lambda_\mu)$ the flow of (4.1). Then the derivative $\mathrm{D}F_{\mathrm{RPO}}$ of

$$F_{\mathrm{RPO}}(x, T, \xi, \lambda_E, \lambda_\mu) = \begin{pmatrix} \alpha\Phi_{\frac{T}{\ell}}(x; \xi, \lambda_E, \lambda_\mu) - x \\ \mathbf{J}_{r+1}(x) - \bar\mu_{r+1} \\ \vdots \\ \mathbf{J}_q(x) - \bar\mu_q \end{pmatrix} = 0 \tag{4.2}$$

where

$$F_{\mathrm{RPO}} : \mathcal{X} \times \mathbb{R}^{2+r+q} \to \mathcal{X} \times \mathbb{R}^{q-r}$$

has full rank at any solution $y = (x, T, \xi, \lambda_E, \lambda_\mu)$ close to $\bar y = (\bar x, \bar T, \bar\xi, 0, 0)$ where $\bar T = \ell\bar\tau$. Moreover, any such solution satisfies $\lambda_E = 0$, $\lambda_\mu = 0$, and hence, determines an RPO of (2.1). Indeed, the solution manifold of (4.2) has dimension $2r + 2$ and is given by $y(s) =$

$(\gamma\Phi^t(x(s)), T(s), \xi(s), 0, 0)$ where $\gamma \in \Gamma^{\rm id}_{(\bar\sigma,\bar\mu)}$, $t \in [0, T(s)]$, $s \in \mathbb{R}^{r+1}$ and $x(s) \in \mathcal{P}(s)$ where $\mathcal{P}(s)$ is the family of RPOs from Section 2.3.

We assume, as before, that the isotropy subgroup $K = \Gamma_{\bar x}$ of the RPO at $\bar x$ is trivial (by restricting to $\mathrm{Fix}_{\mathcal{X}}(K)$) and compute paths of RPOs. For $\mu \in \mathbf{g}^*$ let $\mu_j = \mu(e^\xi_j)$, $j = 1, \ldots, g$, $g = \dim \Gamma$. To obtain a path of RPOs we may fix the energy and $r-1$ out of the first r momentum components, without loss of generality the $r-1$ components $\bar\mu^\flat = (\bar\mu_2, \ldots, \bar\mu_r)$. So we solve (4.2) along with the constraints $\mathbf{J}_j(x) - \bar\mu_j = 0$, $j = 2, \ldots, r$, and $H(x) - \bar E = 0$. This gives a nonlinear equation

$$F^{\bar E, \bar\mu^\flat}_{\rm RPO}(x, T, \xi, \lambda_E, \lambda_\mu) = 0 \tag{4.3}$$

where $F^{\bar E, \bar\mu^\flat}_{\rm RPO} : \mathcal{X} \times \mathbb{R}^{2+r+q} \to \mathcal{X} \times \mathbb{R}^q$. The x-component of the solution $y = (x, T, \xi, 0, 0)$ of (4.3) then lies in

$$\mathcal{X}^{\bar E, \bar\mu^\flat} = \{x \in \mathcal{X}, \;\; H(x) = \bar E, \;\; \mathbf{J}_j(x) = \bar\mu_j, \; j = 2, \ldots, g\},$$

see [35].

Alternatively, to obtain a path of RPOs we may fix the momentum value $\bar\mu$ and continue in energy, i.e., solve the equation (4.2) along with the constraints $\mathbf{J}_j(x) - \bar\mu_j = 0$, $j = 1, \ldots, r$. This gives an equation

$$F^{\bar\mu}_{\rm RPO}(x, T, \xi, \lambda_E, \lambda_\mu) = 0, \tag{4.4}$$

which again maps $\mathcal{X} \times \mathbb{R}^{2+r+q}$ to $\mathcal{X} \times \mathbb{R}^q$. The solutions $y = (x, T, \xi, 0, 0)$ of (4.4) then satisfy $x \in \mathcal{X}^{\bar\mu} = \mathbf{J}^{-1}(\bar\mu)$.

If the RPO $\bar{\mathcal{P}}$ is nondegenerate then a smooth path $x(\mu_1) \in \mathcal{X}^{\bar E, \mu^\flat}$ of points on RPOs exists near $\bar x = x(\bar\mu_1)$ with $\mathbf{J}_1(x(\mu_1)) = \mu_1$, and similarly a path $x(E) \in \mathcal{X}^{\bar\mu}$ of points on RPOs with energy E exists near $x(\bar E) = \bar x$, see Section 2.3. More generally, if the RPO $\mathcal{P}$ is *transversal for* $C(x) := \mathbf{J}_1(x)$ (transversal for $C(x) = H(x)$), as defined in [35], then there is still a smooth path $x(\epsilon) \in \mathcal{X}^{\bar E, \mu^\flat}$ ($x(\epsilon) \in \mathcal{X}^{\bar\mu}$) on RPOs $\mathcal{P}(\epsilon)$, where $\epsilon \in \mathbb{R}$, $\epsilon \approx 0$, such that $\mathcal{P}(0) = \bar{\mathcal{P}}$, but folds in $C(x)$ are possible. Then the solution set of (4.3) and (4.4), respectively, has dimension $r+2$ and is locally of the form $y = (\gamma\Phi^t(x(\epsilon)), T(\epsilon), \xi(\epsilon), 0, 0)$, $\gamma \in \Gamma_{(\bar\sigma,\bar\mu)}$, $t \in [0, T(\epsilon)]$, $\gamma \approx \mathrm{id}$, for details see [35]. Numerically we compute such a path $x(\epsilon)$ either by adding $(r+1)$ further constraints to fix γ and the phase t of the solutions of (4.3) and (4.4), respectively, or by solving those equations by a Gauss-Newton method. In the latter case we choose the continuation tangent $t^{\rm cont}(y) \in \ker \mathrm{D}F^{\bar E, \mu^\flat}_{\rm RPO}(y)$ ($t^{\rm cont}(y) \in \ker \mathrm{D}F^{\bar\mu}_{\rm RPO}(y)$) at a solution $y = (x, T, \xi, 0, 0)$ of $F^{\bar E, \mu^\flat}_{\rm RPO} = 0$ ($F^{\bar\mu}_{\rm RPO} = 0$) such that its x-component $t^{\rm cont}_x(y)$ satisfies $t^{\rm cont}_x(y) \in (\mathbf{g}_{(\bar\sigma,\bar\mu)} x \oplus \mathrm{span}(f_H(x)))^\perp$.

4.2 Detecting Lyapunov centre bifurcations along branches of RPOs

The following proposition collects some facts which we use in the design of numerical methods for detecting relative Lyapunov centre bifurcations during the continuation of RPOs. As before we assume that the isotropy subgroup K of the path of RPOs that we continue is trivial, by restricting the dynamics to $\mathrm{Fix}(K)$.

Proposition 4.1 *Let $\Gamma\bar x$ be a relative Lyapunov centre bifurcation point with velocity-momentum pair $(\bar\xi, \bar\mu)$ and energy $\bar E$ along a path of RPOs $\mathcal{P}(\epsilon)$ with average drift velocity $\xi(\epsilon)$ and drift symmetry α of order ℓ and period $T(\epsilon)$ in its comoving frame $\xi(\epsilon)$ at $x(\epsilon) \in \mathcal{P}(\epsilon)$ such that $y(\epsilon) := (x(\epsilon), T(\epsilon), \xi(\epsilon), 0, 0)$ is smooth with $\bar x = x(0)$. Then under the assumptions of Theorem 3.2 we have the following:*

a) If we continue in energy $C(x) = H(x)$, i.e., $x(\epsilon) \in \mathcal{X}^{\bar\mu}$ and $y(\epsilon)$ solves (4.4), then $C'(x(\epsilon))|_{\epsilon=0} = 0$, $C''(x(\epsilon))|_{\epsilon=0} \neq 0$.

b) *Assume that the condition* $\bar{\xi}_1 \neq 0$ *is satisfied. If we continue in the momentum component* $C(x) = \mathbf{J}_1(x)$, *i.e.,* $x(\epsilon) \in \mathcal{X}^{\bar{E},\bar{\mu}^\flat}$ *and* $y(\epsilon)$ *satisfies (4.3), then again* $C'(x(\epsilon))|_{\epsilon=0} = 0$, $C''(x(\epsilon))|_{\epsilon=0} \neq 0$.

Proof.

a) follows directly from Theorem 3.2.

b) Let $\Gamma x(\chi)$ be the family of relative equilibria from Theorem 3.2 a) and let $E(\chi) = H(x(\chi))$ be their energy. Then

$$\mathrm{D}_{\chi_1} E(\chi)|_{\chi=0} = \bar{\xi}_1. \tag{4.5}$$

This follows from the fact that $x(\chi) \simeq (\mathrm{id}, \tilde{\nu}, \tilde{w}(\tilde{\nu}))$ in the bundle coordinates used in (3.12), with $\chi = \tilde{\nu}$, so that $E(\chi) = \tilde{h}(\tilde{\nu}, \tilde{w}(\tilde{\nu}))$ and

$$\mathrm{D}_\chi E(0) = \partial_{\tilde{\nu}} \tilde{h}(0,0) + \mathrm{D}_{\tilde{w}} \tilde{h}(0,0) \mathrm{D}_{\tilde{\nu}} \tilde{w}(0) = \partial_{\tilde{\nu}} \tilde{h}(0,0) = \bar{\xi}.$$

Here we used that $\tilde{w} = 0$ is an equilibrium of $f_{\widetilde{\mathcal{N}}_1}(0,\cdot)$ so that $\mathrm{D}_{\tilde{w}} \tilde{h}(0,0) = 0$.

Since $\bar{\xi}_1 \neq 0$ we can solve for χ_1 as a function $\chi_1 = \tilde{\chi}_1(E, \chi_2, \ldots, \chi_r)$ and reparametrize $x(\chi)$ as $x(\chi) = \tilde{x}(E, \chi_2, \ldots, \chi_r)$, such that $\tilde{x}(E, 0, \ldots, 0)$ is an isolated relative equilibrium in $\mathrm{Fix}_{\mathcal{X}^{\bar{E},\bar{\mu}^\flat}}(L)$.

Similarly the family of RPOs through $x(\epsilon, \chi)$ from Theorem 3.2 b) has energy $E(\epsilon, \chi)$ and $\mathrm{D}_{\chi_1} E(0,0) = \bar{\xi}_1 \neq 0$ so that we can compute $\chi_1 = \tilde{\chi}_1(\epsilon, E, \chi_2, \ldots, \chi_r)$ and can also parametrize this family as $x(\epsilon, \chi) = \tilde{x}(\epsilon, E, \chi_2, \ldots, \chi_r)$. Then $x(\epsilon) := \tilde{x}(\epsilon, \bar{E}, 0, \ldots, 0) \in \mathcal{X}^{\bar{E},\mu^\flat}$ and $C(x(\epsilon)) = \tilde{\chi}_1(\epsilon, \bar{E}, 0, \ldots, 0)$. Differentiating

$$H(x(\epsilon, \tilde{\chi}_1(\epsilon, \bar{E}, \chi_2, \ldots, \chi_r), \chi_2, \ldots, \chi_r)) = \bar{E}$$

with respect to ϵ we get

$$\mathrm{D}_\epsilon H(x(\epsilon, 0))|_{\epsilon=0} + \mathrm{D}_{\chi_1} H(x(0,\chi))|_{\chi=0} \mathrm{D}_\epsilon C(x(\epsilon))|_{\epsilon=0} = 0.$$

The first term vanishes since $H(x(\epsilon, 0)) = \bar{E} \pm O(\epsilon^2)$ by Theorem 3.2. Hence, due to (4.5) and since $\bar{\xi}_1 \neq 0$ we have $\mathrm{D}_\epsilon C(x(\epsilon))|_{\epsilon=0} = 0$. Differentiating again we get

$$\mathrm{D}^2_\epsilon H(x(\epsilon, 0))|_{\epsilon=0} = -\mathrm{D}_{\chi_1} H(x(0,\chi))|_{\chi=0} \mathrm{D}^2_\epsilon C(x(\epsilon))|_{\epsilon=0} = -\bar{\xi}_1 \mathrm{D}^2_\epsilon C(x(\epsilon))|_{\epsilon=0}$$

and so, since by Theorem 3.2 the first term does not vanish, we get $\mathrm{D}^2_\epsilon C(x(\epsilon))|_{\epsilon=0} \neq 0$ as claimed.

■

Note that in the above setting both choices of C are constant on the loops $\exp(-t\xi(\epsilon))\Phi^t(x(\epsilon))$, $t \in [0, T(\epsilon)]$, $\epsilon > 0$, and hence, up to order two in ϵ, the graphs of these loops form a paraboloid around $\bar{x} = x(0)$, where $\xi(0) = \bar{\xi}$. The same picture applies to the reduced dynamics on $\widetilde{\mathcal{N}}_1$, the symplectic normal space for the symmetry group $\tilde{\Gamma} = \Gamma_{\bar{\sigma}}$ at $\bar{x}$, see Figure 1. Relative Lyapunov centre bifurcations along branches of RPOs can therefore detected by checking whether

$$\langle u(x^{(0)}), u(x^{(1)}) \rangle < 0 \tag{4.6}$$

where $y^{(0)} = (x^{(0)}, T^{(0)}, \xi^{(0)}, 0, 0)$ and $y^{(1)} = (x^{(1)}, T^{(1)}, \xi^{(1)}, 0, 0)$ are two consecutively computed solutions of (4.3) or (4.4). Here

$$u(x) = (\mathrm{id} - \mathrm{Q}(x)) f_H(x) \in \widetilde{\mathcal{N}}_1$$

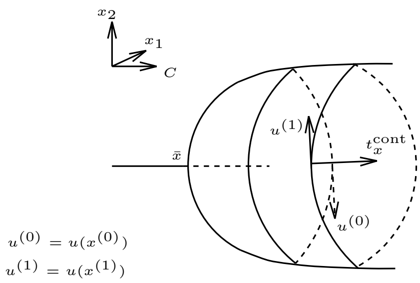

Figure 1: Detection of relative Lyapunov centre bifurcations, where $t_x^{\rm cont} = t_x^{\rm cont}(y^{(1)})$, for more explanations see text.

where $\mathrm{Q}(x)$ is the orthonormal projection onto $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}x$.

To see this note that at the relative Lyapunov centre bifurcation point $\bar{x}$ we have $f_H(\bar{x}) = \bar{\xi}\bar{x}$. Since $\bar{\xi} \in \mathbf{g}_{(\bar{\sigma},\bar{\mu})}\bar{x}$ we have $u(\bar{x}) = 0$. Moreover,

$$\mathrm{D}_\epsilon u(x(\epsilon))|_{\epsilon=0} = (\mathrm{id} - \mathrm{Q}(\bar{x}))\mathrm{D}f_H(\bar{x})x'(0) - \mathrm{DQ}(\bar{x})x'(0)f_H(\bar{x}) = (\mathrm{id} - \mathrm{Q}(\bar{x}))Mx'(0).$$

Here, as before, $M = \mathrm{D}f(\bar{x}) - \bar{\xi}$, and we used that $\mathrm{Q}(x(\epsilon))\bar{\xi}(x(\epsilon)) = \bar{\xi}x(\epsilon)$ so that

$$\mathrm{DQ}(\bar{x})x'(0)\,\bar{\xi}\bar{x} + \mathrm{Q}(\bar{x})\bar{\xi}x'(0) = \bar{\xi}x'(0).$$

We saw in the proof of Theorem 3.2 that $x'(0) = \tilde{w}'(0) = \mathrm{Re}\,\bar{w} \in \widetilde{W}^{\bar{\omega}}$, see (3.17), so that, by (3.11) applied to the symmetry group $\tilde{\Gamma} = \Gamma_{\bar{\sigma}}$ we have $(\mathrm{id} - \mathrm{Q}(\bar{x}))Mx'(0) = \tilde{A}x'(0) = -\bar{\omega}\,\mathrm{Im}\,\bar{w} \neq 0$ where $\tilde{A} = \mathrm{D}f_{\tilde{\mathcal{N}}_1}(0)$. Therefore $\mathrm{D}_\epsilon u(x(\epsilon))|_{\epsilon=0} \neq 0$ and (4.6) holds true as required.

4.3 Computation of relative Lyapunov centre bifurcation points

Assume that along a path of RPOs $\mathcal{P}(\epsilon)$ through the points $x(\epsilon) \in \mathcal{P}(\epsilon)$ a relative Lyapunov centre bifurcation has been detected at $x(0) = \bar{x}$ and that the assumptions of Theorem 3.2 are satisfied. As before, we reduce the dynamics to $\mathrm{Fix}_{\mathcal{X}}(K)$ where $K = \Gamma_{x(\epsilon)}$ so that we can assume that $K = \{\mathrm{id}\}$ and that each RPO $\mathcal{P}(\epsilon)$ corresponds to a periodic orbit through $x(\epsilon)$ with spatio-temporal symmetry group $\mathbb{Z}_\ell(\alpha)$ in its co-moving frame $\xi(\epsilon)$.

We now want to compute the relative Lyapunov centre point along this path of RPOs. The first problem we encounter is that the relative Lyapunov centre point $\bar{x} = x(0)$ might not be an equilibrium in the computed co-moving frame $\xi(0)$ of the RPO through $x(0)$. This is due to the fact that co-moving frames and drift symmetries of RPOs are not unique, as we demonstrate in the following remark:

Remark 4.2 The decomposition (2.8) is in general not unique: let $\bar{x}$ lie on an RPO with drift-symmetry $\bar{\sigma} = \alpha\exp(-\bar{\tau}\bar{\xi})$ as in (2.8) and relative period $\bar{\tau}$ such that $\dim\mathbf{g}_{\bar{\sigma}} \geq 1$ and let $\mathbf{t}_{\bar{\sigma}}$ be the Lie algebra of the maximal torus of $\Gamma_{\bar{\sigma}}$. Let η be an infinitesimal rotation in $\mathbf{t}_{\bar{\sigma}}$ which generates the rotation group $\exp(\phi\eta) = R_\phi$, $\phi \in [0, 2\pi]$. Then other possible choices $\hat{\alpha}$ for α and $\hat{\xi}$ for $\bar{\xi}$ would be $\hat{\alpha} = R_\phi\alpha$ for any $\phi \in \pi\mathbb{Q}$ and $\hat{\xi} = \bar{\xi} + \phi\eta/\bar{\tau}$. So $\hat{\alpha}$ could have arbitrarily high order $\hat{\ell}$. Moreover in a frame moving with velocity $\hat{\xi}$ the RPO would have arbitrarily large period $\hat{\ell}\bar{\tau}$. If $\phi = 2\pi n/\ell$ where $n \in \mathbb{Z}$, then $\ell = \hat{\ell}$. Given an RPO it is natural to assume that ℓ in the decomposition (2.8) is minimal. But note that the co-rotating frame $\bar{\xi}$ such that ℓ is minimal is not invariant under parameter continuation (see Section 5 for a concrete example) and that minimality of ℓ is not guaranteed in the setting of Theorem 3.2.

Due to the non-uniqueness of co-moving frames of RPOs we have to compute the co-moving frame $\bar{\xi}$ in which the relative Lyapunov centre bifurcation point $\bar{x}$ becomes an equilibrium, and the symmetry $\bar{\alpha}$ which lies in the isotropy subgroup $\bar{K}$ of $\bar{x}$, such that the bifurcating RPOs have drift symmetry $\bar{\alpha}$ in the frame $\bar{\xi}$ of the relative equilibrium. In a first step (Section 4.3.1) we construct approximate values of the drift velocity $\bar{\xi}$ of the relative equilibrium, of the drift velocity $\xi(0)$ of the path of RPOs at bifurcation, and of the relative Lyapunov centre point $\bar{x}$ from numerically available data which we then use to find the symmetry $\bar{\alpha}$ (in Section 4.3.2). Then, in Section 4.3.3 we show how to compute the relative Lyapunov centre point using the results from Sections 4.3.1 and 4.3.2. Finally, in Section 4.3.4, we show how to compute the relative normal frequency of the bifurcating branch of RPOs.

4.3.1 Initial approximation for the relative Lyapunov centre point

Assume that along a path $x(\epsilon)$ on points of RPOs $\mathcal{P}(\epsilon)$ which, as in the above sections, is obtained by solving (4.3) or (4.4), a relative Lyapunov centre bifurcation takes place at $\epsilon = 0$ and that the conditions of Theorem 3.2 are satisfied. As before we assume that the isotropy subgroup K of the RPOs is trivial.

In this section we show how to obtain initial approximations for the relative Lyapunov centre point $x(0) = \bar{x}$, and the drift velocity $\xi(0)$ and relative period $\bar{\tau}$ of the family $\mathcal{P}(\epsilon)$ at bifurcation $\epsilon = 0$. These approximations will then be fed into Newton-type methods for the computation of the relative Lyapunov centre bifurcation (see Sections 4.3.3 and 4.3.4).

If a relative Lyapunov centre bifurcation has been detected between two consecutively computed solutions $y^{(0)} = (x^{(0)}, T^{(0)}, \xi^{(0)}, 0, 0)$ and $y^{(1)} = (x^{(1)}, T^{(1)}, \xi^{(1)}, 0, 0)$ of (4.3) or (4.4) then, as for the computation of turning points, see [35], we use Hermite interpolation $(\hat{y}(\epsilon), \hat{C}(\epsilon))$ between $(y^{(0)}, C^{(0)})$ and $(y^{(1)}, C^{(1)})$ and compute $\hat{\epsilon}$ such that $\hat{C}'(\hat{\epsilon}) = 0$, using that, by Proposition 4.1, we have $C'(0) = 0$ at the relative equilibrium and $C''(0) \neq 0$. Let $\hat{y} = (\hat{x}, \hat{T}, \hat{\xi}, 0, 0) = \hat{y}(\hat{\epsilon})$. As initial guess for the relative Lyapunov centre point $\bar{x}$ we then take $\hat{x}$ and as approximation for the relative period at bifurcation we take $\hat{\tau} = \hat{T}/\ell$. Moreover we approximate the drift velocity $\xi(0)$ of the relative equilibrium in the frame of the RPOs as $\hat{\xi}$.

4.3.2 Computing symmetries of the Lyapunov centre relative equilibrium

Along the path $x(\epsilon) \in \mathcal{P}(\epsilon)$ from above we have

$$\alpha \exp(-\xi(\epsilon)\tau(\epsilon))\Phi^{\tau(\epsilon)}(x(\epsilon)) = x(\epsilon) \tag{4.7}$$

where $\xi(\epsilon) \in \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ and $\bar{x} = x(0)$ is on the relative equilibrium. At the relative Lyapunov centre point $\Phi^t(\bar{x}) = \exp(t\bar{\xi})\bar{x}$ holds for all t where $\bar{\xi} \in \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ is determined from

$$f_H(\bar{x}) = \bar{\xi}\bar{x}.$$

We employ this equation to approximate the drift velocity $\bar{\xi}$ of the relative Lyapunov centre point $\bar{x}$ by

$$\hat{\xi}_{\mathrm{RE}} = \sum_{i=1}^{r} (\hat{\xi}_{\mathrm{RE}})_i \hat{e}_i^{\xi} \approx \bar{\xi}$$

where $(\hat{\xi}_{\mathrm{RE}})_i := \langle f_H(\hat{x}), \hat{e}_i^{\xi}\hat{x}\rangle$. Here $\hat{e}_i^{\xi}\hat{x}, i = 1, \ldots, r$, denotes an orthonormal basis of $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}\hat{x}$ and $\hat{x}$ is the approximation of the relative Lyapunov centre point from Section 4.3.1 above. Letting $\epsilon \to 0$ in (4.7) we get

$$\bar{x} = \alpha \exp(-\xi(0)\bar{\tau})\Phi^{\bar{\tau}}(\bar{x}) = \alpha \exp(-\xi(0)\bar{\tau}) \exp(\bar{\tau}\bar{\xi})\bar{x}$$

and so $\bar{x}$ has the symmetry

$$\bar{\alpha} = \alpha \exp(-\xi(0)\bar{\tau}) \exp(\bar{\tau}\bar{\xi}) = \alpha \exp(\bar{\tau}(\bar{\xi} - \xi(0))). \tag{4.8}$$

Here we used that $\mathbf{g}_{(\bar\sigma,\bar\mu)}$ is abelian and that α and $\mathbf{g}_{(\bar\sigma,\bar\mu)}$ commute. This follows from Lemma 3.3 b) since $(\bar\sigma,\bar\mu)$ is a regular drift-momentum pair by Remark 3.5.

Note further that $\bar\alpha$ is the drift symmetry of the RPOs bifurcating from $\bar x$ in the frame moving with the drift velocity $\bar\xi$ of the relative equilibrium through $\bar x$. We now show how to compute $\bar\alpha$ given α using the numerical approximations for $\hat\xi$, $\hat\xi_{\rm RE}$ and $\hat\tau$ for $\xi(0)$, $\bar\xi$ and $\bar\tau$.

Under the assumptions of Theorem 3.2 we have $\bar\alpha^{\bar\ell}=\mathrm{id}$ for some $\bar\ell\in\mathbb{N}$. Since α has finite order ℓ, we know that $\exp(\bar\tau(\bar\xi-\xi(0))$ has finite order $\mathrm{lcm}(\ell,\bar\ell)$, the smallest common multiple of ℓ and $\bar\ell$:

$$\exp(\mathrm{lcm}(\ell,\bar\ell)\bar\tau(\bar\xi-\xi(0))=\mathrm{id}\,. \tag{4.9}$$

Let $\eta=\bar\tau(\bar\xi-\xi(0))$. Since the identity component $\Gamma^{\rm id}_{(\bar\sigma,\bar\mu)}$ of $\Gamma_{(\bar\sigma,\bar\mu)}$ is abelian, it is isomorphic to an r-dimensional torus group: $\Gamma^{\rm id}_{(\bar\sigma,\bar\mu)}\simeq\mathbb{T}^r$. Let, as before, e^ξ_j, $j=1,\dots,r$, be a basis of the Lie algebra $\mathbf{t}^r$ of $\mathbb{T}^r$ such that each e^ξ_j generates a copy of $\mathbb{S}^1\simeq\mathbb{R}/\mathbb{Z}$. From (4.9) we conclude that there are $k_j\in\mathbb{Z}$, $j=1,\dots,r$ such that

$$\eta=\bar\tau(\bar\xi-\xi(0))=\sum_{j=1}^r\eta_j e^\xi_j \quad\text{where}\quad \eta_j=k_j/\mathrm{lcm}(\ell,\bar\ell). \tag{4.10}$$

Using (4.8), (4.9), and (4.10) we can compute $\bar\alpha$ exactly if our approximations for $\hat\xi$, $\hat\xi_{\rm RE}$ and $\hat\tau$ from above are good enough. In practice, we do not know $\bar\ell$. So we replace $\mathrm{lcm}(\ell,\bar\ell)$ by $p\ell$, where $\mathrm{lcm}(\ell,\bar\ell)$ divides $p\ell$. Here p is determined by the action of Γ on $\mathcal{X}$, see Remark 4.3 below, and also the succeeding examples. For $j=1,\dots,r$, we compute the integer $\hat k_j$ closest to $p\ell\hat\eta_j$ where $\hat\eta_j$ is an approximation for η_j obtained from the Hermite approximations of $\hat\xi$ for $\xi(0)$, $\hat\tau$ for $\bar\tau$ and $\hat\xi_{\rm RE}$ for $\bar\xi$. This gives us an algorithmic way to compute $\bar\alpha$. The algorithm then accepts $\bar\alpha$ as a symmetry of the relative Lyapunov centre bifurcation if

$$\langle\bar\alpha x^{(1)}-x^{(1)},\bar\alpha x^{(0)}-x^{(0)}\rangle<0. \tag{4.11}$$

Note that there are small $\epsilon_0,\epsilon_1\in\mathbb{R}$ with $\epsilon_0\epsilon_1>0$ such that

$$x^{(0)}=\bar x+\epsilon_0\,\mathrm{Re}\,\bar w+O(\epsilon_0^2),\quad \bar\alpha x^{(0)}=\bar x+\epsilon_0\,\mathrm{Re}(\mathrm{e}^{-2\pi\mathrm{i}/\bar\ell}\bar w)+O(\epsilon_0^2),$$

and

$$x^{(1)}=\bar x-\epsilon_1\,\mathrm{Re}\,\bar w+O(\epsilon_1^2),\quad \bar\alpha x^{(1)}=\bar x-\epsilon_1\,\mathrm{Re}(\mathrm{e}^{-2\pi\mathrm{i}/\bar\ell}\bar w)+O(\epsilon_1^2).$$

Here, in the notation of Theorem 3.2, $x'(0)=\mathrm{Re}\,\bar w$, see (3.17). Therefore (4.11) is satisfied at a passage through a relative Lyapunov centre bifurcation.

Remark 4.3 Note that (4.8) implies that $\mathrm{e}^{\ell\eta}$ has finite order $p=\mathrm{lcm}(\ell,\bar\ell)/\ell$. Therefore η generates a one-parameter group $\mathbb{S}^1=\mathbb{S}^1_\eta$. Hence we know that the action of $\exp(\theta\eta)$, $\theta\in[0,1]$, on $\mathcal{X}$ has irreducible representations $e^{\pm\mathrm{i}2\pi p_k\theta}$, for some $p_k\in\mathbb{Z}$. Moreover, $\mathrm{e}^{\ell\eta}$ fixes $\bar x$. Therefore p divides at least one p_k and in particular $\mathrm{lcm}_k p_k$, the least common multiple of the numbers p_k. If we a priori know the numbers p_k for all rotations we can set $p=\mathrm{lcm}_k p_k$. In the N-body problem the rotation group acts freely on the position and momentum of each particle (unless both vanish), so that $p=1$ and $\bar\alpha\alpha^{-1}$ has order ℓ, see Section 5.1.

Remark 4.4 If the rotating frame of the RPO is chosen such that ℓ is minimal then $\ell/\bar\ell$: to see this let $j=\gcd(\ell,\bar\ell)$ be the greatest common divisor of ℓ and $\bar\ell$. Then there are $k,\bar k\in\mathbb{Z}$ such that $j=k\ell+\bar k\bar\ell$. Then, as $\bar\alpha=\alpha e^\eta$ we have $\alpha^j=\exp(-\bar k\bar\ell\eta)$, so that $\alpha\exp(-\bar k\bar\ell\eta/j)$ has an order which divides j. As ℓ is minimal, ℓ divides j and so ℓ divides $\bar\ell$.

Example 4.5 Let the identity component of Γ be $\mathrm{SO}(2)$ and assume that it acts freely on $\mathcal{X} \setminus \{0\}$ and that the conditions of Theorem 3.2 are satisfied. Let $R(\phi)$ be a rotation by ϕ. Let $\bar{\omega}^{\mathrm{rot}}$ be the rotation frequency of the relative Lyapunov centre point $\bar{x}$ and let $\pm \mathrm{i}\bar{\omega}$ be the eigenvalues of $\mathrm{D}f_{\mathcal{N}_1}(0)$ corresponding to the computed branch $\mathcal{P}(\epsilon)$ of RPOs bifurcating from $\bar{x}$ at $\epsilon = 0$. Assume that the isotropy subgroup of the relative Lyapunov centre point is $\bar{K} = \mathbb{Z}_{\bar{\ell}}(\bar{\alpha})$. Then $\bar{\alpha}\alpha^{-1}$ has order $\ell = \bar{\ell}$ by Remark 4.3. Let $\bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}} := \omega^{\mathrm{rot}}(0)$ be the rotation frequency of the path of RPOs $\mathcal{P}(\epsilon)$ at $\epsilon = 0$ in their rotating frame and $\bar{\tau} = 2\pi/(\bar{\omega}\ell)$ be their relative period. Then (4.8) and (4.10) give

$$\bar{\alpha} = \alpha R((\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}})\bar{\tau}) = \alpha R(2\pi(\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}})/(\bar{\omega}\ell)) = \alpha R(2\pi j/\ell) \tag{4.12}$$

for some $j \in \mathbb{Z}$. This implies that for some choice of $j \in \mathbb{Z}$

$$\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}} = j\bar{\omega}. \tag{4.13}$$

Note that for $j \neq 0$, the relative equilibrium is not an equilibrium in the frame moving with the drift velocity $\bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}} = \omega^{\mathrm{rot}}(0)$ of the family of RPOs $\mathcal{P}(\epsilon)$ at the relative Lyapunov centre bifurcation $\epsilon = 0$, but a circle which is traversed j times. This is due to the fact that the drift velocity of an RPO is not unique, c.f. Remark 4.2. In Section 5.3 we show that this situation occurs in the continuation of RPOs of the gravitational three-body problem.

Example 4.6 Now assume that $\Gamma^{\mathrm{id}} = \mathrm{SO}(2)$ acts non-freely and that the path of RPOs has a co-moving frame such that ℓ is minimal. Then ℓ divides $\bar{\ell}$ by Remark 4.4, and (4.12) becomes

$$\bar{\alpha} = \alpha R((\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}})\bar{\tau}) = \alpha R(2\pi(\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}})/(\bar{\omega}\bar{\ell})) = \alpha R(2\pi j/\bar{\ell})$$

for some $j \in \mathbb{Z}$ and therefore (4.13) holds again. If ℓ is not minimal and $\mathrm{lcm}(\ell, \bar{\ell}) = p\ell$ then (4.10) implies

$$(\bar{\omega}^{\mathrm{rot}} - \bar{\omega}^{\mathrm{rot}}_{\mathrm{RPO}})p = j\bar{\omega}^{\mathrm{RPO}} \tag{4.14}$$

where $2\pi/(\ell\bar{\omega}^{\mathrm{RPO}}) = \bar{\tau} = 2\pi/(\bar{\ell}\bar{\omega})$. Note that an approximation $2\pi/\hat{T}$ of $\bar{\omega}^{\mathrm{RPO}}$ is numerically accessible during the continuation of the RPO (see Section 4.3.1) whereas an approximation of $\bar{\omega}$ is in general not known (unless we know that $\ell = \bar{\ell}$, see above).

4.3.3 Computation of the Lyapunov centre relative equilibrium

Let $\Gamma\bar{x}$ be a nondegenerate relative equilibrium with regular velocity-momentum pair $(\bar{\xi}, \bar{\mu})$ at $\bar{x}$. Let, as before, $e^{\xi}_1, \ldots, e^{\xi}_r$ denote a basis of $\mathbf{g}_{(\bar{\xi},\bar{\mu})}$, let $e^{\xi}_1, \ldots, e^{\xi}_q$ denote a basis of $\mathbf{g}_{\bar{\xi}} := \mathbf{g}_{(\bar{\xi},0)}$ and identify $(\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$ with $\xi = \sum_{i=1}^q \xi_i e^{\xi}_i$. Moreover as before set $\mathbf{J}_i(x) = \mathbf{J}_{e^{\xi}_i}(x)$, $i = 1, \ldots, q$. Again, typically $q = r$, see [27, 35].

The manifold of relative equilibria $\Gamma x(\chi)$, $\chi \in \mathbf{g}^*_{(\bar{\xi},\bar{\mu})}$, near $\Gamma\bar{x}$ with momentum $\mathbf{J}(x(\chi)) = \bar{\mu} + \chi$ from Section 2.2 can then be computed numerically by solving the under-determined

$$F_{\mathrm{RE}}(x, \xi, \lambda_\mu) = \begin{pmatrix} f_H(x) - \sum_{i=1}^r \lambda_{\mu,i} \nabla \mathbf{J}_i(x) - \sum_{i=1}^q \xi_i e^{\xi}_i x \\ \mathbf{J}_{r+1}(x) - \bar{\mu}_{r+1} \\ \vdots \\ \mathbf{J}_q(x) - \bar{\mu}_q \end{pmatrix} = 0, \tag{4.15}$$

where $F_{\mathrm{RE}} : \mathcal{X} \times \mathbb{R}^{r+q} \to \mathcal{X} \times \mathbb{R}^{q-r}$, see [35]. In particular, any solution $y = (x, \xi, \lambda_\mu)$ of $F_{\mathrm{RE}} = 0$ close to $(\bar{x}, \bar{\xi}, 0)$ satisfies $\lambda_\mu = 0$ and $x = \gamma x(\chi)$ for some $\gamma \in \Gamma_{(\bar{\xi},\bar{\mu})}$, $\gamma \approx \mathrm{id}$ and χ small, hence is a relative equilibrium of (2.1). Moreover $\mathrm{D}F_{\mathrm{RE}}(\bar{x}, \bar{\xi}, 0)$ has full rank, and so (4.15) can, for example, be solved by a Gauss-Newton method.

In this case we want to compute a relative equilibrium in $\mathrm{Fix}_{\mathcal{X}}(L)$ which is a relative Lyapunov centre point along a path of RPOs through $x(\epsilon) \in \mathcal{X}^{\bar{\mu}}$ with trivial isotropy subgroup K which we continue in energy or along a path of RPOs through $x(\epsilon) \in \mathcal{X}^{\bar{E},\bar{\mu}^{\flat}}$ which we continue in the first momentum component μ_1. As before we assume that the RPOs through $x(\epsilon)$ have trivial isotropy subgroup $K = \{\mathrm{id}\}$ and spatio-temporal symmetry group $L = \mathbb{Z}_{\bar{\ell}}(\bar{\alpha})$ in the corotating frame $\bar{\xi}$ of the relative Lyapunov centre point. Let F_{RE}^{L} denote the function F_{RE} from (4.15) with $\mathcal{X}$ replaced by $\mathrm{Fix}_{\mathcal{X}}(L)$, $\mathbf{g}_{\bar{\xi}}$ replaced by $\mathrm{Fix}_{\mathbf{g}_{\bar{\xi}}}(L)$ and $\mathbf{g}_{(\bar{\xi},\bar{\mu})}$ replaced by $\mathrm{Fix}_{\mathbf{g}_{(\bar{\xi},\bar{\mu})}}(L) = \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$, see (3.9). Under the assumptions of Theorem 3.2 the relative equilibrium through $\bar{x}$ is L-nondegenerate with L-regular velocity-momentum pair. So if we continue in energy then we can compute the relative Lyapunov centre point $\bar{x} \in \mathcal{X}^{\bar{\mu}}$ that has been detected along the branch of RPOs by solving $F_{\mathrm{RE}}^{L} = 0$ along with the constraints $\mathbf{J}_j(x) - \bar{\mu}_j = 0$, $j = 1, \ldots, r.$, $r = \dim \mathbf{g}_{(\bar{\sigma},\bar{\mu})}$ We denote the corresponding nonlinear equation as

$$F_{\mathrm{RE}}^{\bar{\mu}}(x, \xi, \lambda_{\mu}) = 0, \tag{4.16}$$

where $F_{\mathrm{RE}}^{\bar{\mu}} : \mathcal{X} \times \mathbb{R}^{q+r} \to \mathcal{X} \times \mathbb{R}^{q}$. Then any solution of $F_{\mathrm{RE}}^{\bar{\mu}} = 0$ near $\bar{y} = (\bar{x}, \bar{\xi}, 0)$ takes the form $y = (x, \xi, 0)$ where the constraints imply that $\chi = 0$ so that $x = \gamma\bar{x} \in \mathcal{X}^{\bar{\mu}}$, $\gamma \in \Gamma_{(\bar{\sigma},\bar{\mu})}$, $\gamma \approx \mathrm{id}$. As before (4.16) can be solved numerically by a Newton type method.

In the second case, when we continue in μ_1 we solve $F_{\mathrm{RE}}^{L} = 0$ along with the constraints $\mathbf{J}_j(x) - \bar{\mu}_j = 0$, $2 = 1, \ldots, r$, $H(x) - \bar{E} = 0$. We denote the corresponding nonlinear equation as

$$F_{\mathrm{RE}}^{\bar{E},\bar{\mu}^{\flat}}(x, \xi, \lambda_{\mu}) = 0, \tag{4.17}$$

and we assume that $\bar{\xi}_1 \neq 0$ as before (see Proposition 4.1). In this case the constraints imply that any solution $y = (x, \xi, 0)$ of (4.17) satisfies $y = (\gamma x(\chi), \xi(\chi), 0)$ with $\gamma \in \Gamma_{(\bar{\sigma},\bar{\mu})}$ and $\chi_2 = \ldots = \chi_r = 0$. Here $\Gamma x(\chi)$ is the family of relative equlibria near $\Gamma\bar{x}$ from Section 2.2. The constraint $H(x) = \bar{E}$ then gives $x \in \mathcal{X}^{\bar{E},\bar{\mu}}$. From (4.5) and $\bar{\xi}_1 \neq 0$ we conclude that $\mathrm{D}F_{\mathrm{RE}}^{\bar{E},\bar{\mu}^{\flat}}(\bar{x}, \bar{\xi}, 0)$ has full rank again and that the x-components of the solutions of (4.17) lie on the relative equilibrium $\Gamma\bar{x}$, as required.

4.3.4 Computation of bifurcating relative normal modes

In this section we show how to compute the relative normal frequency $\bar{\omega}$ and the start off plane $\mathrm{span}(\mathrm{Re}\,\bar{w}, \mathrm{Im}\,\bar{w}) = \mathrm{Fix}_{\mathcal{W}_{\bar{\omega}}}(\Lambda)$ for the bifurcating relative normal modes. As before we assume that the bifurcating RPOs have trivial isotropy subgroup $K = \{\mathrm{id}\}$ (after reduction to $\mathrm{Fix}_{\mathcal{X}}(K)$) and spatio-temporal symmetry group $L = \mathbb{Z}_{\bar{\ell}}(\bar{\alpha})$ in the corotating frame of the relative equilibrium.

Let, as in the proof of Theorem 3.2, $\widetilde{\mathcal{N}}_1$ be the symplectic normal space with respect to the symmetry group $\tilde{\Gamma} = \Gamma_{\bar{\sigma}}$, and let $\mathrm{P}_{\widetilde{\mathcal{N}}_1}$ be an L-equivariant projection onto $\widetilde{\mathcal{N}}_1$ with kernel $\widetilde{\mathcal{T}} \oplus \widetilde{\mathcal{N}}_0$, where $\widetilde{\mathcal{T}} = \mathbf{g}_{\bar{\sigma}}\bar{x}$ and $\widetilde{\mathcal{N}}_0 \simeq \mathbf{g}^{*}_{(\bar{\sigma},\bar{\mu})}$. Let, as before, $\tilde{A} = P_{\widetilde{\mathcal{N}}_1} M P_{\widetilde{\mathcal{N}}_1} = \mathrm{D}f_{\widetilde{\mathcal{N}}_1}(0)$. Here, $M = \mathrm{D}f_H(\bar{x}) - \bar{\xi}$ is as in (3.11).

We first treat the simplest case $\bar{\ell} = 1$ where the spatio-temporal symmetry group $L = \mathbb{Z}_{\bar{\ell}}(\bar{\alpha})$ of the bifurcating RPOs in the co-moving frame of the relative Lyapunov centre point is trivial. Define $F^{(1)} : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+1}$, where $\dim \mathcal{X} = 2d = n$, as

$$F^{(1)}(u, v, \omega, \lambda_E) = \begin{pmatrix} \tilde{A}u - \omega v - \lambda_E \mathbb{J}^{-1}\tilde{A}u \\ \tilde{A}v + \omega u - \lambda_E \mathbb{J}^{-1}\tilde{A}v \\ \langle u, u\rangle + \langle v, v\rangle - 1 \end{pmatrix}. \tag{4.18}$$

Note that $y = (\bar{u}, \bar{v}, \bar{\omega}, 0)$ satisfies the equation $F^{(1)} = 0$ when $\bar{w} = \bar{u} + \mathrm{i}\bar{v}$ is an eigenvector of $\tilde{A}$ to the eigenvalue $\mathrm{i}\bar{\omega}$ as in the proof of Theorem 3.2. Then $F^{(1)} = 0$ can be solved by a Gauss-

Newton method (or by a Newton method if the phase θ is fixed) as the following proposition shows:

Proposition 4.7 *Under the assumptions of Theorem 3.2 and if $\bar{\ell} = 1$ then the derivative $\mathrm{D}F^{(1)}(\bar{u}, \bar{v}, \bar{\omega}, 0)$ of (4.18) has full rank. Moreover any solution y of $F^{(1)} = 0$ close to $\bar{y} = (\bar{u}, \bar{v}, \bar{\omega}, 0)$ has the form $y = (\mathrm{Re}(\mathrm{e}^{i\theta}\bar{w}), \mathrm{Im}(\mathrm{e}^{i\theta}\bar{w}), \bar{\omega}, 0)$ where $\theta \in [0, 2\pi]$.*

Proof. Let $y = (u, v, \omega, \lambda_E)$ be in the kernel of $\mathrm{D}F^{(1)}(\bar{y})$. Then

$$\tilde{A}u - \bar{\omega}v - \omega\bar{v} - \lambda_E\mathbb{J}^{-1}\tilde{A}\bar{u} = 0, \quad \tilde{A}v + \bar{\omega}u + \omega\bar{u} - \lambda_E\mathbb{J}^{-1}\tilde{A}\bar{v} = 0, \quad \langle u, \bar{u}\rangle + \langle v, \bar{v}\rangle = 0. \quad (4.19)$$

Since $\bar{\ell} = 1$ the assumptions of Theorem 3.2 imply that $\mathrm{i}\bar{\omega}$ is a simple eigenvalue of $\tilde{A}$, as shown in the proof of Theorem 3.2. Moreover $\mathbb{J}^{-1}\tilde{A}|_{\mathcal{W}^{\bar{\omega}}} = \bar{\omega}$ or $\mathbb{J}^{-1}\tilde{A}|_{\widetilde{\mathcal{W}}^{\bar{\omega}}} = -\bar{\omega}$. From this we conclude that the first two equations of (4.19) have the two-dimensional solution space spanned by $y = ((\bar{u}, \bar{v}), 0, 0)$ and $y = ((\bar{v}, -\bar{u}), 0, 0)$. Together with the last equation this gives that (u, v) is parallel to $(\bar{v}, -\bar{u})$ and so $\ker \mathrm{D}F(\bar{y})$ is one-dimensional. This implies that $\mathrm{D}F(\bar{y})$ has full rank. Hence there is a one-parameter family of solutions of $F^{(1)} = 0$ near $\bar{y}$ which is given by $y = (\mathrm{Re}(\mathrm{e}^{i\theta}\bar{w}), \mathrm{Im}(\mathrm{e}^{i\theta}\bar{w}), \bar{\omega}, 0)$, $\theta \in [0, 2\pi]$. ∎

When $\bar{\ell} > 1$ then the eigenvalue $\mathrm{i}\bar{\omega}$ of $\mathrm{D}f_{\mathcal{N}_1}(0)$ where $\mathrm{i}\bar{\omega}$ is the relative normal mode frequency of the bifurcating branch of RPOs might be multiple due to symmetry. We show next how to deal with this case. We use (3.17) to construct nonlinear equations $F^{(\bar{\ell})} = 0$ which determine the relative normal mode $\bar{w}$ and the relative normal frequency $\bar{\omega}$ and can be solved by Newton like methods. For this we use results from [34] on the numerical computation of equivariant Hopf points.

When $\bar{\ell} = 2$ we store an L-orthonormal basis (i.e., a basis which is orthonormal with respect to an L-invariant inner product) of $\mathcal{X}_2 := \mathrm{Fix}_{\mathcal{X}}(-\bar{\alpha})$ in the row vectors of $Q^{(2)} \in \mathrm{Mat}(k, n)$ (where $k = \dim \mathrm{Fix}_{\mathcal{X}}(-\bar{\alpha})$ and $n = \dim \mathcal{X}$). When $\bar{\ell} > 2$ let k be the dimension of the space of complex vectors $u + \mathrm{i}v$, $u, v \in \mathcal{X}$ satisfying

$$u + \mathrm{i}v = \bar{\alpha}e^{\frac{\mathrm{i}2\pi}{\ell}}(u + \mathrm{i}v) \quad (4.20)$$

(called the complex isotypic component of $\mathcal{X}$ where $\bar{\alpha}$ acts as $e^{\frac{-\mathrm{i}2\pi}{\ell}}$). Under the assumptions of Theorem 3.2 Equation (4.20), together with the condition that $\bar{w} = \bar{u} + \mathrm{i}\bar{v}$ is an eigenvector of $\tilde{A}$ to $\mathrm{i}\bar{\omega}$, determines $\bar{w}$ uniquely up to a scalar in $\mathbb{C}$. This follows from (3.17) and the L-equivariance of $\tilde{A}$. We store an L-orthonormal basis of this space in the rows of the real-valued $(k, 2n)$-matrix $Q^{(\bar{\ell})} = [Q_1^{(\bar{\ell})}, Q_2^{(\bar{\ell})}]$ where $Q_1^{(\bar{\ell})}$, $Q_2^{(\bar{\ell})}$ are (k, n)-matrices.

Define $F^{(2)} : \mathbb{R}^{2k+2} \to \mathbb{R}^{2k+1}$ as

$$F^{(2)}(u, v, \omega, \lambda_E) = \begin{pmatrix} Q\tilde{A}Q^T u - \omega v - \lambda_E Q\mathbb{J}^{-1}\tilde{A}Q^T u \\ Q\tilde{A}Q^T v + \omega u - \lambda_E Q\mathbb{J}^{-1}\tilde{A}Q^T v \\ \langle u, u\rangle + \langle v, v\rangle - 1 \end{pmatrix} \quad (4.21)$$

where $Q = Q^{(2)}$. For $\bar{\ell} > 2$ define $F^{(\bar{\ell})} : \mathbb{R}^{k+2} \to \mathbb{R}^{k+1}$ as

$$F^{(\bar{\ell})}(u, \omega, \lambda_E) = \begin{pmatrix} Q_1\tilde{A}Q_1^T u - \omega Q_1 Q_2^T u - \lambda_E Q_1\mathbb{J}^{-1}\tilde{A}Q_1^T u \\ \langle u, u\rangle - 1 \end{pmatrix} \quad (4.22)$$

where $Q_j = Q_j^{(\bar{\ell})}$, $j = 1, 2$. Then the following analogue of Proposition 4.7 holds:

Proposition 4.8 *Under the assumptions of Theorem 3.2 the derivatives* $\mathrm{D}F^{(2)}(\bar{u},\bar{v},\bar{\omega},0)$ *of* (4.21) *(for* $\bar{\ell}=2$*) and* $\mathrm{D}F^{(\bar{\ell})}(\bar{u},\bar{\omega},0)$ *of* (4.22) *(for* $\bar{\ell}>2$*) have full rank. Moreover any solution* y *of* $F^{(2)}=0$ *close to* $\bar{y}=(\bar{u},\bar{v},\bar{\omega},0)$ *has the form* $y=(\mathrm{Re}(\mathrm{e}^{i\theta}\bar{w}),\mathrm{Im}(\mathrm{e}^{i\theta}\bar{w}),\bar{\omega},0)$ *where* $\theta\in[0,2\pi]$*. Furthermore, for* $\bar{\ell}>2$*, any solution* y *of* $F^{(\bar{\ell})}=0$ *close to* $\bar{y}=(\bar{u},\bar{\omega},0)$ *has the form* $y=(\mathrm{Re}(\mathrm{e}^{i\theta}\bar{w}),\bar{\omega},0)$ *for some* $\theta\in[0,2\pi]$*.*

Proof. The proof is analogous to the proof of Proposition 4.7, see also [34]. ∎

Remarks 4.9

a) For the numerical computation of $\tilde{A}$ an L-equivariant projector $\mathrm{P}_{\widetilde{\mathcal{N}}_1}$ is needed. For its construction we need an L-invariant inner product. If $L=\mathbb{Z}_{\bar{\ell}}(\bar{\alpha})$ does not act orthogonally with respect to the standard inner product on $\mathcal{X}$ then we can compute an L-invariant inner product on $\mathrm{Fix}(K)$ as
$$\langle x_1,x_2\rangle_L=\frac{1}{\bar{\ell}}\sum_{j=0}^{\bar{\ell}-1}\langle\bar{\alpha}^j x_1,\bar{\alpha}^j x_2\rangle.$$
If $\bar{\ell}\neq\ell$ we may replace $\bar{\ell}$ by $p\ell$, see Section 4.3.2.

b) If the non-resonance condition
$$\dim\mathrm{Fix}_{\mathbf{g}_{\bar{\sigma}}}(K)=r \tag{4.23}$$
holds then we may replace $\tilde{A}$ by M in (4.18), (4.21) and Propositions 4.7 and 4.8 still hold true. This is due to the fact that, because of (4.23), (3.5), (3.6), (3.8) and the L-equivariance of M, Equation (4.20) together with the condition that w is an eigenvector of M to $\mathrm{i}\bar{\omega}$ determines w uniquely up to a scalar in $\mathbb{C}$. Since by (3.11) and condition (4.23) the $\mathcal{N}_0$ and $\mathcal{T}_1$ component of any eigenvector w of M to the eigenvalue $\mathrm{i}\bar{\omega}$ which satisfies (4.20) vanishes, we can then obtain an eigenvector $\bar{w}$ of $\tilde{A}$ satisfying (4.20), as needed for the numerical continuation (see below), from w as $\bar{w}=(\mathrm{id}-\mathrm{P})w$ where P is an orthonormal projection onto $\mathbf{g}_{(\bar{\sigma},\bar{\mu})}\bar{x}$. Note that a perturbation of the relative equilibrium typically destroys a 1 : 1-resonance between the eigenvalues of the blocks $\mathrm{ad}_{\bar{\xi}}$ and $\mathrm{D}f_{\mathcal{N}_1}(0)$ of the linearization M at a relative equilibrium so that (4.23) is generic (see also Remark 3.6). But additional structure in a Hamiltonian system can prevent the resonance from being destroyed, see Remark 5.9.

c) In principle we could also compute $A=\mathrm{D}f_{\mathcal{N}_1}(0)$ numerically as
$$\mathrm{D}f_{\mathcal{N}_1}(\bar{x})=P_{\mathcal{N}_1}MP_{\mathcal{N}_1}$$
Here $P_{\mathcal{N}_1}$ is an L-equivariant projection from $\mathcal{X}$ to $\mathcal{N}_1$. We could then replace $\tilde{A}$ by A in the equations (4.18), (4.21) and (4.22) above and compute an eigenvector w of A to the eigenvalue $\mathrm{i}\bar{\omega}$ in $\mathrm{Fix}_{\mathcal{W}^{\bar{\omega}}}(\Lambda)$. From w we can then construct an eigenvector $\bar{w}$ of $\tilde{A}$ as needed for the numerical continuation, see below. This method is reasonable if $\bar{\mu}$ is regular or if the isotropy subgroup $\bar{K}$ of $\bar{x}$ is known and enforces $\bar{\mu}$ to be non-regular, i.e., if $\bar{\mu}$ is regular for the group $N(\bar{K})/\bar{K}$ (called $\bar{K}$-regular). But if $\bar{\mu}$ is not $\bar{K}$-regular the dimension of $\mathcal{N}_1$ may change under arbitrarily small changes of the relative equilibrium $\Gamma\bar{x}=\Gamma x(0)$ to a nearby relative equilibrium $\Gamma x(\chi)$, $\chi\approx 0$. So this technique is only numerically stable for the computation of relative Lyapunov centre points with momentum values $\bar{\mu}$ which are $\bar{K}$-regular.

We next show how to compute initial guesses $\hat{\omega}$ and $\hat{w} = \hat{u} + \mathrm{i}\hat{v}$ for the normal frequency $\bar{\omega}$ and the eigenvector $\bar{w} = \bar{u} + \mathrm{i}\bar{v}$ of $\tilde{A}$ to the eigenvalue $\mathrm{i}\bar{\omega}$ satisfying (4.20) which can be fed into a Newton-type method to solve the equations $F^{(\ell)} = 0$ from (4.18), (4.21) and (4.22): Let $\hat{t} = (\hat{t}_x, \hat{t}_T, \hat{t}_\xi, \hat{t}_E, \hat{t}_\mu)$ be the derivative of the Hermite interpolation $\hat{y}(\epsilon)$ between the RPOs through $x^{(0)}$ and $x^{(1)}$ from Section 4.3.1 at the relative Lyapunov centre bifurcation $\hat{y}$. We set $\hat{u} = \hat{t}_x$, which, by construction, lies in $\hat{t}_x \in \widetilde{\mathcal{N}}_1$, and set $\hat{\omega} = \sqrt{\|\tilde{A}^2\hat{u}\|/\|\hat{u}\|}$ and $\hat{v} = -\tilde{A}\hat{u}/\hat{\omega}$. Here we compute $\tilde{A}$ from $M = \mathrm{D}f_H(\bar{x}) - \bar{\xi}$, where $\bar{x}$, $\bar{\xi}$ were computed in Section 4.3.3.

As tangent vector t_x to the bifurcating family of RPOs at the relative Lyapunov centre bifurcation we take the projection of $\hat{t}_x$ onto the space spanned by $\bar{u}$ and $\bar{v}$. From Theorem 3.2 we deduce that the tangent vector $t = (t_x, t_T, t_\xi, t_E, t_\mu) \in \ker \mathrm{D}F(\bar{y})$, where $\bar{y} = (\bar{x}, \ell\bar{\tau}, \xi(0), 0, 0)$ and $F = F_{\mathrm{RPO}}^{\bar{E},\bar{\mu}^\flat}$ from (4.3) or $F = F_{\mathrm{RPO}}^{\bar{\mu}}$ from (4.4), satisfies $t_\xi = 0$, $t_T = 0$, $t_E = 0$, $t_\mu = 0$.

5 Lagrangian relative equilibria and rotating choreographies

In this section we use Theorem 3.2 to study relative Lyapunov centre bifurcation from Lagrangian relative equilibria in the N body problem. We then apply our numerical methods from Section 4 to compute the relative Lyapunov-centre bifurcation along the type I family of rotating choreographies of the gravitational three-body problem.

5.1 N-body problems and their symmetries

We consider N identical bodies of mass 1 in $\mathbb{R}^3$ acted on only by the forces they exert on each other. These forces are assumed to be given by $\frac{1}{2}N(N-1)$ identical copies of a potential energy function V (one for each pair of bodies) which depends only on the distance between the bodies. Writing p_j for the momenta conjugate to the positions q_j, $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$, the Hamiltonian is

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{N} |p_j|^2 + V(q), \quad \text{where} \quad V(q) = \sum_{i<j} v(r_{ij}), \quad r_{ij} = |q_i - q_j|. \tag{5.1}$$

We assume that $v'(r) > 0$ for all $r > 0$. In the gravitational N-body problem we have $v(r) = -\frac{1}{r}$. Excluding collisions, the configuration space Q is

$$\mathcal{Q} = \{q = (q_1, \ldots, q_N) \in \mathbb{R}^{3N}, \quad q_i \neq q_j \text{ for } i \neq j\}$$

and the phase space is $\mathcal{Q} \times \mathbb{R}^{3N} \subset \mathbb{R}^{6N}$. The equations of motion are

$$\dot{q}_j = p_j, \qquad \dot{p}_j = \sum_{i \neq j} v'(r_{ij})\frac{q_i - q_j}{r_{ij}}, \qquad j = 1, \ldots, N. \tag{5.2}$$

The angular momentum is $\mathbf{J}(q,p) = \sum_{j=1}^{N} q_j \times p_j$. Without loss of generality, the centre of mass of the systems can be assumed to be fixed at 0 restricting the configuration space to

$$\mathcal{Q}^0 = \{q \in Q \ : \ q_N = -\sum_{j=1}^{N-1} q_j\} \simeq \mathbb{R}^{3(N-1)}$$

with corresponding phase space $\mathcal{X} = \mathcal{Q}^0 \times \mathbb{R}^{3(N-1)} \subseteq \mathbb{R}^{6(N-1)}$.

The N-identical-body Hamiltonian (5.1) has the symmetry group $\Gamma = \mathrm{O}(3) \times S_N$ where $R \in \mathrm{O}(3)$ acts on $x = (q,p)$, by mapping q_j to Rq_j and p_j to Rp_j, $j = 1,\dots,N$. The group of all permutations S_N of the integers $1,\dots,N$ acts on x by re-labeling $x_j = (q_j,p_j)$ as $x_{\pi(j)}$, $j = 1,\dots,N$ for any $\pi \in S_N$, in other words $(\pi x)_{\pi(j)} = x_j$. In the following we will frequently use the notation $\pi = (\pi(1),\dots,\pi(N))$. We let e_j^ξ, $j = 1,2,3$, denote an infinitesimal rotation of unit speed around the e_j-axis, $j = 1,2,3$ and we use the notations of Example 3.7 and 3.9 in this section.

5.2 Rotating choreographies near Lagrangian relative equilibria

A Lagrangian relative equilibrium $\Gamma\bar{x}$ in the N body problem is a configuration where all N bodies lie on a circle with equal distance between them (see, e.g., [20] for $N = 3$, [7] and references there in for general N). We assume in this section that $\bar{x}$ lies in the horizontal plane and that the masses are aligned counter-clockwise, i.e., such that $x_{j+1} = R_3(2\pi/N)x_j$, $j = 1,\dots,N$. Let $\bar{\alpha} = \kappa_3\zeta R_3(2\pi/N)$ where $\zeta = (23\dots N1)$. Then its isotropy subgroup is

$$\Gamma_{\bar{x}} = \bar{K} = \begin{cases} \mathbb{Z}_{2N}(\bar{\alpha}) & \text{for odd } N, \\ \mathbb{Z}_N(\bar{\alpha}) \times \mathbb{Z}_2(\kappa_3) & \text{for even } N. \end{cases} \tag{5.3}$$

5.2.1 Lagrangian relative equilibria of the N-body problem

Restricting (5.2) to $\mathrm{Fix}(\bar{K})$ we

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{p}_1 &= \textstyle\sum_{j=2}^N v'(\|(R_3(2\pi j/N)-1)q_1\|)\frac{(R_3(2\pi j/N)-1)q_1}{\|(R_3(2\pi j/N)-1)q_1\|} \\ &= -\textstyle\sum_{j=1}^{\lfloor (N-1)/2\rfloor} v'(\alpha_{j,N}\|q_1\|)\frac{\alpha_{j,N}q_1}{\|q_1\|} - v'(2\|q_1\|)\frac{\beta_{1,N}q_1}{\|q_1\|}, \end{aligned} \tag{5.4}$$

where q_1, p_1 lie in the (e_1,e_2)-plane,

$$\alpha_{j,N} = \alpha_{N-j,N} = (2-2\cos(2\pi j/N))^{1/2} = 2\sin(\pi j/N), \quad \beta_{k,N} = \begin{cases} 1 & \text{if } N \text{ even}, k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and $\lfloor r\rfloor$ is the smallest integer $\leq r$ of $r \in \mathbb{R}$. Here we identified $\mathbb{R}^2$ with $\mathbb{C}$ and used that

$$\overline{(1-e^{2\pi \mathrm{i} j/N})}(1-e^{2\pi \mathrm{i} j/N}) = (1-e^{-2\pi \mathrm{i} j/N})(1-e^{2\pi \mathrm{i} j/N}) = 2-2\cos(2\pi j/N).$$

The system (5.4) has the symmetry group $N(\bar{K})/\bar{K} = \mathrm{O}_2(2) := \mathrm{SO}_3(2) \ltimes \mathbb{Z}_2(\kappa_1)$.

Remark 5.1 In the case of the gravitational N-body problem (5.4) becomes

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = -\frac{aq_1}{\|q_1\|^3} \tag{5.5}$$

where

$$a = \frac{1}{2}\sum_{j=1}^{\lfloor (N-1)/2\rfloor} \frac{1}{\sin(\pi j/N)} + \frac{\beta_{1,N}}{4}.$$

Hence (5.5) is the Kepler problem.

Let

$$b(r) = v'(2r)\frac{\beta_{1,N}}{r} + \sum_{j=1}^{\lfloor (N-1)/2\rfloor} v'(\alpha_{j,N}r)\frac{\alpha_{j,N}}{r}. \tag{5.6}$$

Relative equilibria with drift velocity $\xi \in \mathrm{so}(2)$ satisfy

$$p_1 = \xi q_1, \quad -b(\|q_1\|)q_1 = \xi p_1$$

and so $\xi^2 q_1 = -b(\|q_1\|)q_1$. Therefore $(\omega^{\mathrm{rot}})^2 = b(\|q_1\|)$ where $\omega^{\mathrm{rot}} = \omega^{\mathrm{rot}}(\|q_1\|)$ is the rotation frequency of the relative equilibrium through q_1, and there is a one-parameter family of relative equilibria parametrized by its scale $c = \|q_1\|$. In particular if $\bar{x}$ lies on a relative equilibrium and has isotropy subgroup $\bar{K}$ and momentum $\bar{\mu} \neq 0$, then there is a family of relative equilibria $\Gamma x(\nu)$ near $\bar{x}$ with $x(0) = \bar{x}$, with isotropy $\bar{K}$ at $x(\nu) = (q(\nu), p(\nu))$, with rotation frequency $\omega^{\mathrm{rot}}(\nu)$ and with momentum

$$\mathbf{J}(x(\nu)) = N(b(\|q_1(\nu)\|))^{1/2}\|q_1(\nu)\|^2 e_3 = \bar{\mu} + \nu e_3 \neq 0. \tag{5.7}$$

To see that $\mathbf{J}(x(\nu)) \neq 0$ (which will be needed below) note first that $q_1(\nu) \neq 0$ since collisions where $q_i = q_j$ for some $i \neq j$ are excluded. Moreover, from (5.6) we see that $b(r) > 0$ for all r. This follows from the fact that $v'(r) > 0$ for all $r \in \mathbb{R}$ and that $\alpha_{j,N} > 0$, $\beta_{j,N} \geq 0$, $j = 1, \ldots, N-1$. This proves that $\mathbf{J}(x(\nu)) \neq 0$ along a Lagrangian relative equilibrium $\Gamma x(\nu)$.

In the following let $\bar{\omega}^{\mathrm{rot}}$ be the rotation frequency of the relative equilibrium through $\bar{x}$ and let $\bar{\xi} = \bar{\omega}^{\mathrm{rot}} e_3^{\xi} \in \mathrm{so}_3(2)$ be its drift velocity.

5.2.2 Relative Lyapunov centre bifurcation from Lagrangian relative equilibria

In this section we prove the existence of smoothly parametrized families of RPOs which bifurcate from Lagrangian relative equilibria by a relative Lyapunov centre bifurcation; in particular we study bifurcation to rotating choreographies as defined below. The RPOs we find have been proved to exist for $N = 3$ by Marchal [18]. Chenciner and Féjoz prove existence results for those RPOs in the gravitational N-body problem using a relative Moser-Weinstein theorem [7], see Remark 5.8 for more details.

Lemma 5.2 *Let $\bar{x} = (\bar{q}, \bar{p})$ lie on a Lagrangian relative equilibrium with drift velocity $\bar{\xi} = \bar{\omega}^{\mathrm{rot}} e_3^{\xi}$ and with isotropy subgroup $\Gamma_{\bar{x}} = \bar{K}$ as in* (5.3)*. Then*

a) $\mathrm{D}^2 V(\bar{q})$ *has positive eigenvalues*

$$\lambda_k^- = \frac{\beta_{k,N} v'(2\|\bar{q}_1\|)}{\|\bar{q}_1\|} + \sum_{m=1}^{\lfloor (N-1)/2 \rfloor} \frac{\alpha_{mk,N}^2 v'(\alpha_{m,N}\|\bar{q}_1\|)}{\alpha_{m,N}\|\bar{q}_1\|}, \quad k = 1, \ldots, \lfloor N/2 \rfloor \tag{5.8}$$

where $\lambda_1^- = b(\|\bar{q}_1\|)$, with b as in (5.6)*. The eigenspace to λ_k^- contains the space $\mathcal{Q}_{k,r}^-$ which is spanned by the real and imaginary part of $\mathcal{Q}_k^-$ where*

$$\mathcal{Q}_k^- = \mathrm{span}\{(q_1, \ldots, q_N), q_j = \mathrm{e}^{2\pi\mathrm{i}(1-j)k/N} e_3, j = 1, \ldots, N\}$$

is the isotypic component in $\mathcal{Q}^- = \mathrm{Fix}_{\mathcal{Q}^0}(-\kappa_3)$ corresponding to the irreducible representation where $\bar{\alpha}$ acts as $-\mathrm{e}^{2\pi\mathrm{i}k/N}$, $k = 1, \ldots, N-1$. The eigenvalue λ_k^- of $\mathrm{D}^2 V(\bar{q})$ gives rise to eigenvalues $\pm\mathrm{i}\omega_k^-$ of $M|_{\mathcal{X}^-}$ where

$$M = \begin{pmatrix} -\bar{\xi} & \mathrm{id}_N \\ -\mathrm{D}^2 V(\bar{q}) & -\bar{\xi} \end{pmatrix}$$

is the linearization along the relative equilibrium and $\mathcal{X}^- = \mathrm{Fix}_{\mathcal{X}}(-\kappa_3)$. Here $\omega_{N-k}^- = \omega_k^- = (\lambda_k^-)^{1/2} > 0$ and $\mathcal{X}_k^-$ is M-invariant with eigenvalues $\pm\mathrm{i}\omega_k^-$, $k = 1, \ldots, N-1$. As before, $\mathcal{X}_k^-$, $k = 1, \ldots, N-1$, is the isotopic component of $\mathcal{X}^-$ where $\bar{\alpha}$ acts as $-\mathrm{e}^{2\pi\mathrm{i}k/N}$. Moreover, $\omega_1^- = |\bar{\omega}^{\mathrm{rot}}|$ and $(e_1^{\xi} + \mathrm{i}e_2^{\xi})\bar{x} \in \mathcal{X}_1^-$.

b) Let $\mathcal{X}_k^+$ be the isotypic component in $\mathcal{X}^+ = \mathrm{Fix}(\kappa_3)$ corresponding to the irreducible representation where $\bar{\alpha}$ acts as $\mathrm{e}^{2\pi \mathrm{i}k/N}$, $k = 0, 1, 2, \ldots, N-1$. Then $\mathcal{X}_k^+$ is M-invariant.

Proof.

a) Part a) was shown by Chenciner and Féjoz [7] for the gravitational N body problem. The $(3,3)$-matrices $(\mathrm{D}^2 V(\bar{q}))_{ij} := \mathrm{D}^2_{q_i q_j} V(\bar{q})$ for $i, j = 1, \ldots, N-1$, are given by

$$\mathrm{D}^2_{q_i q_j} V(\bar{q}) = \begin{cases} i = j: & \sum_{k \neq j} \frac{v'(\bar{r}_{kj})}{\bar{r}_{kj}} + \sum_{k \neq j} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v'(r)}{r}\right)|_{r=\bar{r}_{kj}} (\bar{q}_j - \bar{q}_k)(\bar{q}_j - \bar{q}_k)^T / \bar{r}_{kj} \\ i \neq j: & -\frac{v'(\bar{r}_{ij})}{\bar{r}_{ij}} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v'(r)}{r}\right)|_{r=\bar{r}_{ij}} (\bar{q}_j - \bar{q}_i)(\bar{q}_j - \bar{q}_i)^T / \bar{r}_{ij} \end{cases} \tag{5.9}$$

where $\bar{r}_{ij} = \|\bar{q}_i - \bar{q}_j\|$. Since the the relative equilibrium through $\bar{x} = (\bar{q}, \bar{p})$ lies in $\mathrm{Fix}(\kappa_3)$, we have by (5.9)

$$\left(\mathrm{D}^2 V(\bar{q})|_{\mathcal{Q}^-}\right)_{\ell,j} = \begin{cases} \ell = j: & \sum_{k \neq j} \frac{v'(\alpha_{k-j,N}\|\bar{q}_1\|)}{\alpha_{k-j,N}\|\bar{q}_1\|} \\ \ell \neq j: & -\frac{v'(\alpha_{\ell-j,N}\|\bar{q}_1\|)}{\alpha_{\ell-j,N}\|\bar{q}_1\|} \end{cases}, \quad \bar{\xi}|_{\mathcal{X}^-} = 0.$$

Due to the $\bar{K}$-equivariance of $\mathrm{D}^2 V(\bar{q})$ the spaces $\mathcal{Q}_k^-$ are invariant under $\mathrm{D}^2 V(\bar{q})$. Hence, $\mathrm{D}^2 V(\bar{q})$ has eigenvalues

$$\lambda_k^- = \sum_{m=2}^{N} (1 - \mathrm{e}^{\mathrm{i}2\pi(m-1)k/N}) \frac{v'(\alpha_{m-1,N}\|\bar{q}_1\|)}{\alpha_{m-1,N}\|\bar{q}_1\|} = \frac{\beta_{k,N} v'(2\|\bar{q}_1\|)}{\|\bar{q}_1\|} + \sum_{m=1}^{\lfloor (N-1)/2 \rfloor} \frac{\alpha^2_{mk,N} v'(\alpha_{m,N}\|\bar{q}_1\|)}{\alpha_{m,N}\|\bar{q}_1\|}$$

with eigenspace $\mathcal{Q}_{k,r}^-$ which shows (5.8). the symmetry mode $(e_1^\xi + \mathrm{i}e_2^\xi)\bar{x}$ is an eigenvector of M to $\mathrm{i}\bar{\omega}^{\mathrm{rot}}$ which lies in $\mathcal{X}_1^-$ since

$$\bar{\alpha}(e_1^\xi + \mathrm{i}e_2^\xi)\bar{x} = \mathrm{Ad}_{\bar{\alpha}}(e_1^\xi + \mathrm{i}e_2^\xi)\bar{x} = (-\mathrm{Ad}_{R_3(2\pi/N)}(e_1^\xi + \mathrm{i}e_2^\xi))\bar{x} = -\mathrm{e}^{2\pi\mathrm{i}/N}(e_1^\xi + \mathrm{i}e_2^\xi)\bar{x}. \tag{5.10}$$

Hence $\omega_1^- = |\bar{\omega}^{\mathrm{rot}}| = \sqrt{b(\|\bar{q}_1\|)}$ by (5.6).

b) follows from equivariant bifurcation theory [11].

■

Note that $\mathcal{X}_0^- = \mathcal{Q}_0^- = \{0\}$ due to the centre of mass fixed at 0. Note further that by part a) of this lemma the space $\mathcal{X}_{k,r}^-$, spanned by the real and imaginary part of $\mathcal{X}_k^-$, is contained in the real eigenspace corresponding to $\pm\mathrm{i}\omega_k^-$ and all eigenvalues $\pm\mathrm{i}\omega_k^-$ with $k \neq N/2$ of M have an algebraic multiplicity of at least 2. In the following we denote by $\pm\mathrm{i}\omega_\ell^-(\nu)$ the eigenvalues of $M(\nu)|_{\mathcal{X}_\ell^-}$, where $M(\nu)$ is the linearization at the relative equilibrium through $x(\nu)$, c.f., (5.7).

Remark 5.3 Let $\mathcal{X}_{k,r}^+$ be the space spanned by the real and imaginary parts of $\mathcal{X}_k^+$, $k = 1, 2, \ldots N$. In the gravitational N-body problem the eigenvalues λ_j^- are decreasing in j, see [7]. In this case the linearization $M_{0,r}^+ := M|_{\mathcal{X}_{0,r}^+}$ at the relative equilibrium of (5.5), where $\mathcal{X}_{0,r}^+ = \mathrm{Fix}(\bar{K})$, has an eigenvalue 0 of algebraic multiplicity two and geometric multiplicity one with eigenvector $\bar{\xi}\bar{x}$ and left eigenvector $\mathrm{D}\mathbf{J}(\bar{x})$ and a pair of eigenvalues $\pm\mathrm{i}\bar{\omega}^{\mathrm{rot}}$. As shown in [21] for the gravitational N body problem with $N < 6$ the matrices $M|_{\mathcal{X}_{k,r}^+}$, $k = 1, \ldots, N-1$, do not have imaginary eigenvalues, but there are numbers $N \geq 6$, $k \in \{1, \ldots, N-1\}$ and $\omega_k^+ > 0$ such that $\pm\mathrm{i}\omega_k^+$ is an eigenvalue of $M|_{\mathcal{X}^+}$.

Definition 5.4 *A periodic orbit of (5.2) is a* choreography *if all the bodies follow the same path in* $\mathbb{R}^3$*, separated only by a (possibly vanishing) phase shift. This is equivalent to requiring that* $\zeta \in L$ *where* L *is the spatio-temporal symmetry group of the periodic orbit. Similarly an RPO of (5.2) is a rotating choreography if it is a choreography in its co-rotating frame.*

Note that we allow a vanishing phase shift in our definition of a choreography. Thus our definition is different from the one in [7] who impose a non-zero phase shift (and talk of partial choreographies if $\zeta \in L$ such that ζ^m for some $m \neq 1$ dividing N corresponds to a zero phase shift).

Theorem 5.5 *Let* $\Gamma\bar{x}$ *be a Lagrangian relative equilibrium of the* N*-body problem,* $N \geq 2$*, with isotropy subgroup* $\Gamma_{\bar{x}} = \bar{K}$ *as in* (5.3) *and with drift velocity* $\bar{\xi} = \bar{\omega}^{\mathrm{rot}} e_3^{\xi}$ *and momentum* $\bar{\mu}||e_3$*. Let* $j \in \{1, \ldots, \lfloor N/2 \rfloor\}$*, let* $m = \gcd(N, j)$*, and set* $N' = N/m$*,* $j' = j/m$*. Then the following hold true:*

a) Let N' *be odd and* $j \neq 1$*; then, under the non-resonance conditions* (5.12) *below, there are two smooth two parameter families of non-planar rotating choreographies* $\mathcal{P}_j(\epsilon, \nu)$*,* $\mathcal{P}_{N-j}(\epsilon, \nu)$ *through* $x_j(\epsilon, \nu)$*,* $x_{N-j}(\epsilon, \nu)$ *with* $x_j(0, \nu) = x_{N-j}(0, \nu) = x(\nu)$*, with*

$$\partial_\epsilon x_j(\epsilon, \nu)|_{\epsilon=0} \in \mathcal{X}_{j,r}^-, \quad \partial_\epsilon x_{N-j}(\epsilon, \nu)|_{\epsilon=0} \in \mathcal{X}_{j,r}^-,$$

and with isotropy subgroup

$$K_j = K_{N-j} = \mathbb{Z}_m(R_3(2\pi/m)\zeta^{N'}).$$

The families of RPOs $\mathcal{P}_j(\epsilon, \nu)$*,* $\mathcal{P}_{N-j}(\epsilon, \nu)$ *have drift symmetry* $\alpha_j = \bar{\alpha}^{k_j}$ *and* $\alpha_{N-j} = \bar{\alpha}^{-k_j}$ *and spatio-temporal symmetry group* $L = \bar{K}$ *in their co-rotating frame* $\xi_j(\epsilon, \nu)$ *and* $\xi_{N-j}(\epsilon, \nu)$ *where* $\xi_j(0, 0) = \xi_{N-j}(0, 0) = \bar{\xi}$ *and* $L/K_j = \mathbb{Z}_{2N'}(\alpha_j)$*; here* $k_j \in \mathbb{Z}$ *is such that*

$$(2j' + N')k_j = -1 \mod 2N'; \tag{5.11}$$

their relative periods $\tau_j(\epsilon, \nu)$*,* $\tau_{N-j}(\epsilon, \nu)$ *satisfy*

$$\tau_j(0, \nu) = \tau_{N-j}(0, \nu) = \pi/(N'\omega_j^-(\nu)).$$

The non-resonance conditions are

$$\omega_{m\ell'}^-/\omega_j^- \neq \pm k_j(N' + 2\ell') \mod 2N', \ \ell' = 0, \ldots, \lfloor N'/2 \rfloor, \quad \ell' \neq j', \tag{5.12a}$$

with the notation from Lemma 5.2, and

$$\omega_{m\ell'}^+/\omega_j^- \neq \pm 2k_j\ell' \mod 2N'. \tag{5.12b}$$

The last condition has to hold for all eigenvalues $\mathrm{i}\omega_\ell^+$*,* $\omega_\ell^+ \in \mathbb{R}$ *of* $M|_{\mathcal{X}_\ell^+}$ *where* $\ell = m\ell'$ *and* $\ell' = 1, \ldots, \lfloor N'/2 \rfloor$*.*

b) Let $N' \neq 2$ *be even and let* $j \neq 1$*; then, under the non-resonance conditions* (5.14) *below, there are two smooth two parameter families of non-planar rotating choreographies* $\mathcal{P}_j(\epsilon, \nu)$*,* $\mathcal{P}_{N-j}(\epsilon, \nu)$ *through* $x_j(\epsilon, \nu)$*,* $x_{N-j}(\epsilon, \nu)$ *with* $x_j(0, \nu) = x_{N-j}(0, \nu) = x(\nu)$*, with*

$$\partial_\epsilon x_j(\epsilon, \nu)|_{\epsilon=0} \in \mathcal{X}_{j,r}^-, \quad \partial_\epsilon x_{N-j}(\epsilon, \nu)|_{\epsilon=0} \in \mathcal{X}_{j,r}^-,$$

and with isotropy

$$K_j = K_{N-j} = \mathbb{Z}_{2m}(R_3(\pi/m)\zeta^{N'/2}\kappa_3).$$

The families of RPOs $\mathcal{P}_j(\epsilon,\nu)$ and $\mathcal{P}_{N-j}(\epsilon,\nu)$ have drift symmetry $\alpha_j = (\bar{\alpha}\kappa_3)^{k_j}$ and $\alpha_{N-j} = (\bar{\alpha}\kappa_3)^{-k_j}$ and spatio-temporal symmetry group $L = \bar{K}$ in their co-rotating frame $\xi_j(\epsilon,\nu)$ and $\xi_{N-j}(\epsilon,\nu)$ where $\xi_j(0,0) = \xi_{N-j}(0,0) = \bar{\xi}$; here $L/K_j = \mathbb{Z}_{N'}(\alpha_j)$ and $k_j \in \mathbb{Z}$ is such that

$$j'k_j = -1 \mod N'; \tag{5.13}$$

their relative periods $\tau_j(\epsilon,\nu)$ and $\tau_{N-j}(\epsilon,\nu)$ satisfy

$$\tau_j(0,\nu) = \tau_{N-j}(0,\nu) = 2\pi/(N'\omega_j^-(\nu)).$$

The non-resonance conditions are

$$\omega_{m\ell'}^-/\omega_j^- \neq \pm k_j\ell' \mod N', \tag{5.14a}$$

for all odd numbers $\ell' \neq j'$ between 1 and $\lfloor N'/2 \rfloor$, and

$$\omega_{m\ell'}^+/\omega_j^- \neq \pm k_j\ell' \mod N'. \tag{5.14b}$$

The last condition has to hold for all eigenvalues $\mathrm{i}\omega_\ell^+$, $\omega_\ell^+ \in \mathbb{R}$ of $M|_{\mathcal{X}_\ell^+}$ where $\ell = m\ell'$ and ℓ' is any even number between 0 and $\lfloor N'/2 \rfloor$.

c) *If $j = 1$ then there is exactly one family $\mathcal{P}_1(\epsilon,\nu)$ with properties as specified in a) and b) und under the conditions given there.*

d) *If $j = N/2$ so that $N' = 2$ then b) is still true, but there is only one family $\mathcal{P}_{N/2}(\epsilon,\nu)$ of RPOs.*

Proof.

a) Let N' be odd and $j \neq 1$; then $\pm\mathrm{i}\omega_j^-$ is an eigenvalue of $M|_{\mathcal{X}_{j,r}^-}$ of multiplicity 2 by Lemma 5.2. Here, as before, $\mathcal{X}_{j,r}^- \simeq \mathbb{R}^4$ is spanned by the real and imaginary part of $\mathcal{X}_j^-$ or equivalently $\mathcal{X}_{N-j}^-$. Note that the cyclic permutation ζ has order N' on $\mathcal{X}_{j,r}^-$ and that the action of $\bar{\alpha}$ on $\mathcal{X}_{j,r}^-$ has order $2N'$. Any non-zero element of $\mathcal{X}_{j,r}^-$ has isotropy $K_j = \mathbb{Z}_m(R_3(2\pi/m)\zeta^{N'})$.

So a drift symmetry of any family of bifurcating periodic orbits of the $\dot{w}$ system of (2.7) with tangent in $\mathcal{X}_{j,r}^- \cap \mathcal{N}_1$ at bifurcation corresponds to a phase shift $\theta = \frac{1}{2N'}$, see (3.2) and Theorem 3.2. Let $\bar{w}_j \in \mathcal{X}_j^-$ be an eigenvector of M (or equivalently of $A = \mathrm{D}f_{\mathcal{N}_1}(0)$) to $\mathrm{i}\omega_j^-$. To compute the drift symmetry $\alpha_j = \bar{\alpha}^k$ of a family of RPOs $\mathcal{P}_j$ through $x_j(\epsilon,\nu)$ with $x_j(0,0) = \bar{x}$ and $\partial_\epsilon x_j(0) = \mathrm{Re}\,\bar{w}_j$ we need to find $k = k_j$ such that

$$\bar{w}_j = \bar{\alpha}^k \mathrm{e}^{\frac{\pi\mathrm{i}}{N'}}\bar{w}_j.$$

Since $\bar{\alpha}$ acts as $-\mathrm{e}^{2\pi\mathrm{i}j/N} = -\mathrm{e}^{2\pi\mathrm{i}j'/N'}$ on $\mathcal{X}_j^-$ this becomes

$$\bar{w}_j = (-\mathrm{e}^{2\pi\mathrm{i}j'/N'})^k \mathrm{e}^{\frac{\pi\mathrm{i}}{N'}}\bar{w}_j = \mathrm{e}^{\mathrm{i}2\pi(k/2 + kj'/N' + 1/(2N'))}\bar{w}_j$$

and hence

$$\frac{k}{2} + \frac{kj'}{N'} + \frac{1}{2N'} = 0 \mod 1 \quad \Longleftrightarrow \quad k(N'+2j') = -1 \mod 2N'.$$

This shows (5.11) for the family of RPOs $\mathcal{P}_j$. For the family $\mathcal{P}_{N-j}$ we have $\partial_\epsilon x_{N-j}(0) = \mathrm{Re}\,\bar{w}_{N-j}$ where $\bar{w}_{N-j} \in \mathcal{X}_{N-j}^-$ is an eigenvector of M to $\mathrm{i}\omega_j = \mathrm{i}\omega_{N-j}$. Replacing j by

$N-j$ in the above computation shows that this family of RPOs has the drift symmetry $\alpha_{N-j} = \bar{\alpha}^{-k_j} = (\alpha_j)^{-1}$. Note that the $1:1$ resonance on $\mathcal{X}^-_{j,r}$ can only violate condition (3.6) of Theorem 3.2 if the action of $\bar{\alpha}$ on $\mathcal{X}^-_j$ and $\mathcal{X}^-_{N-j}$ are identical. But this cannot happen when N' is odd.

Next we check that conditions (3.5) and (3.6) are satisfied by investigating resonances on $\mathcal{X}^-_\ell$, $\ell \neq j, N-j$ and on $\mathcal{X}^+$. Note that $\mathcal{X}^-_\ell$ is fixed by K_j iff $\ell = m\ell'$ for some $\ell' \in \{1,\ldots,N'\}$. Assume that $\omega^-_\ell \in \omega^-_j\mathbb{Z}$ where $\ell = m\ell'$. For the family of RPOs $\mathcal{P}_j$ we then need to ensure that

$$w = \alpha_j \exp\left(\frac{\mathrm{i}\pi\omega^-_\ell}{N'\omega^-_j}\right) w \quad \Longrightarrow \quad w = 0$$

for all $w \in \mathcal{X}^-_\ell \cap \mathcal{N}_1$. Since α_j acts as $(-\mathrm{e}^{2\pi\mathrm{i}\ell/N})^{k_j}$ on $\mathcal{X}^-_\ell$ this amounts to the condition

$$1 \neq \mathrm{e}^{k_j(\pi\mathrm{i}+2\pi\mathrm{i}\ell'/N')+\mathrm{i}\pi\frac{\omega^-_\ell}{\omega^-_j N'}}$$

or $k_j(1/2+\ell'/N') + \frac{\omega^-_\ell}{\omega^-_j 2N'} = 0 \mod 1$. The corresponding condition for the family $\mathcal{P}_{N-j}$ is obtained by exchanging k_j with $-k_j$. Both non-resonance conditions together are equivalent to (5.12a).

Note that $\mathcal{X}^+_\ell$ is fixed by K_j iff $\ell = m\ell'$ for some $\ell' \in \{1,\ldots,N'\}$. Assume that $\omega^+_\ell \in \omega^-_j\mathbb{Z}$ where $\pm\mathrm{i}\omega^+_\ell$ is an eigenvalue of $M|_{\mathcal{X}^+_\ell}$ with ℓ specified above. For the family of RPOs $\mathcal{P}_j$ we then need to ensure that

$$w = \alpha_j \exp\left(\frac{\mathrm{i}\pi\omega^+_\ell}{N'\omega^-_j}\right) w \quad \Longrightarrow \quad w = 0$$

holds for all $w \in \mathcal{X}^+_\ell \cap \mathcal{N}_1$. Since $\alpha_j = \bar{\alpha}^{k_j}$ acts as $\mathrm{e}^{2\pi\mathrm{i}k_j\ell'/N'}$ on $\mathcal{X}^+_\ell$ this and the corresponding condition for the family $\mathcal{P}_{N-j}$ amount to the non-resonance condition (5.12b).

Condition (3.8) of Theorem 3.2 holds with $r=1$ since the momentum value $\bar{\mu} = \mathbf{J}(x(\nu))$ of the Lagrangian relative equilibrium, given by (5.7), is non-zero (see Section 5.2.1) and so $\dim \mathbf{g}_{\bar{\mu}}(L) = 1$. Hence Theorem 3.2 implies the existence of two families of RPOs $\mathcal{P}_j(\epsilon,\nu)$ and $\mathcal{P}_{N-j}(\epsilon,\nu)$ with the symmetries specified above.

We now show that $\mathcal{P}_j$, $\mathcal{P}_{N-j}$ are families of rotating choreographies. With α_j also

$$(\alpha_j)^2 = R_3\left(\frac{4\pi k_j}{N}\right)\zeta^{2k_j}$$

is a spatio-temporal symmetry of $\mathcal{P}_j(\epsilon,\nu)$ in its co-rotating frame $\xi_j(\epsilon,\nu)$. So in a suitable co-rotating frame the element ζ^{2k_j} is a spatio-temporal symmetry of a periodic orbit through $x_j(\epsilon,\nu)$ within $\mathrm{Fix}_{\mathcal{X}}(K_j)$ (note that $\mathrm{Fix}_{\mathcal{X}}(K_j)$ is invariant under $R_3(\phi)$, $\phi \in [0,2\pi]$). Since $\gcd(2k_j,N') = \gcd(k_j,N') = 1$ due to (5.11) we see that the cyclic permutation ζ is a spatio-temporal symmetry of $\mathcal{P}_j(\epsilon,\nu)$ in a suitable rotating frame too and so $\mathcal{P}_j(\epsilon,\nu)$ is a family of rotating choreographies. The same argument applies to the family $\mathcal{P}_{N-j}(\epsilon,\nu)$.

b) Let $N' \neq 2$ be even and let $j \neq 1$; then any non-zero element of $\mathcal{X}^-_{j,r}$ has isotropy subgroup $K_j = \mathbb{Z}_{2m}(R_3(\frac{\pi}{m})\zeta^{N'/2}\kappa_3)$, and the action of $\bar{\alpha}$ on $\mathcal{X}^-_{j,r}$ has order N' if $N'/2$ even and $N'/2$ if it is odd. But $\kappa_3\bar{\alpha}$ has order N' on $\mathcal{X}^-_{j,r}$.

So a drift symmetry of any family of bifurcating periodic orbits of the $\dot{w}$ system of (2.7) with tangent in $\mathcal{X}_{j,r}^- \cap \mathcal{N}_1$ at bifurcation corresponds to a phase shift $\theta = \frac{1}{N'}$, see (3.2) and Theorem 3.2. Let $\bar{w}_j \in \mathcal{X}_j^-$ be an eigenvector of M to $\mathrm{i}\omega_j^-$. For the drift symmetry α_j of a family of RPOs $\mathcal{P}_j$ through $x_j(\epsilon,\nu)$ with $\partial_\epsilon x_j(0) = \mathrm{Re}\,\bar{w}_j$ we make the ansatz $\alpha_j = (\kappa_3\bar{\alpha})^k$ for some $k \in \mathbb{Z}$. (Note that $\bar{\alpha}^{N'/2} \in K_j$ if $N'/2$ is odd, so that we can not construct α_j as a power of $\bar{\alpha}$ as in a) in this case). We then need to find $k = k_j$ such that

$$\bar{w}_j = (\kappa_3\bar{\alpha})^k \mathrm{e}^{\frac{2\pi\mathrm{i}}{N'}} \bar{w}_j.$$

Since $\kappa_3\bar{\alpha}$ acts as $\mathrm{e}^{2\pi\mathrm{i}j/N} = \mathrm{e}^{2\pi\mathrm{i}j'/N'}$ on $\mathcal{X}_j^-$ this becomes

$$\bar{w}_j = \mathrm{e}^{2\pi\mathrm{i}kj'/N'} \mathrm{e}^{\frac{2\pi\mathrm{i}}{N'}} \bar{w}_j = \mathrm{e}^{\mathrm{i}2\pi(kj'+1)/N'} \bar{w}_j$$

and hence shows (5.13) for the family of RPOs $\mathcal{P}_j$. For the family $\mathcal{P}_{N-j}$ we have $\partial_\epsilon x_{N-j}(0) = \mathrm{Re}\,\bar{w}_{N-j}$ where $\bar{w}_{N-j} \in \mathcal{X}_{N-j}^-$ is an eigenvector of M to $\mathrm{i}\omega_{N-j}^- = \mathrm{i}\omega_j^-$. Replacing j by $N-j$ in the above computation shows that this family of RPOs has the drift symmetry $\alpha_{N-j} = (\alpha_j)^{-1}$. Note that the $1:1$ resonance on $\mathcal{X}_{j,r}^-$ can only violate condition (3.6) of Theorem 3.2 if the action of $\kappa_3\bar{\alpha} = \zeta R_3(2\pi/N)$ on $\mathcal{X}_j^-$ and $\mathcal{X}_{N-j}^-$ are identical. But this cannot happen when $N' \neq 2$.

Next we check that conditions (3.5) and (3.6) are satisfied by investigating resonances on $\mathcal{X}_\ell^-$, $\ell \neq j, N-j$ and on $\mathcal{X}^+$. Note that $\mathcal{X}_\ell^-$ is fixed by K_j iff $\ell = m\ell'$ for some odd $\ell' \in \{1,\ldots,N'\}$. Assume that $\omega_\ell^- \in \omega_j^-\mathbb{Z}$ where $\ell = m\ell'$. For the family of RPOs $\mathcal{P}_j$ we then need to ensure that

$$w = \alpha_j \exp\left(\frac{2\mathrm{i}\pi\omega_\ell^-}{N'\omega_j^-}\right) w \quad \Longrightarrow \quad w = 0$$

for all $w \in \mathcal{X}_\ell^- \cap \mathcal{N}_1$. Since α_j acts as $\mathrm{e}^{k_j 2\pi\mathrm{i}\ell/N}$ on $\mathcal{X}_\ell^-$ this amounts to the condition

$$1 \neq \mathrm{e}^{2\pi\mathrm{i}\ell' k_j/N' + \frac{2\mathrm{i}\pi\omega_\ell^-}{\omega_j^- N'}}$$

or $\ell' k_j/N' + \frac{\omega_\ell^-}{\omega_j^- N'} = 0 \mod 1$, which, together with the corresponding condition for the family $\mathcal{P}_{N-j}$, is equivalent to (5.14a).

Now assume that $\omega_\ell^+ \in \omega_j^-\mathbb{Z}$ where $\pm\mathrm{i}\omega_\ell^+$, $\omega_\ell^+ \in \mathbb{R}$, is an eigenvalue of $M|_{\mathcal{X}_\ell^+}$. Here $\ell = m\ell'$ for some even $\ell' \in \{1,\ldots,N'\}$ so that $\mathcal{X}_\ell^+$ is fixed by K_j. For the family of RPOs $\mathcal{P}_j$ we need to ensure that

$$w = \alpha_j \exp\left(\frac{2\mathrm{i}\pi\omega_\ell^+}{\omega_j^- N'}\right) w \quad \Longrightarrow \quad w = 0$$

holds for all $w \in \mathcal{X}_\ell^+ \cap \mathcal{N}_1$. Since $\alpha_j = (\kappa_3\bar{\alpha})^{k_j}$ acts as $\mathrm{e}^{2\pi\mathrm{i}k_j\ell/N}$ on $\mathcal{X}_\ell^+$ this, together with the corresponding condition for the family of RPOs $\mathcal{P}_{N-j}$ amounts to the non-resonance condition (5.14b).

Finally $\mathcal{P}_j$ is a family of rotating choreographies since $\alpha_j = R_3(2k_j\pi/N)\zeta^{k_j}$ is its drift symmetry in its co-rotating frame. Since due to (5.13) we have $\gcd(k_j, N') = 1$ the cyclic permutation ζ is a spatio-temporal symmetry of a periodic orbit through $x_j(\epsilon,\nu)$ on $\mathrm{Fix}(K_j)$ in a suitable corotating frame and so $\mathcal{P}_j(\epsilon,\nu)$ is a family of rotating choreographies. The same argument applies to $\mathcal{P}_{N-j}$.

c) When $j = 1$ then, since $\mathcal{X}_1^- \subseteq \mathcal{T} = \mathbf{g}\bar{x}$ by Lemma 5.2, there is only one family of RPOs in this case.

d) When $N' = 2$ so that $j = N/2$ then $\mathcal{X}_{N/2}^- \simeq \mathbb{R}^2$ and there is only one family of RPOs $\mathcal{P}_{N/2}$ under the non-resonance conditions specified in part b).

■

Remark 5.6 In the gravitational N body problem there is also a smooth two parameter planar family of periodic orbits $\mathcal{P}_{\rm pl}(\epsilon, \nu)$ through $x_{\rm pl}(\epsilon, \nu)$ with isotropy $K_{\rm pl} = \bar{K}$, with momentum $\bar{\mu} + \nu e_3^{\mu}$, and period $T_{\rm pl}(\epsilon, \nu)$ such that $T_{\rm pl}(0, \nu) = 2\pi/\omega^{\rm rot}(\nu)$ and $x_{\rm pl}(0, 0) = \bar{x}$. It is called homographic family, see e.g. [6]. Its existence follows from Remark 5.3 where we saw that the Lagrangian relative equilibrium $\Gamma x(\nu)$ has planar normal modes which give a two parameter familiy of periodic orbits of (5.5) (not RPOs, since (5.5) is Kepler's problem), and hence a two-parameter family of periodic orbits $\mathcal{P}_{\rm pl}(\epsilon, \nu)$ of (5.2) with isotropy subgroup $\bar{K}$.

Remark 5.7 Chenciner and Féjoz proved the existence of a smoothly parametrized non-planar family of rotating choreographies for the gravitational three-body problem using a higher order normal form [6]. This is a special case of the above theorem. To see this, notice that in this case the vertical space $\mathcal{X}^- = \mathrm{Fix}(-\kappa_3)$ is four-dimensional, and the linearization at the relative equilibrium when restricted to $\mathcal{X}^-$ has double eigenvalues $\pm \mathrm{i}\bar{\omega}^{\rm rot}$ with one eigenvector given by $(e_1^{\xi} \pm \mathrm{i} e_2^{\xi})\bar{x}$. So the Lagrangian relative equilibrium has exactly one nonlinear relative normal mode in $\mathcal{N}_1 \cap \mathcal{X}^-$ with relative normal frequency $\bar{\omega} = \bar{\omega}^{\rm rot}$. Since $\bar{\alpha}$ acts on $\mathcal{X}_{N-1}^-$ as $-\mathrm{e}^{-2\pi i/3} = e^{\pi i/3}$, the bifurcating relative normal modes have trivial isotropy and spatio-temporal symmetry $\mathbb{Z}_6(\bar{\alpha})$ in the rotating frame in which the Lagrangian relative equilibrium is stationary (see Theorem 5.5 c) and a)). In this case $m = 1$, $j = k = 1$, and the non-resonance conditions (5.12a) are void. By Remark 5.6, the Lagrangian relative equilibrium has eigenvalues $\pm \mathrm{i}\omega_0^+$ with $\omega_0^+ = \bar{\omega}^{\rm rot} = \omega_1^-$ which satisfies the required non-resonance condition (5.12b) (for $m = 1$, $k = 1$, $\ell' = 0$), and, by Remark 5.3, the linearization M at the relative equilibrium does not have any other purely imaginary eigenvalues with eigenvectors in $\mathcal{X}^+$. So the non-resonance condition (5.12b) is also satisfied and Theorem 3.2 can be applied. The bifurcating relative normal mode is the type I family of rotating Eights, which we study numerically in the next section (Section 5.3).

Remark 5.8 In [7] Chenciner and Féjoz show for the gravitational N body problem that $\mathrm{D}^2 V(\bar{q})$ when restricted to the space of vertical directions $\mathcal{Q}^-$ is positive definite (see also (5.8)) and invoke the Moser-Weinstein theorem to obtain the existence of two families of RPOs for each double eigenvalue λ_k of $\mathrm{D}^2 V(\bar{q})|_{\mathcal{Q}^-}$ provided that there are no resonances with frequencies in $\mathcal{X}^+$. They prove that there are no resonances on the whole of $\mathcal{X}$ for $N \leq 6$ building on results of [21], c.f. Remark 5.3. Our contribution here is to notice that in the cases described in the proposition we can, by exploiting spatio-temporal symmetries of the relative normal modes, obtain smoothly parametrized families of RPOs under suitable non-resonance conditions. The non-resonance conditions which pose on $\mathcal{X}^+$ are weaker than those non-resonance conditions in [7].

Remark 5.9 We see from Lemma 5.2 a) and in particular from (5.10) that in the case $j = 1$ of Theorem 5.5 the stronger non-resonance condition (4.23) is violated if we decide to ignore the spatio-temporal symmetries of the bifurcating RPOs, i.e., set $\alpha_1 = \mathrm{id}$, see also Remark 3.8.

5.3 Computation of relative Lyapunov centre bifurcation along the type I rotating eights

We now restrict attention to the gravitational three body problem. It is well-known that the non-planar family of RPOs bifurcating from the Lagrangian relative equilibrium connects to the famous Figure Eight solution [4, 18]. The Figure Eight of Chenciner and Montgomery [4] is a choreography of the 3-body system (5.2) with $V(r) = -1/r$. Assume that the Eight lies in the plane perpendicular to e_3 aligned along the e_2 axis with both e_2 axis and e_1 axis as symmetry axes. The purely spatial symmetry group of the Figure Eight choreography is the group $K = \mathbb{Z}_2(\kappa_3)$ and the drift symmetry of the Eight is $\alpha := \kappa_1(231)$ and has order $\ell = 6$. When angular momentum is switched on, three families of rotating choreographies bifurcate from the Figure Eight, rotating about the e_i-axis, $i = 1, 2, 3$, discovered by Marchal (type $i = 1$, [18]), Chenciner et al (type $i = 2$, [5]) and Henon (type $i = 3$, [13]). In [35] we showed numerically that the type 2 family of rotating Figures of Eight bifurcates to the type 1 family which ends at the Lagrangian relative equilibrium.

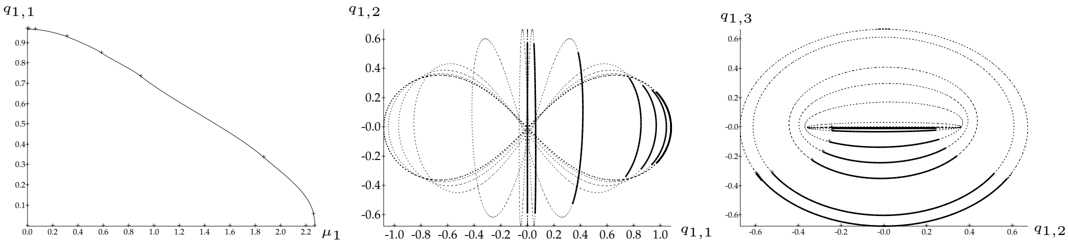

Figure 2: Relative Lyapunov centre bifurcation of the type I rotating Eights to the Lagrangian relative equilibrium; for explanations see text.

In this section we show how to apply our numerical techniques from Section 4 to detect the relative Lyapunov centre bifurcation during the numerical continuation of the type 1 family of rotating Eights $\mathcal{P}^I(\nu)$ in its angular momentum $\mu_1 = \nu$ starting at the Figure Eight at $\nu = 0$ and fixing its energy E to the energy of the Figure Eight. As shown in [5], see [35] for our notation, the drift symmetry α_I of the family of RPOs in their co-rotating frame $\omega^{\text{rot}}(\nu)$ satisfies $\alpha_I = \kappa_1(231)$. In its co-rotating frame the type I family of rotating Eights lifts its centre and sinks its sides as angular momentum μ_1 is increased and finally ends on the Lagrangian relative equilibrium as a doubly traversed circle. Let $\bar{K} = \mathbb{Z}_6(\bar{\alpha})$ be the isotropy subgroup of the Lagrangian relative equilibrium rotating around the e_1-axis. Note that it is customary to choose coordinates such that the original Figure Eight lies in the horizontal plane, but then the type I family of rotating Eights connects to a Lagrangian relative equilibrium which lies in the (e_2, e_3)-plane.

By Theorem 5.5 c), with $N = 3$, $j = k_j = 1$, the drift symmetry of the family of RPOs bifurcating from the Lagrangian relative equilibrium is

$$\bar{\alpha} = R_1(2\pi/3)\kappa_1(231) = \alpha_I R_1(2\pi/3) = \alpha_I R_1(\bar{\omega}^{\text{rot}} - \bar{\omega}^{\text{rot}}_{\text{RPO}})2\pi/(\bar{\ell}\bar{\omega})) = \alpha_I R_1(\bar{\omega}^{\text{rot}} - \bar{\omega}^{\text{rot}}_{\text{RPO}})\pi/(3\bar{\omega})),$$

see Example 4.5. We checked numerically that at bifurcation $\nu = \bar{\nu}$

$$(\bar{\omega}^{\text{rot}} - \bar{\omega}^{\text{rot}}_{\text{RPO}})/\bar{\omega} = 2. \tag{5.15}$$

Here $\bar{\omega}^{\text{rot}}_{\text{RPO}} = \omega^{\text{rot}}_{\text{RPO}}(\bar{\nu}) < 0$ is the rotating frame of the RPOs at bifurcation. Because of (5.15) in the rotating frame $\bar{\omega}^{\text{rot}}_{\text{RPO}}$ of the RPOs at bifurcation the relative equilibrium is a doubly traversed circle. Since in this case $\bar{\omega} = \bar{\omega}^{\text{rot}} > 0$ we have $\bar{\omega}^{\text{rot}}_{\text{RPO}} = -\bar{\omega}^{\text{rot}}$ at bifurcation.

SYMPERCON [29] computes the angular momentum value of the relative Lyapunov centre bifurcation as $\mu_1 \approx 1.87$ (up to 3 digits of accuracy) and its rotation frequency as $\bar{\omega}^{\rm rot}_{\rm RPO} \approx -1.38$.

The first plot of Figure 2 shows the dependence of the $q_{1,1}$ component of the type I rotating choreography (where $q_{ij} = \langle q_i, e_j \rangle$) on the momentum component μ_1, the second plot shows the projection into the $(q_{1,1}, q_{1,2})$-plane of the position q_1 of the first particle of RPOs near the bifurcation in their respective co-rotating frame. Here the relative equilibrium appears as vertical line as it lies in $\mathrm{Fix}(\bar{\alpha}) \subseteq \mathrm{Fix}(\kappa_1)$ and hence has component $\bar{q}_{1,1} = 0$. The last plot shows computed RPOs near the relative Lyapunov centre bifurcation in the $(q_{1,2}, q_{1,3})$-plane. In this projection the bifurcation point becomes a doubly traversed circle. Note that all RPOs in that plot are doubly-traversed since they satisfy $x(t+T/2) = \kappa_1 x(t)$ in their co-rotating frame, where T is the period of the RPO in its co-rotating frame, and κ_1 acts trivially in the (e_2, e_3)-plane.

Remark 5.10 If we continue the Figure Eight in the second component $\nu = \mu_2$ of the momentum along the family of type II rotating choreographies $\mathcal{P}_{II}(\nu)$, then their drift symmetry in the co-rotating frame is $\alpha_{II} = R_2(\pi)(231)$, see [5], and the review in [35] for our notation). As found in [35], this family bifurcates to the type I rotating choreographies in a relative period halving bifurcation; the bifurcating family has positive rotation frequency and drift symmetry $\tilde{\alpha}_I = R_2(\pi/2)\kappa_2(312)$ of order $\tilde{\ell}_I = 12$ in its co-moving frame. The bifurcating family of RPOs ends at the Lagrangian relative equilibrium rotating around the e_2-axis at $\mu_2 \approx 1.87$. Its masses are aligned clockwise. Hence the isotropy subgroup $\bar{K} = \mathbb{Z}_6(\bar{\alpha})$ of the relative equilibrium is generated by $\bar{\alpha} = (231)R_2(-2\pi/3)\kappa_2$ or, equivalently, by $\bar{\alpha}_I = \bar{\alpha}^{-1} = (312)R_2(2\pi/3)\kappa_2$. Then

$$\bar{\alpha}_I \tilde{\alpha}_I^{-1} = R_2(2\pi/3)R_2(-\pi/2) = R_2(\pi/6),$$

so that $j = 1 \mod 12$ in (4.14) as $p = 1$ (in the notation of Example 4.6). Numerically we checked that $j = 1$. Since $\ell = \tilde{\ell}_I = 2\bar{\ell}$ we have $\bar{\tau} = 2\pi/(\ell\bar{\omega}_{\rm RPO}) = 2\pi/(\bar{\ell}\bar{\omega})$ and so $\bar{\omega}_{\rm RPO} = \bar{\omega}/2$. So the period $T = 12\bar{\tau}$ of the family of RPOs $\mathcal{P}_I$ at the Lagrangian relative equilibrium in the rotating frame $\bar{\omega}^{\rm rot}_{\rm RPO}$ of the RPO at bifurcation satisfies $T = 2\bar{T}$ where $\bar{T} = 2\pi/\bar{\omega}$ and $\bar{\omega}$ is the frequency of the relative normal mode of the relative equilibrium. Since $j = 1$ and $\bar{\omega} = \bar{\omega}^{\rm rot}$ we deduce from (4.14) that $\bar{\omega}^{\rm rot}_{\rm RPO} = \bar{\omega}^{\rm rot}/2$, and the Lagrangian relative equilibrium is a simply traversed circle in the rotating frame of the RPOs at bifurcation.

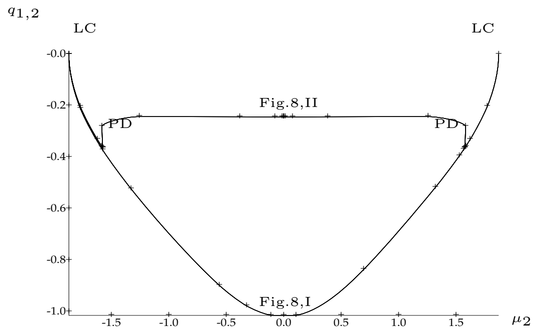

Figure 3: Bifurcation diagram of the type II rotating Eights from [35]; for explanations see text.

Figure 3 shows the complete bifurcation diagram of the type II rotating Figure Eight. The labels "Fig. 8 I" and "Fig. 8 II" denote the starting point of the type I rsp. II rotating Eights at the original Figure 8 solution, "PD" stands for relative period doubling bifurcation and "LC" for relative Lyapunov centre bifurcation.

So we see that the rotating frame of the RPOs at the relative Lyapunov centre bifurcation depends on how we approach the Lagrangian relative equilibrium in the bifurcation diagram, Figure 3.

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