Relative periodic points of symplectic maps: persistence and bifurcations

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Dedicated to the 60th birthday of Andre Vanderbauwhede

Abstract

In this paper we study symplectic maps with a continuous symmetry group arising by periodic forcing of symmetric Hamiltonian systems. By Noether's Theorem, for each continuous symmetry the symplectic map has a conserved momentum. We study the persistence of relative periodic points of the symplectic map when momentum is varied and also treat subharmonic persistence and relative subharmonic bifurcations of relative periodic points.

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1 Introduction

In [1] subharmonic bifurcations of periodic points of parameter dependent symplectic maps are studied, i.e., bifurcations to periodic points whose period is a multiple of the period of the original periodic point. In this paper we also consider symplectic maps, but we do not assume that they depend on an external parameter. Instead we consider symplectic maps with a continuous symmetry group and study bifurcations when the corresponding conserved quantities, the momenta of the continuous symmetry group, are varied. Such symplectic maps arise as time 1 maps of periodically forced Hamiltonian systems (with forcing period 1). A generalization of periodic points are relative periodic points (RPPs) which are periodic points after symmetry reduction. In [7] persistence results for relative periodic orbits (RPOs) of symmetric Hamiltonian systems to nearby energy-momentum level sets are obtained under the conditions that the RPOs are nondegenerate and have a regular drift-momentum pair. In this paper we extend these results to periodically forced Hamiltonian systems and RPPs of symplectic maps. In addition we also treat subharmonic bifurcations of RPPs extending results of [1] to symplectic maps with continuous symmetries. We will see that there are two scenarios for subharmonic bifurcations. In the first scenario, also treated in [7] for RPOs, which we denote *subharmonic persistence*, the nondegenerate RPP persists to certain nearby momenta only as RPP with higher relative period. In other words, the original RPP of relative period 1 *does not persist* to those nearby momenta. In the second scenario, generalizing [1], the subharmonic bifurcation is caused by the RPP becoming *degenerate* when considered as an RPP of relative period k. In other words, the linearization of the periodic point of the symmetry reduced map inside its momentum level set has an eigenvalue which is a kth root of unity. In this case both the original RPP with relative period 1 and the bifurcating RPPs of relative period k may coexist.

2 Relative fixed points and relative periodic points

Let M be a manifold with a proper action of a Lie group Γ on it and let $\Psi : M \to M$ be a diffeomorphism which is Γ-equivariant, i.e.,

$$\gamma\Psi(x) = \Psi(\gamma x), \quad \text{for all} \;\; \gamma \in \Gamma, \; x \in M.$$

We call a point x_0 of a Γ-equivariant map Ψ a *relative periodic point (RPP)* of relative period n if there is some $\sigma_0 \in \Gamma$ such that

$$\Psi^n(x_0) = \sigma_0 x_0.$$

If $n = 1$ we also call x_0 a *relative fixed point.* In this paper we assume the action of Γ to be free, i.e., $\gamma x = x$ for some $\gamma \in \Gamma$ and $x \in M$ implies $\gamma = \text{id}$.

2.1 Symmetry adapted coordinates near relative fixed points

x

Γx_0 γ

x_0 v

Figure 1: Palais coordinates near Γx_0

Let U be a Γ-invariant neighbourhood of Γx_0 in M and let $N \subset T_{x_0}M$ be a normal space to the group orbit $\mathbf{g}x_0$ at x_0, so that $\mathbf{g}x_0 \oplus N = T_{x_0}M$. Using Palais slice coordinates [5] we can write every $x \in U$ as $x \simeq (\gamma, v)$ where $\gamma \in \Gamma$ and $v \in U_N$, see Figure 1. Here $U_N \subseteq N$ is a neighbourhood of 0 in N and $x_0 \simeq (\mathrm{id}, 0)$.

Indeed, let S be a submanifold of M transversal to Γx_0 at x_0 with tangent space $T_{x_0}S = N$ at x_0 and let $\psi : U_N \to S$, $\psi(0) = x_0$, be a local chart near x_0. Then we identify $x \in U$ with $x = \gamma\psi(v) \simeq (\gamma, v)$. We call N a *linear slice* and S a *nonlinear slice* at x_0. Due to Γ-equivariance in these coordinates Ψ takes the form

$$\Psi(\gamma, v) = \begin{pmatrix} \gamma\Psi_\Gamma(v) \\ \Psi_N(v) \end{pmatrix} \tag{2.1}$$

for smooth functions $\Psi_\Gamma : U_N \to \Gamma$, $\Psi_N : U_N \to N$. We call Ψ_N the *slice map*. For a relative fixed point x_0 with drift symmetry $\sigma_0 \in \Gamma$ we have $\Psi_\Gamma(0) = \sigma_0$ and $\Psi_N(0) = 0$. For $x = \gamma\psi(v) \in U$ the group element $\Psi_\Gamma(v) \in \Gamma$ is defined by the condition

$$(\Psi_\Gamma(v))^{-1}\gamma^{-1}\Psi(x) \in S,$$

and the vector $\Psi_N(v) \in N$ is then determined by the equation

$$\psi(\mathrm{id}, \Psi_N(v)) = (\Psi_\Gamma(v))^{-1}\gamma^{-1}\Psi(x).$$

Any RPP x near x_0 of relative period n then becomes an n-periodic point of the map Ψ_N. Moreover if $x_0 = \sigma_0^{-1}\Psi(x_0)$ is a relative fixed point of Ψ then in the coordinates $x \simeq (\gamma, v)$ the linearization $\sigma_0^{-1}\mathrm{D}\Psi(x_0)$ takes the form

$$\sigma_0^{-1}\mathrm{D}\Psi(x_0) = \begin{pmatrix} \mathrm{Ad}_{\sigma_0}^{-1} & \sigma_0^{-1}\mathrm{D}_v\Psi_\Gamma(0) \\ 0 & \mathrm{D}_v\Psi_N(0) \end{pmatrix}. \tag{2.2}$$

2.2 Equivariant symplectic maps

Now we assume that M is a symplectic manifold. Recall that a symplectic manifold is a manifold with a symplectic form ω defined on it, see [4]. We also assume that Ψ is a symplectic map, i.e., that for all $x \in M$,

$$\omega_{\Psi(x)}(\mathrm{D}\Psi(x)v, \mathrm{D}\Psi(x)w) = \omega_x(v, w), \quad \text{for all } \ v, w \in T_xM, \tag{2.3}$$

and that Γ acts symplectically (which means that each $\gamma \in \Gamma$ is a symplectic map). As before, Ψ is assumed to be Γ-equivariant. Such symplectic maps arise for example as time evolutions of periodically forced Γ-equivariant Hamiltonian systems

$$\dot{x} = f_H(t, x) \tag{2.4}$$

where the Hamiltonian vector field $f_H(t, x)$ is defined by the condition

$$\omega(f_H(t, x), v) = \mathrm{D}H(t, x)v, \quad \text{for all } \ v \in T_xM$$

and its Hamiltonian $H(t, x)$ is 1-periodic in t and Γ-invariant: $H(t, \gamma x) = H(t, x)$ for all $\gamma \in \Gamma$, $x \in M$. Let $\Phi_{t,t_0}(x_0)$ be the time-evolution of (2.4), so that $x(t) = \Phi_{t,t_0}(x_0)$ is the solution of (2.4) with $x(t_0) = x_0$. Let $\Psi(x) = \Phi_{1,0}(x)$. Then $\Psi(x)$ is a Γ-equivariant symplectic map. Moreover, by Noether's Theorem, locally near each $x_0 \in M$ there is, for each continuous symmetry $\xi \in \mathbf{g}$ of (2.4), a conserved quantity $\mathbf{J}_\xi$, and the map $\mathbf{J}_\xi$ is linear in ξ, so that $\mathbf{J}$ maps into $\mathbf{g}^*$, see e.g. [4]. Here $\mathbf{g} = T_{\mathrm{id}}\Gamma$ is the Lie algebra of Γ. The map $\mathbf{J}$ is called *momentum map* and for simplicity we assume in this paper that $\mathbf{J}$ is defined globally. So the map $\Psi(x) = \Phi_{1,0}(x)$ conserves the momentum map $\mathbf{J}$.

We assume in the following that $\mathbf{J}$ commutes with $\gamma \in \Gamma$

$$\mathbf{J}(\gamma x) = \gamma \mathbf{J}(x), \quad \gamma \in \Gamma$$

where the action of Γ on momentum space $\mathbf{g}^*$ is the coadjoint action

$$\gamma \mathbf{J}(x) = (\mathrm{Ad}_\gamma^*)^{-1} \mathbf{J}(x), \quad \gamma \in \Gamma.$$

The coadjoint action of Γ on $\mathbf{g}^*$ is given by $\gamma\mu := (\mathrm{Ad}_\gamma^*)^{-1}\mu$ where $\mathrm{Ad}_\gamma : \mathbf{g} \to \mathbf{g}$ defined by $\mathrm{Ad}_\gamma \eta = \gamma\eta\gamma^{-1}$ is the adjoint action. We also define the adjoint action of $\mathbf{g}$ on itself by $\mathrm{ad}_\xi \eta = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\xi)}\eta|_{t=0} = [\xi, \eta]$ and the coadjoint action of $\mathbf{g}$ on $\mathbf{g}^*$ by $\xi\mu = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}^*_{\exp(-t\xi)}\mu|_{t=0} = -\mathrm{ad}_\xi^*\mu$. We denote by $\Gamma_\mu = \{\gamma \in \Gamma, \mathrm{Ad}_\gamma^*\mu = \mu\}$ the isotropy subgroup of $\mu \in \mathbf{g}^*$ and by $\mathbf{g}_\mu = T_{\mathrm{id}}\Gamma_\mu$ its Lie algebra. Note that in the case of a compact symmetry group Γ the momentum map $\mathbf{J}$ can always be chosen equivariant with respect to the coadjoint action of Γ on $\mathbf{g}^*$ using averaging, see e.g. [6].

Example 2.1 The time 1 map $\Psi(x)$ of an SO(3)-symmetric Hamiltonian system, periodically forced with period 1 is an SO(3)-equivariant symplectic map and conserves the angular momentum $\mathbf{J}(x) \in \mathrm{so}(3)^* \simeq \mathbb{R}^3$.

2.3 Drift-momentum pairs

Let $\mu_0 = \mathbf{J}(x_0)$ be the momentum value of the RPP x_0 of relative period n and let σ_0 be its drift symmetry. Then, by Γ-equivariance of the momentum map $\mathbf{J}$ and the fact that $\mathbf{J}$ is conserved by the symplectic map Ψ,

$$\sigma_0\mu_0 = \sigma_0\mathbf{J}(x_0) = \mathbf{J}(\sigma_0 x_0) = \mathbf{J}(\Psi^n(x_0)) = \mathbf{J}(x_0) = \mu_0. \tag{2.5}$$

We call elements $(\sigma, \mu) \in \Gamma \times \mathbf{g}^*$ with $\sigma\mu = \mu$ *drift-momentum pairs* and denote by

$$(\Gamma \times \mathbf{g}^*)^c = \{(\sigma, \mu) \in \Gamma \times \mathbf{g}^*, \quad \sigma_0\mu_0 = \mu_0\}$$

the space of drift-momentum pairs [7]. We define an action of Γ on the space of drift-momentum pairs as follows:

$$\gamma(\sigma, \mu) = (\gamma\sigma\gamma^{-1}, (\mathrm{Ad}_\gamma^*)^{-1}\mu), \quad \gamma \in \Gamma, \quad (\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c.$$

For later purposes we define the isotropy subgroup $\Gamma_{(\sigma,\mu)}$ of $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ with respect to this action as

$$\Gamma_{(\sigma,\mu)} = \{\gamma \in \Gamma, \quad \gamma(\sigma, \mu) = (\sigma, \mu)\}$$

and denote its Lie algebra by $\mathbf{g}_{(\sigma,\mu)}$. Moreover we define $\Gamma_\sigma = \Gamma_{(\sigma,0)} = \{\gamma \in \Gamma, \quad \gamma\sigma\gamma^{-1} = \sigma\}$.

2.4 Symmetry adapted coordinates for symplectic maps

Let, as before, $\mu_0 = \mathbf{J}(x_0)$ be the momentum of a relative fixed point x_0 with drift symmetry σ_0. Let Γ_{μ_0} be its isotropy with respect to the coadjoint group action with Lie algebra $\mathbf{g}_{\mu_0} = T_{\mathrm{id}}\Gamma_{\mu_0}$. In this paper we assume that μ_0 is *strongly split* (see [7]), i.e., there is a Γ_{μ_0}-invariant complement $\mathbf{n}_{\mu_0}$ to $\mathbf{g}_{\mu_0}$ in $\mathbf{g}$. This can always be achieved if the symmetry group Γ is compact, cf. [7]. The slice N to Γx_0 at x_0 from Section 2.1 has the decomposition $N = N_0 \oplus N_1$. Let $\mathrm{ann}(\mathbf{n}_{\mu_0})$ denote the annihilator of $\mathbf{n}_{\mu_0}$ in $\mathbf{g}^*$. The space N_0 is isomorphic to $\mathbf{g}^*_{\mu_0} \simeq \mathrm{ann}(\mathbf{n}_{\mu_0})$ and can be identified with an affine subspace of $\mathbf{g}^*$ through μ_0 which is transverse to the momentum group orbit $\Gamma\mu_0$ at μ_0. The space $N_1 = T_{x_0}S_1$ is the tangent space to the slice $S_1 = S \cap \mathbf{J}^{-1}(\mu_0)$

transverse to $\Gamma_{\mu_0} x_0$ in the momentum level set $\mathbf{J}^{-1}(\mu_0)$. The symplectic form on $T_{x_0}M$ restricts to a symplectic form on N_1, which is therefore called the *symplectic normal space* at x_0. As shown in [2, 3], there is a symplectomorphism between a Γ-invariant neighbourhood U of Γx_0 and $\Gamma \times U_N$ and the slice S can be chosen such that the momentum map in these coordinates is given by

$$\mathbf{J}(\gamma, \nu, w) = \gamma(\mu_0 + \nu), \tag{2.6}$$

and such that the symplectic form on $\Gamma \times N$ takes the form

$$\omega_{\Gamma \times N} = \omega_{\Gamma \times \mathbf{g}^*_{\mu_0}} + \omega_{N_1}. \tag{2.7}$$

Here $\omega_{\Gamma \times \mathbf{g}^*_{\mu_0}}$ is the restriction to $\Gamma \times \mathbf{g}^*_{\mu_0} \simeq \Gamma \times \mathrm{ann}(\mathbf{n}_{\mu_0}) \subseteq \Gamma \times \mathbf{g}^* \simeq T^*\Gamma$ of the symplectic form on $T^*\Gamma$ and ω_{N_1} is the symplectic form on N_1.

Proposition 2.2 *If $\mu_0 = \mathbf{J}(x_0)$ is strongly split then the map Ψ_N from (2.1) takes the form*

$$\Psi_N(\nu, w) = \begin{pmatrix} \Psi_{N_0}(\nu, w) \\ \Psi_{N_1}(\nu, w) \end{pmatrix}, \quad \textit{where} \;\; \Psi_{N_0}(\nu, w) = \gamma(\nu, w)\nu, \quad \gamma(\nu, w) = (\Psi_\Gamma(\nu, w))^{-1} \in \Gamma_{\mu_0},$$

and $\Psi_{N_1}(\nu, \cdot)$ is a symplectic map on N_1.

Proof. From the decomposition $v = (\nu, w) \in N_0 \oplus N_1 = N$ we can decompose Ψ_N into a component $\Psi_{N_0} : U_N \to N_0$ and a component $\Psi_{N_1} : U_N \to N_1$. Since $\omega_{\Gamma \times N}$ splits into the form $\omega_{\Gamma \times \mathbf{g}^*_{\mu_0}}$ depending on $\nu \in N_0$ and $\gamma \in \Gamma$ only and the form ω_{N_1} depending on $w \in N_1$ only, see (2.7), the symplecticity of Ψ, (2.3), implies that $\Psi_{N_1}(\nu, \cdot)$ is a symplectic map on N_1. From the conservation of momentum, $\mathbf{J}(\Psi(x)) = \mathbf{J}(x)$ for $x \in M$, and (2.6) we get

$$\gamma(\mu_0 + \nu) = \mathbf{J}(\gamma, \nu) = \mathbf{J}(\Psi(\gamma, \nu, w)) = \gamma \Psi_\Gamma(\nu, w)(\mu_0 + \Psi_{N_0}(\nu, w))$$

and so

$$\Psi_{N_0}(\nu, w) = \gamma(\nu, w)(\mu_0 + \nu) - \mu_0 \tag{2.8}$$

where $\gamma(\nu, w) = (\Psi_\Gamma(\nu, w))^{-1}$. We conclude that $\gamma = \gamma(\nu, w)$ satisfies the equation

$$\gamma(\mu_0 + \nu) - \mu_0 \in \mathrm{ann}(\mathbf{n}_{\mu_0}). \tag{2.9}$$

For $\nu = 0$ the set of $\gamma \in \Gamma$ solving this equation is Γ_{μ_0} and since μ_0 is strongly split the solution set of (2.9) is given by Γ_{μ_0} also for $\nu \neq 0$, $\nu \approx 0$, see also [7, Lemma 3.8]. This proves the second statement of the proposition. ∎

Remarks 2.3

a) An analogous result for symmetric Hamiltonian systems near group orbits in symmetry adapted coordinates was proved in [6]; in [8] corresponding equations near RPOs of symmetric Hamiltonian systems in symmetry-adapted coordinates were derived.

b) The proof above shows that if μ_0 is not strongly split, then Proposition 2.2 remains true, except that Ψ_{N_0} in this case satisfies (2.8) and we have $\gamma(\nu, w) \in Z_{\mu_0,\nu}$ where $Z_{\mu_0,\nu}$ is the set of γ satisfying (2.9).

3 Persistence of relative periodic points

In this section we first study persistence of relative fixed points (Section 3.1) and then persistence as RPPs with higher relative period (Section 3.2) . For the proofs of the results we employ the symmetry-adapted coordinates near relative fixed points from the previous section.

3.1 Persistence of relative fixed points

In this section we study persistence of RPPs to nearby momentum level sets under a nondegeneracy condition specified below. Corresponding results for RPOs of symmetric Hamiltonian systems can be found in [7].

Definition 3.1 *A relative fixed point of a Γ-equivariant symplectic map $\Psi : M \to M$ which conserves the momentum map $\mathbf{J} : M \to \mathbf{g}^*$ is called non-degenerate if 0 is a nondegenerate fixed point of the map $\Psi_{N_1}(0, \cdot)$, i.e., if $\mathrm{D}_w \Psi_{N_1}(0)$ does not have an eigenvalue 1.*

Definition 3.2 *A pair $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ is called* regular *if $(\Gamma \times \mathbf{g}^*)^c$ is a manifold near (σ, μ). In this case let $r = r_{(\sigma,\mu)}(\Gamma)$ be such that $(\Gamma \times \mathbf{g}^*)^c$ has dimension $\dim_{(\sigma,\mu)}(\Gamma \times \mathbf{g}^*)^c = \dim \Gamma + r$ near (σ, μ).*

In the following we present some sufficient conditions for a drift-momentum pair to be regular. We start with some definitions.

Definition 3.3 *[6, 7]*

a) We call $\mu \in \mathbf{g}^*$ regular *if its isotropy subgroup Γ_μ for the coadjoint action of Γ on $\mathbf{g}^*$ has minimal dimension $r_\mu(\Gamma)$.*

b) We call $\sigma \in \Gamma$ regular *if the dimension $r_\sigma(\Gamma)$ of its isotropy Γ_σ is locally minimal.*

Proposition 3.4 *[7, Proposition 3.11]*

a) If $\mu \in \mathbf{g}^*$ is regular then (σ, μ) is regular and minimal for every $\sigma \in \Gamma_\mu$, and $r_{(\sigma,\mu)}(\Gamma) = \dim \mathbf{g}_{(\sigma,\mu)}$.

b) If $\sigma \in \Gamma$ is regular then $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ is regular for every $\mu \in \mathbf{g}^*$ with $\mathrm{Ad}^*_\sigma \mu = \mu$, and $r_\sigma(\Gamma) = \dim \mathbf{g}_{(\sigma,\mu)} = r_{(\sigma,\mu)}(\Gamma)$.

c) If μ is strongly split and $\mathbf{n}_\mu$ is chosen to be invariant under Γ_μ then $(\sigma, \mu) \in (\Gamma \times \mathbf{g}^*)^c$ is regular if and only if $\dim \mathbf{g}_{(\sigma,\mu)}$ is locally constant in $(\Gamma \times \mathbf{g}^*)^c$. Furthermore (σ, μ) is regular and if and only if σ is regular in Γ_μ and then $r_{(\sigma,\mu)}(\Gamma) = \dim \mathbf{g}_{(\sigma,\mu)}$.

If the relative fixed point x_0 is nondegenerate then it is an isolated fixed point of the symmetry reduced dynamics inside its momentum level set $\mathbf{J}^{-1}(\mu_0)$, where $\mu_0 = \mathbf{J}(x_0)$. Also note that the derivative of the slice map Ψ_N at the fixed point 0 corresponding to the relative fixed point x_0 is

$$\mathrm{D}\Psi_N(0) = \begin{pmatrix} \mathrm{D}_\nu \Psi_{N_0}(0) & 0 \\ \mathrm{D}_\nu \Psi_{N_1}(0) & \mathrm{D}_w \Psi_{N_1}(0) \end{pmatrix} \tag{3.1}$$

where

$$\mathrm{D}_\nu \Psi_{N_0}(0) = \sigma_0^{-1} = \mathrm{Ad}^*_{\sigma_0}|_{\mathbf{g}^*_{\mu_0}}.$$

Therefore the eigenvalue 1 of the derivative of the slice map $\mathrm{D}\Psi_N(0)$ at a nondegenerate relative fixed point has multiplicity $\dim \mathbf{g}_{(\sigma_0,\mu_0)}$. Since relative fixed points near x_0 are fixed points of the slice map Ψ_N one would expect that typically there are r-dimensional manifolds of fixed points of Ψ_N near 0 which give r-dimensional manifolds of relative fixed points near x_0 where $r = \dim \mathbf{g}_{(\sigma_0,\mu_0)}$. In the following persistence result we specify the conditions when this is true.

Theorem 3.5 *Let $x_0 = \sigma_0^{-1}\Psi(x_0)$ be a relative fixed point of the symplectic Γ-equivariant map Ψ which is non-degenerate and has a regular drift-momentum pair (σ_0, μ_0). Let $r = r_{(\sigma_0,\mu_0)}(\Gamma)$ and assume that μ_0 is strongly split. Then there is an r-dimensional smooth family of relative fixed points $x(\lambda)$ with drift symmetry $\sigma(\lambda)$ close to σ_0 and momentum $\mu(\lambda) = \mathbf{J}(x(\lambda)) = \mu_0 + \lambda$ with $\lambda \in \mathrm{ann}(\mathbf{n}_{\mu_0}) \cap \mathrm{Fix}_{\mathbf{g}^*}(\sigma_0) \simeq \mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\sigma_0) \simeq \mathbf{g}^*_{(\sigma_0,\mu_0)} \simeq \mathbb{R}^r$, such that $x(0) = x_0$, $\sigma(0) = \sigma_0$.*

Here $\mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\sigma_0) = \{\nu \in \mathbf{g}^*_{\mu_0}, \quad \sigma_0\nu = \nu\}$ denotes the fixed point space of σ_0 in $\mathbf{g}^*_{\mu_0}$. An analogous result for RPOs of symmetric Hamiltonian systems is [7, Theorem 4.2]. Before we prove the theorem we present an example of it for which the proof of the theorem is particularly simple.

Example 3.6 Let the momentum $\mu_0 = \mathbf{J}(x_0)$ of the relative fixed point x_0 be regular and assume that μ_0 is strongly split. Then the fact that μ_0 is regular implies that every $\nu \in \mathrm{ann}(\mathbf{n}_{\mu_0}) \simeq \mathbf{g}^*_{\mu_0}$ near $\nu = 0$ has isotropy of dimension $\dim \mathbf{g}_{\mu_0+\nu} = \dim \mathbf{g}_{\mu_0}$ and the fact that μ_0 is strongly split implies that $\mathbf{g}_{\mu_0+\nu} \subseteq \mathbf{g}_{\mu_0}$. Thus, $\mathbf{g}_{\mu_0+\nu} = \mathbf{g}_{\mu_0}$ and

$$\gamma\nu = \nu \quad \text{for} \quad \gamma \in \Gamma^{\mathrm{id}}_{\mu_0}, \; \nu \in \mathbf{g}^*_{\mu_0}, \quad \nu \approx 0.$$

Here $\Gamma^{\mathrm{id}}_{\mu_0}$ denotes the identity component of Γ_{μ_0}. In particular, we see that $\Gamma^{\mathrm{id}}_{\mu_0}$ is abelian if μ_0 is regular. Assume that the drift symmetry $\sigma_0 \in \Gamma_{\mu_0}$ of x_0 lies in the centralizer of $\Gamma^{\mathrm{id}}_{\mu_0}$, i.e., $\mathbf{g}_{\mu_0} = \mathbf{g}_{(\sigma_0,\mu_0)}$ (e.g. choose $\sigma_0 \in \Gamma^{\mathrm{id}}_{\mu_0}$). Then

$$\Psi_{N_0}(\nu, w) \equiv \nu.$$

As a consequence all RPPs near x_0 are periodic points of the ν-dependent symplectic map $\Psi_{N_1}(\nu, \cdot)$. Since x_0 is a nondegenerate relative fixed point the matrix $\mathrm{D}_w\Psi_{N_1}(0)$ has no eigenvalue 1. Therefore by the implicit function theorem there is unique fixed point $w(\nu)$ of $\Psi_{N_1}(\nu, \cdot)$ for each $\nu \in \mathbf{g}^*_{\mu_0}$, $\nu \approx 0$, and this gives an r-dimensional manifold $x(\nu) \simeq (\mathrm{id}, \nu, w(\nu))$ of relative fixed points near x_0 with momentum $\mathbf{J}(x(\nu)) = \mu_0 + \nu$. Here $r = \dim \mathbf{g}_{\mu_0} = \dim \mathbf{g}_{(\sigma_0,\mu_0)} = r_{(\sigma_0,\mu_0)}(\Gamma)$. Note that in this case x_0 persists as relative fixed point to each nearby momentum.

The proof of Theorem 3.5 reduces to the case treated in the example above if we replace the symmetry group Γ by Γ_{σ_0}:

Proof of Theorem 3.5. We treat Ψ as a symplectic map with symmetry group $\widetilde{\Gamma} = \Gamma_{\sigma_0}$. We denote by $\widetilde{\mathbf{g}} = \mathbf{g}_{\sigma_0}$ the Lie algebra of $\widetilde{\Gamma}$ and by $\widetilde{N}$ a slice for the action of $\widetilde{\Gamma}$ which has the decomposition $\widetilde{N} = \widetilde{N}_0 \oplus \widetilde{N}_1$. Here $\widetilde{N}_1 \supseteq N_1$ is the symplectic normal space for the action of $\widetilde{\Gamma}$ on M and $\widetilde{N}_0 \subseteq N_0 \simeq \mathbf{g}^*_{\mu_0}$ is defined as $\widetilde{N}_0 \simeq \mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\sigma_0) \simeq \widetilde{\mathbf{g}}^*_{\mu_0}$. Since μ_0 is strongly split and (σ_0, μ_0) is regular we conclude from Proposition 3.4 c) that $\dim \mathbf{g}_{(\sigma_0,\mu_0+\nu)} = r = \dim \mathbf{g}_{(\sigma_0,\mu_0)}$ for every $\nu \in \mathrm{Fix}_{\mathbf{g}^*}(\sigma_0) \cap \mathrm{ann}(\mathbf{n}_{\mu_0}) \simeq \mathbf{g}^*_{(\sigma_0,\mu_0)}$ with $\nu \approx 0$. Moreover $\mathbf{g}_{\mu_0+\nu} \subseteq \mathbf{g}_{\mu_0}$ since μ_0 is strongly split. Consequently $\mathbf{g}_{(\sigma_0,\mu_0+\nu)} = \mathbf{g}_{(\sigma_0,\mu_0)}$ and so, similarly as in Example 3.6,

$$\gamma\lambda = \lambda \quad \text{for} \quad \gamma \in \widetilde{\Gamma}_{\mu_0} = \Gamma_{(\sigma_0,\mu_0)}, \quad \gamma \approx \sigma_0, \quad \lambda \in \widetilde{\mathbf{g}}^*_{\mu_0} \simeq \mathbf{g}^*_{(\sigma_0,\mu_0)}, \quad \lambda \approx 0.$$

Thus, $\Psi_{\widetilde{N}_0}(\lambda, w) \equiv \lambda$. The relative fixed point x_0 of Ψ is a fixed point of $\sigma_0^{-1}\Psi$ and a fixed point of the slice map $\Psi_{\widetilde{N}}$. Since x_0 is nondegenerate and because of (2.2) the eigenvalue 1 of $\sigma_0^{-1}\mathrm{D}\Psi(x_0)$ has exactly multiplicity $\dim \mathbf{g}_{(\sigma_0,\mu_0)} + \dim \mathbf{g}_{\sigma_0}$, and all eigenvalues 1 of $\sigma_0^{-1}\mathrm{D}\Psi(x_0)$ are caused by symmetry and momentum conservation for the symmetry group $\widetilde{\Gamma}$. Hence the matrix $\mathrm{D}_{\tilde{w}}\Psi_{\widetilde{N}_1}(0)$ has no eigenvalue 1 and so we can solve the equation $\Psi_{\widetilde{N}_1}(\lambda, \tilde{w}) = \tilde{w}$ for each $\lambda \approx 0$ by the implicit function theorem. This gives an r-dimensional manifold $(\lambda, \tilde{w}(\lambda))$ of periodic points of $\Psi_{\widetilde{N}}$ which are relative fixed points $x(\lambda) \simeq (\mathrm{id}, \lambda, \tilde{w}(\lambda))$ of Ψ with drift symmetry $\sigma(\lambda) = (\Psi_{\widetilde{\Gamma}}(\lambda, \tilde{w}(\lambda)))^{-1}$ (see Proposition 2.2) and momentum $\mu(\lambda) = \mu_0 + \lambda$ where $\lambda \in \mathbf{g}^*_{(\sigma_0,\mu_0)}$. ■

Remark 3.7 A persistence result similar to Theorem 3.5 is also valid in the non-split case, but then the proof is more complicated. The main issue is the construction of a suitable slice transverse to the group orbit $\Gamma(\sigma_0, \mu_0)$ in the drift-momentum space $(\Gamma \times \mathbf{g}^*)^c$, see [7, Lemma 3.10].

Example 3.8 Consider a symplectic map $\Psi : M \to M$ with symmetry $\Gamma = \mathrm{SO}(3)$ which is the time-1 map of an SO(3)-symmetric periodically forced Hamiltonian system with forcing period 1. Then Ψ conserves the angular momentum $\mathbf{J} : M \to \mathrm{so}(3)^*$, see Example 2.1. Let x_0 be a nondegenerate relative fixed point with drift symmetry σ_0 and momentum μ_0. Typically $\mu_0 \neq 0$ or $\sigma_0 \neq \mathrm{id}$. In this case (σ_0, μ_0) is regular with $r_{(\sigma_0,\mu_0)}(\Gamma) = 1$ and by Theorem 3.5 there is a one-dimensional family of relative fixed points nearby.

3.2 Subharmonic persistence

We start this section with an example to illustrate the concept of subharmonic persistence.

Example 3.9 Let $\Gamma = \mathrm{O}(2)$ and let x_0 be a nondegenerate relative fixed point of Ψ with drift symmetry $\sigma_0 \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$, i.e., σ_0 is a reflection. In Section 2.3 we saw that $\sigma_0\mu_0 = \mu_0$, where $\mu_0 = \mathbf{J}(x_0)$ is the momentum of the RPP. Hence $\mu_0 = 0$ in this case. Moreover $r_{(\sigma_0,\mu_0)}(\Gamma) = r_{\sigma_0}(\Gamma) = 0$ since Ad_{σ_0} acts on $\mathbf{g} = \mathrm{so}(2)$ as multiplication by -1 (c.f. Proposition 3.4 b)). Hence the RPP does not persist to any nearby momentum as RPP with relative period 1. Now let us treat x_0 as RPP with relative period 2 and drift symmetry $\tilde\sigma_0 = \sigma_0^2 = \mathrm{id}$. For the symmetry group $\Gamma = \mathrm{O}(2)$ all momenta $\mu \in \mathrm{so}(2)^*$ are regular with $r_\mu(\Gamma) = 1$. Hence the drift momentum pair $(\tilde\sigma_0, \tilde\mu_0) = (\mathrm{id}, 0)$ of x_0 considered as RPP of relative period 2 is regular. Replacing Ψ by $\tilde\Psi := \Psi^2$ we obtain $\tilde\Psi_{N_0}(\nu) \equiv \nu$. Hence if x_0 is nondegenerate when considered as RPP of relative period 2, i.e., when $\mathrm{D}\Psi_{N_1}(0)$ has no eigenvalues ± 1 then there is a one-dimensional family of RPPs with relative period 2 nearby. We call this phenomenon *subharmonic persistence*.

We generalize the above example in the following theorem.

Theorem 3.10 *Let $x_0 = \sigma_0^{-1}\Psi(x_0)$ be a relative fixed point of the symplectic Γ-equivariant map Ψ with drift symmetry σ_0 and strongly split momentum $\mu_0 = \mathbf{J}(x_0)$. Let $\ell \in \mathbb{N}$, assume that* $\exp(2\pi \mathrm{i} p/\ell)$*, $p \in \mathbb{N}$, $\gcd(\ell, p) = 1$, $\ell \geq 2$, is an eigenvalue of $\mathrm{D}_\nu\Psi_{N_0}(0) = \mathrm{Ad}^*_{\sigma_0}|_{\mathbf{g}^*_{\mu_0}}$ and that $(\tilde\sigma_0, \mu_0)$ is a regular drift-momentum pair where $\tilde\sigma_0 = \sigma_0^\ell$. Let $\tilde r = r_{(\tilde\sigma_0,\mu_0)}(\Gamma)$. If $\mathrm{D}_w\Psi_{N_1}(0)$ does not have an ℓ-th root of unity as eigenvalue then there is an $\tilde r$-dimensional smooth family of RPPs $x(\lambda)$ near $x_0 = x(0)$, with relative period ℓ, with drift symmetry $\sigma(\lambda)$ close to $\tilde\sigma_0$ and with momentum $\mu(\lambda) = \mathbf{J}(x(\lambda)) = \mu_0 + \lambda$, $\lambda \in \mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\tilde\sigma_0) \simeq \mathbf{g}^*_{(\tilde\sigma_0,\mu_0)} \simeq \mathbb{R}^{\tilde r}$, such that $x(0) = x_0$, $\sigma(0) = \tilde\sigma_0$.*

The proof follows from Theorem 3.5 applied to the symplectic map $\tilde\Psi = \Psi^\ell$ if we replace σ_0 by $\tilde\sigma_0 = \sigma_0^\ell$ and use that due to our assumptions x_0 is nondegenerate when considered as a relative fixed point of Ψ^ℓ.

Remark 3.11 Note that for μ strongly split the condition that (σ^ℓ, μ) is regular for some $\ell \in \mathbb{N}$ implies that (σ, μ) is regular. To see this note that, by Proposition 3.4 c), (σ, μ) is regular if and only if σ is regular in Γ_μ, i.e., if the multiplicity of the eigenvalue 1 of $\mathrm{Ad}_\sigma|_{\mathbf{g}_\mu}$ is locally constant when σ is varied in Γ_μ. Similarly (σ^ℓ, μ) is regular if the number of eigenvalues (counting multiplicity) of Ad_σ which are ℓth roots of unity is locally constant. From this it follows that regularity of (σ^ℓ, μ) for some $\ell \in \mathbb{N}$ implies regularity of (σ, μ) for μ strongly split. Since $r_{(\sigma,\mu)} \leq r_{(\sigma^\ell,\mu)}$ the manifold of relative fixed points near x_0 is in general a submanifold of the manifold of RPPs near x_0 with relative period dividing ℓ, see Example 3.9.

4 Relative subharmonic bifurcations

We saw in the previous section that relative period multiplying can be caused by non-persistence of a relative fixed point to nearby momentum values. In this case $\mathrm{D}_\nu\Psi_{N_0}(0)$ has an eigenvalue of the form $\exp(2\pi \mathrm{i} p/\ell)$, $p \in \mathbb{N}$, $\gcd(\ell, p) = 1$, $\ell \geq 2$.

Since the relative fixed point x_0 of Ψ corresponds to the fixed point 0 of the slice map Ψ_N we expect (under reasonable assumptions) subharmonic branches to bifurcate whenever $\mathrm{D}\Psi_N(0)$ has a root of unity as eigenvalue. From (3.1) we see that eigenvalues of the linearization $\mathrm{D}\Psi_N(0)$ of the slice map at 0 are eigenvalues of $\mathrm{D}_\nu\Psi_{N_0}(0)$ or eigenvalues of $\mathrm{D}_w\Psi_{N_1}(0)$. Hence another mechanism for relative period multiplying is caused by a subharmonic bifurcation of the map Ψ_{N_1}. We call this phenomenon *relative subharmonic bifurcation*. In this case $\mathrm{D}_w\Psi_{N_1}(0)$ has eigenvalues which are k-th roots of unity for some $k \in \mathbb{N}$, and so the relative fixed point x_0 is degenerate when considered as RPP of relative period k. In this section we show that then under some (generic) conditions several branches of RPPs may coexist at nearby momentum values.

We assume that $\mathrm{D}_w\Psi_{N_1}(0)$ has an eigenvalue $\beta_0 = e^{2\pi \mathrm{i} j/k}$ such that $\beta_0^k = 1$ for some $k \in \mathbb{N}$, $k \geq 3$, where $j \in \mathbb{N}$, $\gcd(k,j) = 1$, and that there are no other eigenvalues β of $\mathrm{D}_w\Psi_{N_1}(0)$ with $\beta^k = 1$. Let $w(\nu)$ be the fixed point of $\Psi_{N_1}(\nu,\cdot)$ and let $\beta(\nu)$ be the smooth path of eigenvalues of $\mathrm{D}_w\Psi_{N_1}(\nu, w(\nu))$ such that $\beta(0) = \beta_0$. We assume that

$$\frac{\partial\beta}{\partial\nu}(0) \neq 0 \tag{4.1}$$

and choose coordinates for $\nu \in \mathbf{g}^*_{\mu_0} \simeq \mathbb{R}^m$ such that

$$\beta(\nu)|_{\nu_1=0} = \beta(0). \tag{4.2}$$

We first start with a special case.

Example 4.1 Let the momentum $\mu_0 = \mathbf{J}(x_0)$ of the relative fixed point x_0 be regular and assume that μ_0 is strongly split. If the drift symmetry σ_0 of x_0 satisfies $\sigma_0 \in \Gamma^{\mathrm{id}}_{\mu_0}$ (or more generally, for σ_0 in the centralizer of $\Gamma^{\mathrm{id}}_{\mu_0}$ so that $\mathbf{g}_{\mu_0} = \mathbf{g}_{(\sigma_0,\mu_0)}$) then, as in Example 3.6,

$$\Psi_{N_0}(\nu, w) \equiv \nu.$$

Denote by $w(\nu)$ the fixed point of $\Psi_{N_1}(\nu,\cdot)$ from Example 3.6 corresponding to the relative fixed point $x(\nu)$ of Ψ with momentum $\mu_0+\nu$. Let $\tilde\nu = (\nu_2,\ldots,\nu_m)$. We can now apply the results from [1] onto Ψ_{N_1} with ν_1 acting as bifurcation parameter. By [1] two branches $(w^{(i)}(\rho,\tilde\nu), \nu_1^{(i)}(\rho,\tilde\nu))$, $i = 1,2$, of periodic points of $\Psi_{N_1}(\nu,\cdot)$ with period k bifurcate from 0, i.e.,

$$w^{(i)}(0,\tilde\nu) = w(0,\tilde\nu), \quad \nu_1^{(i)}(0) = 0, \quad i = 1,2.$$

These give two branches $x^{(i)}(\rho,\tilde\nu)$, $i = 1,2$, of RPPs of Ψ with relative period k and momentum $\mu^{(i)}(\rho,\tilde\nu) = \mu_0 + \nu^{(i)}(\rho,\tilde\nu)$ where $\nu^{(i)}(\rho,\tilde\nu) = (\nu_1^{(i)}(\rho,\tilde\nu),\tilde\nu)$ (in [1] subharmonic bifurcations of parameter dependent symplectic maps were studied for scalar parameters. But the proof of the results in [1] remain true in the case of several parameters as studied here).

We now consider the effect of subharmonic bifurcations of Ψ_{N_1} under more general assumptions on the drift momentum pair of the relative fixed point x_0, similarly as in the previous section.

Theorem 4.2 *Let x_0 be nondegenerate relative fixed point of Ψ with regular drift-momentum pair (σ_0,μ_0) and an eigenvalue $\beta_0 = \exp(2\pi \mathrm{i} j/k)$ of $\mathrm{D}_w\Psi_{N_1}(0)$ with $j \in \mathbb{N}$, $\gcd(k,j) = 1$, $k \geq 3$. Assume that $\mathrm{D}_w\Psi_{N_1}(0)$ has no other k-th root of unity as eigenvalue and that the transversality condition* (4.1) *holds. Let μ_0 be strongly split, let $r = r_{(\sigma_0,\mu_0)} = \dim \mathbf{g}_{(\sigma_0,\mu_0)}$ and assume that $\dim \mathbf{g}_{(\sigma_0,\mu_0)} = \dim \mathbf{g}_{(\sigma_0^k,\mu_0)}$. Choose coordinates on $\mathbf{g}^*_{\mu_0} \simeq \mathbb{R}^m$ such that* (4.2) *holds, denote by e_1^ν the first unit vector of $\mathbf{g}^*_{\mu_0}$ and assume that*

$$\langle e_1^\nu, \mathbf{g}_{(\sigma_0,\mu_0)}\rangle \neq \{0\}.$$

Then there are two r-dimensional smooth manifolds of RPPs $x^{(i)}(\rho,\tilde\lambda)$ of Ψ, $i=1,2$, with relative period k, drift symmetry $\sigma^{(i)}(\rho,\tilde\lambda)$ and momentum $\mu^{(i)}(\rho,\tilde\lambda)$, where $\mu^{(i)}(\rho,\tilde\lambda)-\mu_0\in\mathbf{g}^*_{(\sigma_0,\mu_0)}$, $\tilde\lambda\in\mathbb{R}^{\tilde r-1}$, $\rho\in\mathbb{R}^+_0$, $i=1,2$, such that $x^{(i)}(0)=x_0$, $\sigma^{(i)}(0)=\sigma_0^k$, $\mu^{(i)}(0)=\mu_0$.

Note that the condition $\dim\mathbf{g}_{(\sigma_0,\mu_0)}=\dim\mathbf{g}_{(\sigma_0^k,\mu_0)}$ means that the number of eigenvalues of the derivative of the slice map $\mathrm{D}\Psi_N(0)$ which are kth roots of unity is exactly $r+2$. Here 2 roots are eigenvalues of $\mathrm{D}_w\Psi_{N_1}(0)$ and the other r roots are eigenvalues 1 of $\mathrm{D}_\nu\Psi_{N_0}(0)$. Therefore we expect two r-dimensional families of subharmonic periodic points of Ψ_N to bifurcate from the fixed point 0 of Ψ_N, and this is proved below.

Proof of Theorem 4.2. The proof is similar to the proof of Theorem 3.5. We will replace the symmetry group Γ of Ψ by a smaller symmetry group $\tilde\Gamma$ so that μ_0 is a regular momentum value for this smaller symmetry group. In the proof of Theorem 3.5 we chose $\tilde\Gamma=\Gamma_{\sigma_0}$. But the matrix $\mathrm{Ad}_{\sigma_0}|_{\mathbf{n}_{\mu_0}}$ might have eigenvalues which are k-th roots of unity and different from 1. These would be resonant eigenvalues on the symplectic normal space for the symmetry reduction with respect to Γ_{σ_0} so that the results on subharmonic bifurcations from [1] could not be applied.

Therefore we consider Ψ on the manifold $\widetilde M=\Gamma_{\mu_0}\times U_N$ which is invariant under Ψ by Proposition 2.2 and invariant under Γ_{μ_0}. Since the tangent spaces

$$T_{(\gamma,\nu,w)}\widetilde M=T_\gamma\Gamma_{\mu_0}\oplus N\simeq T_\gamma\Gamma_{\mu_0}\oplus\mathbf{g}^*_{\mu_0}\oplus N_1=T_{(\gamma,\nu)}T^*\Gamma_{\mu_0}\oplus N_1$$

at each $(\gamma,v)\in\widetilde M$ are symplectic the manifold $\widetilde M$ is symplectic, see [2, 3, 6]. Furthermore the momentum map for the Γ_{μ_0}-action on $\widetilde M$ is just the momentum map for the action of Γ_{μ_0} on $T^*\Gamma_{\mu_0}\simeq\Gamma_{\mu_0}\times\mathbf{g}^*_{\mu_0}$ and therefore $\mathbf{J}_{\widetilde M}(\gamma,\nu,w)=\nu$, see e.g. [4]. We now treat Ψ as symplectic map on $\widetilde M$ with symmetry group $\widetilde\Gamma=\Gamma_{(\sigma_0,\mu_0)}\subseteq\Gamma_{\mu_0}$. We denote by $\widetilde{\mathbf{g}}$ the Lie algebra of $\widetilde\Gamma$ and choose a slice $\widetilde N=\widetilde N_0\oplus\widetilde N_1$ for this group action on $\widetilde M$. We choose this slice such that it contains $N_0\oplus N_1$. Here, as in the proof of Theorem 3.5, $\widetilde N_1\supseteq N_1$ is the symplectic normal space for the action of $\widetilde\Gamma$ on $\widetilde M$, and we define $\widetilde N_0\simeq\widetilde{\mathbf{g}}^*$ as $\widetilde N_0:=\mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\sigma_0)\subseteq\mathbf{g}^*_{\mu_0}\simeq N_0$. The momentum map $\widetilde{\mathbf{J}}$ for the $\widetilde\Gamma$ action on $(\gamma,\lambda,\tilde w)\in\tilde\Gamma\times\widetilde N_0\oplus\widetilde N_1\simeq\widetilde M$ is then given by $\widetilde{\mathbf{J}}(\gamma,\lambda,\tilde w)=\lambda\in\widetilde N_0$. For $\lambda\in\widetilde{\mathbf{g}}^*=\mathbf{g}^*_{(\sigma_0,\mu_0)}$ we get, as in the proof of Theorem 3.5, that

$$\Psi_{\widetilde N_0}(\lambda,w)\equiv\lambda.$$

Denote by $\tilde w(\lambda)$ the fixed point of $\Psi_{\widetilde N_1}(\lambda,\cdot)$ such that $\tilde w(0)=0$, i.e., $\tilde w(\lambda)$ corresponds to the relative fixed point $x(\lambda)$ of Ψ with momentum $\mu_0+\lambda$, c.f. Theorem 3.5, and let $\beta(\lambda)$ be the eigenvalue of $\mathrm{D}_w\Psi_{\widetilde N_1}(\lambda,w(\lambda))$ such that $\beta(\lambda)$ is smooth in λ and $\beta(0)=\beta_0$. Let e_i^ν be the i-th unit vector of $\mathbf{g}^*_{\mu_0}\simeq\mathbb{R}^m$ and let ξ_j, $j=1,\dots,m=\dim\mathbf{g}_{\mu_0}$ be a basis of $\mathbf{g}_{\mu_0}$ with $\langle e_i^\nu,\xi_j\rangle=\delta_{ij}$. Since by assumption $\langle e_1^\nu,\mathbf{g}_{(\sigma_0,\mu_0)}\rangle\neq\{0\}$ we can choose coordinates on $\mathbf{g}^*_{(\sigma_0,\mu_0)}\simeq\mathbb{R}^r$ such that the first eigenvector $e_1^\lambda\in\mathbf{g}^*_{(\sigma_0,\mu_0)}\simeq\mathrm{Fix}_{\mathbf{g}^*_{\mu_0}}(\sigma_0)\subseteq\mathbf{g}^*_{\mu_0}$ of $\mathbf{g}^*_{(\sigma_0,\mu_0)}$ satisfies $\langle e_1^\lambda,\xi_1\rangle\neq0$ and such that $\beta(0,\tilde\lambda)=\beta_0$ for all $\tilde\lambda\approx0$. Here $\tilde\lambda=(\lambda_2,\dots,\lambda_r)$ so that $\lambda=(\lambda_1,\tilde\lambda)$. We then have $\frac{\partial\beta}{\partial\lambda_1}(0)\neq0$.

We now want to apply the results from [1] onto $\Psi_{\widetilde N_1}(\lambda,\cdot)$ with λ_1 acting as bifurcation parameter. Due to the assumption that β_0 and β_0^{-1} are the only eigenvalues of $\mathrm{D}_w\Psi_{N_1}(0)$ which are kth roots of unity, due to the restriction of Ψ to $\widetilde M$ and the assumption that $\dim\mathbf{g}_{(\sigma_0,\mu_0)}=\dim\mathbf{g}_{(\sigma_0^k,\mu_0)}$ the matrix $\mathrm{D}_{\tilde w}\Psi_{\widetilde N_1}(0)$ has exactly one kth root of unity as eigenvalue. Moreover, $\frac{\partial\beta}{\partial\lambda_1}(0)\neq0$. Hence, by [1] two branches $(\tilde w^{(i)}(\rho,\tilde\lambda),\lambda_1^{(i)}(\rho,\tilde\lambda))$, $i=1,2$, of periodic points of $\Psi_{\widetilde N_1}(\lambda,\cdot)$ with period k bifurcate from 0, i.e.,

$$\tilde w^{(i)}(0,\tilde\lambda)=\tilde w(0,\tilde\lambda),\quad\lambda_1^{(i)}(0,\tilde\lambda)=0,\quad i=1,2.$$

These give two branches $x^{(i)}(\rho,\tilde\lambda)$, $i=1,2$, of RPPs of Ψ with relative period k which have momentum $\mu^{(i)}(\rho,\tilde\lambda)=\mu_0+\lambda^{(i)}(\rho,\tilde\lambda)$ where $\lambda^{(i)}(\rho,\tilde\lambda)=(\lambda_1^{(i)}(\rho,\tilde\lambda),\tilde\lambda)\in\mathbf{g}^*_{(\sigma_0,\mu_0)}$ and drift symmetry $\sigma^{(i)}(\rho,\tilde\lambda)=(\Psi_{\widetilde\Gamma}(\lambda^{(i)}(\rho,\tilde\lambda),\tilde w^{(i)}(\rho,\tilde\lambda)))^{-1}$ (by Proposition 2.2). ■

Subharmonic persistence and relative subharmonic bifurcation may also occur simultaneously. We consider this situation in the following corollary under some non-resonance conditions. This corollary is a generalization of Theorem 4.2, similarly as Theorem 3.10 extends Theorem 3.5. For an application of this corollary see Example 4.4 below.

Corollary 4.3 *Let x_0 be a relative fixed point with drift-momentum pair (σ_0,μ_0). Let μ_0 be strongly split, assume that $\mathrm{D}_\nu\Psi_{N_0}(0)=\mathrm{Ad}_{\sigma_0}|_{\mathbf{g}^*_{\mu_0}}$ has an eigenvalue of the form $\exp(2\pi\mathrm{i}p/\ell)$, $\ell\ge 2$, $p\in\mathbb{N}$, $\gcd(p,\ell)=1$, and that $(\tilde\sigma_0,\mu_0)\in(\Gamma\times\mathbf{g}^*)^c$ is regular where $\tilde\sigma_0=\sigma_0^\ell$ for some $\ell\in\mathbb{N}$. Let $\tilde r=r_{(\tilde\sigma_0,\mu_0)}$. Furthermore assume that $\mathrm{D}_w\Psi_{N_1}(0)$ has an eigenvalue $\beta_0=\exp(2\pi\mathrm{i}j/k)$, $\gcd(j,k)=1$, and that $k/\gcd(\ell,k)\ge 3$. Let $\tilde\ell=\ell/\gcd(\ell,k)$, and assume that $\mathrm{D}_w\Psi_{N_1}(0)$ has no other $(k\tilde\ell)$-th root of unity as eigenvalue. Let the transversality condition (4.1) hold, choose coordinates on $\mathbf{g}^*_{\mu_0}\simeq\mathbb{R}^m$ such that (4.2) holds, denote by e_1^ν the first unit vector of $\mathbf{g}^*_{\mu_0}$ and assume that*

$$\langle e_1^\nu,\mathbf{g}_{(\tilde\sigma_0,\mu_0)}\rangle\neq\{0\}.$$

Then there are two $\tilde r$-dimensional smooth manifolds of RPPs $x^{(i)}(\rho,\tilde\lambda)$ of Ψ, $i=1,2$, with relative period $k\tilde\ell$ and with momentum $\mu^{(i)}(\rho,\tilde\lambda)$, where $\mu^{(i)}(\rho,\tilde\lambda)-\mu_0\in\mathbf{g}^*_{(\tilde\sigma_0,\mu_0)}$, $\tilde\lambda\in\mathbb{R}^{\tilde r-1}$, $\rho\in\mathbb{R}_0^+$, $i=1,2$, such that $x^{(i)}(0)=x_0$, $\mu^{(i)}(0)=\mu_0$.

Note that in addition to those subharmonic branches of relative period $k\tilde\ell$ from Corollary 4.3 there is an $\tilde r$-dimensional manifold of RPPs of relative period ℓ which persists subharmonically from x_0 by Theorem 3.10.

Proof of Corollary 4.3. The proof follows from Theorem 4.2 replacing Ψ by Ψ^ℓ. We just have to check the eigenvalue condition for the matrix $\mathrm{D}_w\Psi^\ell_{N_1}(0)$. By assumption, $\mathrm{D}_w\Psi^\ell_{N_1}(0)$ has an eigenvalue $\beta_0^\ell=\exp(2\pi\mathrm{i}j\ell/k)=\exp(2\pi\mathrm{i}j\tilde\ell/\tilde k)=\exp(2\pi\mathrm{i}\tilde j/\tilde k)$ where $\tilde j=j\tilde\ell$, $\tilde k=k/\gcd(\ell,k)\ \ge 3$, $\gcd(\tilde j,\tilde k)=1$. Furthermore $\mathrm{D}_w\Psi^\ell_{N_1}(0)$ has no other eigenvalue which is $\tilde k$-th root of unity because this would be an eigenvalue of $\mathrm{D}_w\Psi_{N_1}(0)$ which is an m-th root of unity where $m=\tilde k\ell=k\tilde\ell$ and this is ruled out by the assumptions of the corollary. Since the drift-momentum pair (σ_0^ℓ,μ_0) is assumed to be regular Ψ^ℓ satisfies all assumptions of Theorem 4.2, and so the result follows. ■

Example 4.4 As in Example 3.9 we consider the symmetry group $\Gamma=\mathrm{O}(2)$, but now we assume that x_0 is a nondegenerate relative fixed point undergoing a relative subharmonic bifurcation, i.e., that $\mathrm{D}_w\Psi_{N_1}(0)$ has a simple eigenvalue β_0 which is a kth root of unity and the the transversality condition (4.1) is satisfied.

First assume that the drift symmetry σ_0 of the relative fixed point is a rotation, i.e., $\sigma_0\in\mathrm{SO}(2)$. If σ_0 is a nontrivial rotation or if the momentum μ_0 of the relative fixed point x_0 does not vanish then the drift-momentum pair (σ_0,μ_0) is regular, and so ,by Theorem 3.5, x_0 persists as relative fixed point to all nearby momenta. If $k\ge 3$, if σ_0 is not a rotation about an angle which is a multiple of $2\pi/k$ and if $\mathrm{D}_w\Psi_{N_1}(0)$ has no other kth root of unity then by Theorem 4.2, there are two branches of RPPs with relative period k either at nearby positive momenta or at nearby negative momenta.

Now consider the case where the drift symmetry σ_0 of the relative fixed point x_0 of Ψ is a reflection, i.e., $\sigma_0\in\mathrm{O}(2)\setminus\mathrm{SO}(2)$. Then the momentum $\mu_0=\mathbf{J}(x_0)$ of the relative fixed point vanishes, $\mu_0=0$. By Theorem 3.5, x_0 does not persist as relative fixed point. By Theorem

3.10 (with $\ell = 2$) there is a branch of RPPs with relative period 2 which persists to all nearby momentum values, provided that $D_w\Psi_{N_1}(0)$ has no eigenvalues ± 1. Next we study relative subharmonic bifurcation for this case.

If k is even then $\ell = 2$, $\gcd(\ell, k) = 2$ and $\tilde{\ell} = 1$ in the notation of Corollary 4.2. If $k \geq 6$ and $D_w\Psi_{N_1}(0)$ has no eigenvalues which are kth roots of unity apart from β_0 and β_0^{-1} then by Corollary 4.2 there are two branches of RPPs with relative period k bifurcating from x_0 either for nearby positive angular momenta $\mu \in \mathrm{so}(2)^*$ or for nearby negative momenta. If k is odd, then $\ell = 2$, $\gcd(\ell, k) = 1$, $\tilde{\ell} = 2$ in the notation of Corollary 4.3. Hence if $k \geq 3$ and $D_w\Psi_{N_1}(0)$ has no eigenvalues which are $(2k)$-th roots of unity other than β_0 and β_0^{-1} then two branches of RPPs with relative period $2k$ bifurcate from x_0 either for nearby positive or nearby negative angular momenta.

Acknowledgement

This work has partially been supported by the Nuffield Foundation. The author thanks the Freie Universität Berlin for their hospitality during a visit in December 2005/January 2006 when the main part of this paper was written.

References

[1] M.-C. Ciocci and A. Vanderbauwhede. Bifurcation of periodic orbits for symplectic mappings. *J. Difference Eq. Appl.*, 3:485 – 500, 1998.

[2] V. Guillemin and S. Sternberg. A normal form for the moment map. In *Differential geometric methods in mathematical physics*, (S.Sternberg, editor), D.Reidel, Dordrecht, 1984.

[3] C.-M. Marle. Modèle d'action Hamiltonienne d'un groupe de Lie sur une variété symplectique. *Rendiconti del Seminario Matematico, Università e Politechnico, Torino*, 43:227-251, 1985.

[4] J.E. Marsden and T.S. Ratiu. Introduction to Mechanics and Symmetry. Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[5] R.S. Palais. On the existence of slices for actions of noncompact Lie groups. *Ann. of Math.* **73** : 295–323, 1961.

[6] R.M. Roberts, C. Wulff, and J. Lamb. Hamiltonian systems near relative equilibria. *J. Differential Equations*, 179, 562–604, 2002.

[7] C. Wulff. Persistence of Hamiltonian relative periodic orbits. *J. Geom. Phys.* 48: 309-338, 2003.

[8] C. Wulff and M. Roberts. Hamiltonian systems near relative periodic orbits. SIAM J. Dyn. Syst.,1:1–43, 2002.