Bifurcation Theory of Meandering Spiral Waves

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Abstract. Spiral waves are a typical phenomenon of spatio-temporal pattern formation. They are observed in various biological and chemical systems, for example in the catalysis on platinum surfaces and in the Belousov-Zhabotinsky reaction. We develop a mathematical theory for the Hopf bifurcation from rigidly rotating spiral waves to meandering spiral waves; we prove the transition to drifting spiral waves if the rotation frequency of the rigidly rotating spiral wave is a multiple of the module of the Hopf eigenvalue and we study the parameter-dependence of the drift velocity near the bifurcation from rigidly rotating spiral waves. Furthermore we prove that analogous phenomena occur if a rigidly rotating spiral wave is subjected to external periodic forcing. Our results hold for a general class of reaction-diffusion systems and provide a rigorous mathematical explanation of experiments on the meandering transition in autonomous and periodically forced systems.

1 Introduction

Spiral waves have been observed in many chemical and biological systems, see for example Müller et al. ed. (1994), Kapral and Showalter ed. (1995).

We are interested in the transition from rigidly rotating to meandering spiral waves. The tip of a rigidly rotating spiral wave moves on a circle whereas the tip of a meandering spiral wave performs a quasiperiodic "meandering" motion. Meandering spiral waves sponteneously bifurcate from rigidly rotating spiral waves and the transition from rigidly rotating to meandering spiral waves can also be forced by a periodic modulation of the excitability of the medium. Sponteneous and forced meandering transitions have been investigated in the Belousov-Zhabotinsky medium, see e.g. Jahnke et al. (1989), Skinner and Swinney (1991), Braune and Engel (1993a), (1993b), Ungvarai-Nagy et al. (1993), Zykov et al. (1994), Müller and Zykov (1994), and in catalytic reactions on surfaces, see e.g. Nettesheim et al. (1993). Furthermore many numerical studies on the meandering transition have been carried out, see Lugosi (1989), Karma (1990), Jahnke and Winfree (1991), Winfree (1991), Barkley (1992), (1994), Bär and Eiswirth (1993), Bär et al. (1994).

The meandering spiral waves found in experiments and simulations are proper quasiperiodic solutions; frequency locking effects do not occur.

Moreover, in simulations it turned out that the sponteneous transition from rigidly rotating spiral waves to meandering spiral waves is a supercritical

Hopf bifurcation in a frame rotating with the frequency ω of the rigidly rotating spiral wave. Let (after some scaling) $\pm\mathrm{i}$ be the imaginary eigenvalues of the linearization of the corotating system which lead to Hopf bifurcation. We will see that $\pm\mathrm{i}\omega$ are also eigenvalues of the linearization of the corotating system in the rigidly rotating spiral wave. If the rotation frequency ω is an integer then the rigidly rotating spiral wave is a resonant Hopf point. In numerical simulations of a two-parameter-system Barkley (1994) found that a curve of drifting spiral waves emerges from such a 1:1-resonant Hopf point, see Fig. 1. This curve separates meandering spiral wave states with inward petals and outward petals.

As mentioned above, also by periodic forcing of rigidly rotating spiral waves meandering spiral waves are generated and also here drifting spiral waves are generated by resonances: Let Ω be the frequency of the external forcing and let μ be its amplitude. If the frequency ω of the rigidly rotating wave at $\mu = 0$ is a multiple of the external frequency Ω of the system then a curve of drifting spiral waves in the (Ω, μ)-plane emanates. In the case of external periodic forcing also drifting caused by higher resonances has been observed by Nettesheim et al. (1993) whereas in experiments on the spontaneous bifurcation of drifting spiral waves only the case of 1:1-resonance has been found yet.

The transition to meandering spiral waves has also been investigated by use of the kinematical theory, see Mikhailov et al. (1994), and by means of a free boundary value formulation for the spiral interfaces cf. Pelce and Sun (1993), Kessler et al. (1994). But these methods are only applicable if the underlying reaction-diffusion system satisfies additional conditions: the kinematical theory only applies to weakly excitable media, and the free boundary value approach can only be used in systems with different time-scales.

We use a completely different method for our theory of spiral waves than these authors. We treat a general reaction-diffusion system – without the assumption of the kinematical theory or the assumption of different time-scales – and we only use the symmetry properties of the reaction-diffusion system in order to develop a bifurcation theory for spiral waves. Our results hold for a general class of reaction-diffusion equations including systems of activator-inhibitor-type like the Oregonator model of the Belousov-Zhabotinsky-reaction and model equations for catalytic surface reactions. In contrast to the authors mentioned above we do not examine the shape of the spiral wave and the bifurcating modes in our theory. Our results are based on the concepts of bifurcation theory and symmetry which can be found for example in the books of Guckenheimer and Holmes (1990), Chow and Hale (1982), Golubitsky and Schaeffer (1985), Golubitsky et al. (1988).

Barkley (1994) has given a first heuristic explanation for the effects which can be found near a rigidly rotating spiral wave which undergoes a resonant Hopf bifurcation. He found a 5-dimensional system of ordinary differential equations which shows the same qualitative behaviour as the reaction-

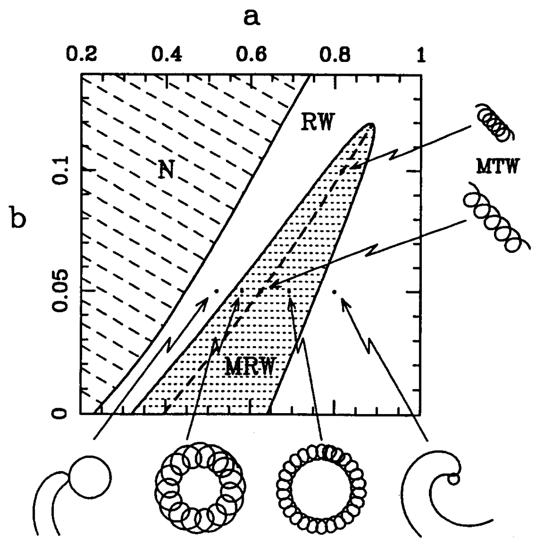

Fig. 1. Phase diagram for the spiral wave dynamics depending on the parameters a, b, from numerical computations of Barkley (1994). Shown are regions containing N: no spiral waves, RW: stable rotating waves, MRW: modulated rotating waves, MTW: modulated travelling waves (dashed curve). Spiral tip paths illustrate states at 6 points.

diffusion system. But he did not show any relationship between this system of ordinary differential equations and the original reaction-diffusion system. However, he emphasized the importance of the Euclidean symmetry of the underlying reaction-diffusion system on which our theory of spiral wave dynamics is based. The Euclidean group of the plane consists of translations in both space dimensions, rotations and reflections. It is a *non-compact* group.

2 Mathematical Modelling

We consider a reaction-diffusion system on the plane

$$\begin{aligned} \frac{\partial u_1}{\partial t}(t,x) &= \Delta u_1(t,x) + f_1(u(t,x),t,\mu), \\ \frac{\partial u_2}{\partial t}(t,x) &= \delta \Delta u_2(t,x) + f_2(u(t,x),t,\mu). \end{aligned} \tag{1}$$

Here $u_1, u_2 : \mathbb{R}_0^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are space-dependent concentrations of chemical species, $f_1(u,t,\mu)$, $f_2(u,t,\mu)$ are reaction-terms depending on the concentration vector $u = (u_1, u_2)$, on t and on a parameter $\mu \in \mathbb{R}^n$ and $\delta \geq 0$ is the diffusion-coefficient. In the well-known two-variable Oregonator model of the Belousov-Zhabotinsky reaction and in the model for the CO-oxidation on Pt(110), see Bär and Eiswirth (1993), the variable u_1 is the activator-concentration and the variable u_2 is the inhibitor-concentration. Since the diffusion in System (1) is isotropic (1) has the symmetry of the Euclidean group $E(2)$ of the plane. The Euclidean group operates on the functions $u(x) = (u_1(x), u_2(x))$ as a rotation, translation or reflection of the space variable $x = (x_1, x_2)$: Let $\tau_{a,\phi}x := m_\phi x + a$ where m_ϕ denotes a rotation in the plane by the angle ϕ, let $\kappa x = (-x_1, x_2)$ and define

$$(T_{a,\phi}u)(x) := u(\tau_{a,\phi}^{-1}x), \ , \ (T_\kappa u)(x) = u(\kappa x).$$

Equation (1) is *equivariant* under the Euclidean group, i.e., with $u(t,x)$ also each rotated, translated or reflected function $(T_{a,\phi}u)(t,x)$ respectively $(T_\kappa u)(t,x)$ is a solution of (1). The infinitesimal generators of $E(2)$ are the partial derivatives $R_1 = -\frac{\partial}{\partial x_1}$, $R_2 = -\frac{\partial}{\partial x_2}$ and the angle-derivative $R_3 = -\frac{\partial}{\partial \phi}$ where we consider $u(x)$ in polar coordinates $u(x) = u(r\cos\phi, r\sin\phi)$.

Abstract Setting

In order to treat Equation (1) mathematically we rewrite (1) as abstract differential equation on the function space $Y = BC_{\mathrm{Eucl}}$. The elements of this function space are the concentration vectors $u(x) = (u_1(x), u_2(x))$. The space consists of all functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ which are bounded, continuous and satisfy

$$\sup_{x\in\mathbb{R}^2} |(T_{a,\phi}u)(x) - u(x)| \to 0 \text{ as } a, \phi \to 0.$$

The last condition means that a rotation by a small angle or a translation by a small amount leads to a small perturbation of the function u even far away from the origin. So we will exclude planar waves of the form $u(x_1, x_2) := \cos(\xi_1 x_1 + \xi_2 x_2 + \theta_0)w$, $w \in \mathbb{R}$, because even an arbitrarily small rotation of such a wave yields a perturbation of order 1 far away from the origin. Spiral waves are elements of our function space Y because they look like circles far away from the origin. The norm on Y is canonically defined by $\|u\| := \sup_{x\in\mathbb{R}^2} |u(x)|$, $u \in Y$.

On the infinite-dimensional function space Y Equation (1) looks like an ordinary differential equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(u, t, \mu), \tag{2}$$

where

$$-A = \begin{pmatrix} \Delta & 0 \\ 0 & \delta\Delta \end{pmatrix}$$

and $f(u,t,\mu) = (f_1(u,t,\mu), f_2(u,t,\mu))$. It can be shown that there is a semi-flow $\Phi_{t,t_0}(u;\mu)$ to (2) on our function space Y which describes the time-evolution of system (1). Note that the $E(2)$-action on BC_{Eucl} is not differentiable in general because the R_i, $i = 1,2,3$, are unbounded operators: Not for every $u \in BC_{\mathrm{Eucl}}$ the derivative with respect to x and ϕ exist.

In Sect. 3, 4 we will assume that the nonlinearity f in (2) is independent of t. In Sect. 5 we will treat the case when f is periodic in t.

3 Spiral Waves

Let f be independent of t and let $\Phi_t(\cdot;\mu)$ be the autonomous time-evolution to (2). Suppose that for the parameter $\mu = 0$ Equation (2) has a rigidly rotating spiral wave solution $u^* \in Y$ rotating with the frequency ω^*. From a symmetry point of view a rigidly rotating spiral wave is a *rotating wave*, that is, it is stationary in a frame rotating with the frequency ω^*

$$-Au^* + f(u^*,0) - \omega^* R_3 u^* = 0. \tag{3}$$

In mathematical terms a meandering spiral wave is a *modulated rotating wave*, this means, it is periodic in a corotating frame. A drifting spiral wave is a *modulated travelling wave*, i.e., it is periodic in a comoving frame. A *modulated wave* is a solution of the fixed point equation

$$T_{a,\phi}^{-1}\Phi_p(u;\mu) = u. \tag{4}$$

Equation (4) defines a modulated travelling wave u drifting with velocity $v := \frac{a}{p}$ if $\phi = 0 \bmod 2\pi$ and $a \neq 0$. If $\phi \neq 0 \bmod 2\pi$, $a = 0$, it defines a modulated rotating wave. In general, a modulated rotating wave is a quasiperiodic solution of (2) with the frequencies $\Omega := 2\pi/p$ and $\omega := \frac{\phi}{p}$. If the rotation number $N_{\mathrm{rot}} := \omega/\Omega$ is rational then the modulated rotating wave is time-periodic. If $\frac{\phi}{2\pi} \notin \mathbb{Z}$ and u is a solution of (4) with $a \neq 0$ then $T_{(m_\phi-1)^{-1}a,0}u$ satisfies

$$T_{0,-\phi}\Phi_p(T_{(m_\phi-1)^{-1}a,0}u) = T_{(m_\phi-1)^{-1}a,0}u$$

because $T_{0,\phi}T_{a,0} = T_{m_\phi a,0}T_{0,\phi}$. Thus, $T_{(m_\phi-1)^{-1}a,0}u$ is a modulated rotating wave, which "meanders" around 0 and u is a modulated rotating waves meandering around $(1-m_\phi)^{-1}a$. So modulated waves are modulated rotating or modulated travelling waves. A modulated wave is fully characterized by the space-dependent concentration vector u, by the angle ϕ, the translation vector a, by its period p and the parameter μ for which it occurs in equation (2).

Define the $E(2)$-orbit of some $u \in Y$ as

$$O_{E(2)}(u) = \{T_{a,\phi}u, T_{a,\phi}T_\kappa u\}_{a\in\mathbb{R}^2,\phi\in[0,2\pi)}.$$

Modulated waves are also called *relative periodic orbits* because they are time-periodic solutions in the space of $E(2)$-orbits. A rotating wave is a *relative steady state* because it is stationary in the space of $E(2)$-orbits.

In order to fix one modulated wave out of its $E(2)$-orbit we define three linear conditions

$$l_i(u) = l_i(u^*),\ i = 1, 2, 3, \text{ with } \{l_i(R_j\, u^*)\}_{i,j} = \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker-symbol. These conditions determine a hyperplane Σ_l which has codimension 3, contains u^* and has the property that the $E(2)$-orbits through u^* are crossing Σ_l transversely near u^*. For instance we can choose the l_i as linear combinations of the functionals $w(x) \to \int_K R_i u^*(x) w(x) \mathrm{d}x$, $w \in BC_{\mathrm{Eucl}}$, where K is some bounded domain in $\mathbb{R}^2$.

4 Hopf Bifurcation to Modulated Waves

In this section we again require that f is independent of t and k-times continuously differentiable where $k \geq 2$. Moreover, we assume that μ is scalar and that (2) has a rotating wave solution u^* at $\mu = 0$. Let

$$L := -A + \frac{\partial f(u^*, 0)}{\partial u} - \omega^* R_3$$

denote the linearization of (3). Due to the Euclidean symmetry L always has eigenvalues on the imaginary axis:

$$L R_3 u^* = 0,\ L(R_1 + \mathrm{i} R_2) u^* = \mathrm{i}\omega^*(R_1 + \mathrm{i} R_2) u^*.$$

This follows from

$$L R u^* = R(-A u^* + f(u^*, 0)) - \omega^* R_3 R u^* = \omega^* [R, R_3] u^*$$

where R is R_1, R_2 or R_3 and $[\ ,\ ]$ is the commutator. Now we require that L has additional imaginary eigenvalues and that time is scaled such that the additional imaginary eigenvalues are lying on the unit circle. More precisely, we assume:

(i) $\pm\mathrm{i}$ are isolated eigenvalues of L and are simple unless $\omega^* = \pm 1$. If $\omega^* = \pm 1$ then $\pm\mathrm{i}$ have algebraic multiplicity 2. Furthermore 0 is a simple eigenvalue of L and the center-unstable eigenspace of L is 5-dimensional. Let P denote the spectral projection onto the center-unstable eigenspace of L. The spectrum of $L|_{(1-P)Y}$ has negative real part and is bounded away from the imaginary axis so that there are constants $\zeta > 0$, $M \geq 1$ with $\|\mathrm{e}^{Lt}|_{(1-P)Y}\| \leq M \mathrm{e}^{-\zeta t}$.

According to condition (i) we allow $j : 1$-resonances caused by symmetry where $j = \omega^* \in \mathbb{Z}$. If $\omega^* = 1$ then $\pm\mathrm{i}$ are double eigenvalues of L and generically L has Jordan blocks with eigenvalues $\pm\mathrm{i}$.

It can be proved that under this assumption we get a path rotating waves $u_{\text{rig}}(\mu)$ with frequency $\omega_{\text{rig}}(\mu)$ such that $u_{\text{rig}}(0) = u^*$, $\omega_{\text{rig}}(0) = \omega^*$ which is k-times continuously differentiable in μ and satisfies

$$-Au_{\text{rig}}(\mu) - \omega_{\text{rig}}(\mu) R_3 u_{\text{rig}}(\mu) + f(u_{\text{rig}}(\mu), \mu) = 0.$$

Also the center-unstable eigenvalues $\pm \mathrm{i}$, $\pm \mathrm{i}\omega^*$, 0 of L can be continued to eigenvalues $\beta(\mu)$, $\overline{\beta(\mu)}$, $\pm \mathrm{i}\omega_{\text{rig}}(\mu)$, 0 of

$$L(\mu) := -Au - \omega_{\text{rig}}(\mu) R_3 + \partial_u f(u_{\text{rig}}(\mu), \mu)$$

and $\pm \mathrm{i}\omega_{\text{rig}}(\mu)$, $\beta(\mu)$, $\overline{\beta(\mu)}$ are $(k-1)$-times continuously differentiable in μ. We need one more assumption before we state the theorem.

(ii) We require the usual transversality condition $\Re \beta'(0) \neq 0$.

Theorem 1 *Under conditions (i), (ii) a path of relative periodic orbits*

$$(u(s), a(s), \phi(s), p(s))$$

with the parameter $\mu(s)$, $s \in \mathbb{R}$, $|s|$ small, bifurcates from the rotating wave (u^*, ω^*) such that

$$u(0) = u^*, p(0) = 2\pi, \mu(0) = 0, a(0) = 0, \phi(0) = \omega^* 2\pi$$

and

$$l_i(z(s)) = 0,\ i = 1, 2, 3,\ z(s) = u(s) - u_{\text{rig}}(\mu(s)).$$

$a(s)$, $\phi(s)$, $p(s)$, $\mu(s)$ are $(k-1)$-times continuously differentiable in s and $T_{a,\phi} u(s)$ is $(k-1)$-times continuously differentiable in s, a, ϕ in the Y-norm. The relative periodic orbits are solutions of (4) and are locally unique except for the non-uniqueness caused by time-shift symmetry. The emanating modulated waves are asymptotically stable provided that $\Re \beta'(0) > 0$ and $s\mu'(s) > 0$. Furthermore $\mu(s)$, $p(s)$ and $\phi(s)$ are even in s and in general $s \approx |\mu|^{\frac{1}{2}}$ if $k > 2$.

In particular, the rotation number $N_{\text{rot}}(s) = \omega(s)/\Omega(s)$ where $\Omega(s) := 2\pi/p(s)$, $\omega(s) := \phi(s)/p(s)$ depends in a differentiable way on the parameter s, and therefore frequency-locking-effects do not occur which is in agreement with the observations from experiments, see Braune and Engel (1993a), Müller and Zykov (1994).

We use the canonical concept of nonlinear stability in autonomous equivariant systems, see Golubitsky et al. (1988): A solution $\{\Phi_t(u^*)\}_{t \in \mathbb{R}}$ is *stable* if for any $\epsilon > 0$ there is some $\vartheta > 0$ such that for each $u \in Y$ with $||u - u^*|| \leq \vartheta$ and for each $t \geq 0$

$$\inf_{(a,\phi,t_0) \in \mathbb{R}^4} ||T_{a,\phi} \Phi_t(u) - \Phi_{t+t_0}(u^*)|| \leq \epsilon.$$

The orbit $\{\Phi_t(u^*)\}_{t\in\mathbb{R}}$ is called asymptotically stable if it is stable and if there is some $\vartheta > 0$ such that for all $u \in Y$ with $\|u - u^*\| \leq \vartheta$

$$\inf_{(a,\phi,t_0)\in\mathbb{R}^4} \|T_{a,\phi}\Phi_t(u) - \Phi_{t+t_0}(u^*)\| \to 0 \text{ as } t \to \infty.$$

There is numerical evidence, see Barkley (1992), that the purely imaginary eigenvalues at the Hopf bifurcation point and the symmetry eigenvalues on the imaginary axis are isolated. Moreover, in numerical simulations the rest of the spectrum seems to be in the half plane $\Re\lambda < 0$ and is bounded away from the imaginary axis, even in an infinitely extended medium. Thus, our condition (i) seems to be reasonable.

The Hopf bifurcation to modulated waves is an ordinary Hopf bifurcation in the space of $E(2)$-orbits. Define the $E(2)$-reduced semiflow $\Psi_t(\cdot)$ by

$$\Psi_t(z,\mu) := T^{-1}_{a(z,t,\mu),\phi(z,t,\mu)}\Phi_t(u_{\text{rig}}(\mu) + z, \mu) - u_{\text{rig}}(\mu), \tag{6}$$

where $l_i(z) = 0$, $i = 1, 2, 3$, and $a(z, t, \mu)$, $\phi(z, t, \mu)$ are satisfying $a(0, t, \mu) = 0$, $\phi(0, t, \mu) = \omega_{\text{rig}}(\mu)t$ and are chosen such that $l_i(\Psi_t(z, \mu)) = 0$, $i = 1, 2, 3$. Under hypotheses (i), (ii) the semiflow $\Psi_t(z, \mu)$ undergoes an ordinary Hopf bifurcation in $\mu = 0$.

The theorem is proved by solving (4) with u near u^*, p near 2π, μ small. The main difficulty in the proof of the theorem is the non-differentiability of the group action: the ϕ-derivative and the a-derivative of (4) do not exist unless $R_i u \in Y$, $i = 1, 2, 3$. This is due to the fact that the time-evolution $\Phi_t(\cdot)$ only smoothens the solutions in space if $\delta > 0$ and it does not smoothen the action of the rotation at all. So we can not use a standard infinite-dimensional Lyapunov-Schmidt-reduction to resolve (4). To show the theorem a hierarchy of Banach spaces with increasing regularity of the $E(2)$-action is defined, see Wulff (1996). The details of the proof are technical and will be published elsewhere.

The Radius of the Spiral Tip Trajectory

There are various definitions of the spiral tip around, see e.g. Lugosi (1989), Jahnke et al. (1989). From the symmetry point of view the position of the spiral tip $x_{\text{tip}}(\cdot)$ is an $E(2)$-equivariant map from the function space Y into $\mathbb{R}^2$. The radius $r_{\text{tip}}(u)$ of the tip trajectory of a modulated spiral wave u is the radius of the polygone generated by

$$x_{\text{tip}}(\Phi_{\ell p}(u)),\ \ell = 0, 1, 2, \ldots.$$

Let m_ϕ be a rotation in the plane by the angle ϕ. For a modulated wave (u, a, ϕ) we have

$$r_{\text{tip}}(u) = |x_{\text{tip}}(u) + (m_\phi - 1)^{-1}a|. \tag{7}$$

From this equation we see that the radius of the tip motion tends to infinity if ϕ tends to zero and a keeps bounded away from 0. Moreover, outside a resonance the radius $r_{\text{tip}}(u(s))$ of the tip trajectory of the modulated waves obtained in Theorem 1 satisfies to first order in s

$$r_{\text{tip}}(u(s)) \leq r_{\text{tip}}(u^*) + s\left(c + \frac{\hat{c}}{\sin(\pi\omega^*)}\right) + O(s^2).$$

Bifurcation of Modulated Travelling Waves from a Resonant Hopf Point

In this section we study System (2) again, but now we assume that $\mu \in \mathbb{R}^2$. Let hypotheses (i), (ii) of Theorem 1 be satisfied and let $\omega^* \in \mathbb{Z}$. There is a k-times continuously differentiable surface $(u_{\text{rig}}(\mu), \omega_{\text{rig}}(\mu)) \subset Y \times \mathbb{R}$ of rotating waves near the rotating wave (u^*, ω^*), and the eigenvalue i of L can be continued in a $(k-1)$-times continuously diffrentiable way to an eigenvalue $\beta(\mu)$ of $L(\mu)$. Therefore (u^*, ω^*) lies on a k-times continuously differentiable curve of Hopf-points. We assume that after a transformation of the parameter vector μ the curve of Hopf-points in the μ-plane is equal to $\mu_1 = 0$.

Theorem 2 *Under the above assumptions and under the conditions*

$$\frac{\partial}{\partial\mu_1}\Re\beta(0) \neq 0, \quad \frac{\partial}{\partial\mu_2}\left(\frac{\omega_{\text{rig}}(0)}{\Im\beta(0)}\right) \neq 0 \tag{8}$$

there is a path $(u(s), a(s), p(s))$, $s \in \mathbb{R}$, $|s|$ small, of modulated travelling waves to the parameters $\mu(s)$ in the (μ_1, μ_2)-parameter plane satisfying

$$u(0) = u^*, \mu(0) = 0, p(0) = 2\pi, a(0) = 0$$

and

$$l_i(z(s)) = 0, \quad i = 1, 2, 3, \quad z(s) = u(s) - u_{\text{rig}}(\mu(s)).$$

The modulated travelling waves are solutions of (4) with $\phi \equiv 0$. Apart from the non-uniqueness caused by time-shift-symmetry the modulated travelling waves are locally unique. Moreover, $a(s)$, $p(s)$, $\mu(s)$ are $(k-1)$-times continuously differentiable in s and $T_{a,\phi}u(s)$ is $(k-1)$-times continuously differentiable in a, ϕ, s in the Y-norm. The functions $\mu(s)$ and $p(s)$ are even in s and $\frac{\partial^i}{\partial s^i}a(0) = 0$ for $i < \omega^*$, $i \leq k-1$.

The transversality conditions (8) are quite natural hypotheses because they have the following meaning: The first condition is the usual Hopf condition that the eigenvalues at the resonant Hopf point u^* are crossing the imaginary axis with non-zero speed; the second condition means that the imaginary eigenvalues on the Hopf-curve are passing through the resonance $\omega^*/\Im\beta(0) \in \mathbb{Z}$ with non-zero speed.

Proof of the Scaling Law for the Drift Velocity

We parametrize the modulated waves locally by circles, i.e.,

$$u(s_1, s_2) = u_{\text{rig}}(\mu(s)) + z(s_1, s_2),\ \ s_1 = s\cos t, s_2 = s\sin t,$$

where

$$z(s_1, s_2) = \Psi_{tp(s)/2\pi}(z(s); \mu(s))\ \ l_i(z(s_1, s_2)) = 0,\ i = 1, 2, 3.$$

There are $(k-1)$-times differentiable functions $\tilde{a}(t, s)$, $\tilde{\phi}(t, s)$ such that

$$u(s_1, s_2) = T_{\tilde{a}(t,s),\omega^* t + \tilde{\phi}(t,s)} \Phi_{tp(s)/2\pi}(u(s), \mu(s)).$$

Obviously, $\tilde{a}(t, 0) = 0$, $\tilde{\phi}(t, 0) = 0$. Define $a(s_1, s_2)$ by the equation

$$\Phi_{tp(s)/2\pi}(u(s_1, s_2), \mu(s)) = T_{a(s_1,s_2)} u(s_1, s_2)$$

$u(s_1, s_2)$ and also the corresponding translation vector $a(s_1, s_2)$ are $(k-1)$-times continuously differentiable in s_1, s_2. It holds

$$a(s_1, s_2) = m_{\omega^* t + \tilde{\phi}(t,s)} a(s). \tag{9}$$

From the Taylor-expansion of (9)

$$a(s_1, s_2) = m_{\omega^* t + s\hat{\phi}(t,s)} \sum_{i=1}^{k-1} a_i s^i,\ \ \ \tilde{\phi}(t, s) = s\hat{\phi}(t, s)$$

we conclude that $\frac{\partial^i}{\partial s^i} a(0) = 0$, for $i < \omega^*$, $i \leq k-1$. □

5 Periodic Forcing of Rotating Waves

In this section we assume that the nonlinearity from (2) is of the form $f(u, t, \mu) = \mu \hat{f}(u, t)$ where $\mu \in \mathbb{R}$ and $f(u, \cdot)$ is 2π-periodic in t and k-times differentiable in its variables, $k \geq 1$. Denote by $\Phi_{t,t_0}(\cdot; \Omega, \mu)$ the time-evolution to the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + \mu \hat{f}(u, \Omega t). \tag{10}$$

Clearly μ is the amplitude of the periodic forcing and Ω is its frequency. Similarly as in autonomous systems the equation

$$T^{-1}_{a,\phi} \Phi_{\frac{2\pi}{\Omega},0}(u; \Omega, \mu) = u \tag{11}$$

defines a modulated wave (u, a, ϕ) of (10) to the parameters (Ω, μ). Assume that (10) has a rotating wave solution (u^*, ω^*) to the parameter value $\mu = 0$ which is linearly stable, i.e., the center-unstable eigenspace of L is spanned by $R_i u^*$, $i = 1, 2, 3$ and there are some constants $\zeta > 0$, $M \geq 1$ such that

$\|e^{Lt}|_{(1-P)Y}\| \leq M e^{-\zeta t}$ where P is the spectral projection onto the center-unstable eigenspace of L. The following theorem shows that a periodic forcing of rotating waves generates modulated waves and that modulated travelling waves emanate if the frequency of the rotating wave is a multiple of the external frequency. This is in agreement with experiments, see Braune and Engel (1993b), Nettesheim et al. (1993).

Theorem 3 *For μ near 0 there is a k-times differentiable surface*

$$(u(\Omega,\mu), a(\Omega,\mu), \phi(\Omega,\mu))$$

of asymptotically stable modulated waves of (10) to the parameters (Ω,μ) such that $u(\Omega,0) = u^*$, $a(\Omega,0) = 0$, $\phi(\Omega,0) = 2\pi\frac{\omega^*}{\Omega^*}$, $l_i(u(\Omega,\mu) - u^*) = 0$, $i = 1,2,3$. The modulated waves are solutions of (11).

Let ω^* be a multiple of Ω^*. Then for small μ there is a k-times differentiable path

$$(u(\mu), a(\mu), \Omega(\mu))$$

of asymptotically modulated travelling waves of (10) to the parameters μ and $\Omega(\mu)$ such that $\Omega(0) = \Omega^*$, $u(\Omega^*,0) = u^*$, $a(\Omega^*,0) = 0$, $l_i(u(\mu) - u^*) = 0$, $i = 1,2,3$. The modulated travelling waves are solutions of (11) with $\phi \equiv 0$.

The proof of the theorem is similar to the proof of Theorem 1 and will be published elsewhere. If we choose a harmonic periodic forcing $f(u,\Omega t) = h(u)\cos\Omega t$ then the angle $\phi(\mu)$ of the modulated waves is even in μ. The drift velocity $v(\mu) = a(\mu)\Omega(\mu)/2\pi$ of the emanating modulated travelling waves grows like μ if $\Omega^* = \omega^*$ and like μ^2 if $\omega^*/\Omega^* \geq 2$. This can be proved by differentiating $T^{-1}_{a(\mu),0}\Phi_{\frac{2\pi}{\Omega(\mu)},0}(u(\mu);\Omega(\mu),\mu) = u(\mu)$ once respectively twice with respect to μ. Outside a resonance the radius $r_{\rm tip}(u(\mu))$ of the spiral tip trajectory satisfies to first order in μ

$$r_{\rm tip}(u(\mu)) \leq r_{\rm tip}(u^*) + \mu\left(c + \frac{\hat{c}}{\sin(\frac{\pi\omega^*}{\Omega^*})}\right) + O(\mu^2).$$

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