MAT3011 Galois Theory Revision Questions

The test will cover Chapters 1 and 2 of the notes.
The questions will be largely similar to ones on this sheet.

1. Let $\mathrm{f} = t^3 - 2t + 2 \in \mathbb{Q}[t]$.

   (a) Write down the Vieta substitution which transforms the equation $\mathrm{f}(x) = 0$ into a quadratic in z^3.

   (b) Without solving the equation, state how many real zeros f has.

   (c) State what is meant by the *splitting field* of f over $\mathbb{Q}$. Determine whether this field can be obtained by adjoining any one zero of f to $\mathbb{Q}$.

   (d) State whether f has a constructible real zero.

   (e) Find the sum of the squares of the zeros of f.

2. You are given (and will be given in the test and exam, if needed) that the cubic resolvent of t^4+ct^2+dt+e is $t^3-2ct^2+(c^2-4e)t+d^2$, and that $t^4 + ct^2 + dt + e = (t^2 + kt + \ell)(t^2 - kt + m)$ where $\ell = \frac{1}{2}\left(k^2 + c - \frac{d}{k}\right)$, $m = \frac{1}{2}\left(k^2 + c + \frac{d}{k}\right)$ and $-k^2$ is a zero of the cubic resolvent.

   Let $\mathrm{f} = t^4 - 2t^2 + 12t - 18 \in \mathbb{Q}[t]$ and let ρ be the cubic resolvent of f.

   Find ρ and show that $\rho(-2) = 0$. Hence find expressions in terms of radicals for the four zeros of f in $\mathbb{C}$. State how many of these zeros are in $\mathbb{R}$.

3. Let $\mathrm{f} = t^n - nt^{n-1} - n \in \mathbb{Q}[t]$, where $n \in \mathbb{N}$. Use the formal derivative Df to show that f has no repeated zeros.

4. Give an example of each of the following. Justify your answers clearly.

   (a) Finite fields L and K such that $L : K$ is a finite extension.

   (b) A field $L \subset \mathbb{C}$ such that $L : \mathbb{Q}$ is not an algebraic extension.

   (c) A field $L \subset \mathbb{C}$ such that $[L : \mathbb{Q}] = 3$ and $L : \mathbb{Q}$ is not a normal extension.

5. Let α be the real number $\sqrt{\sqrt{11}-1}$.

   (a) Find μ, the minimal polynomial of α over $\mathbb{Q}$.

   (b) State the value of $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and give a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

   (c) Show that $\mathbb{Q}(\alpha)$ is not the splitting field of μ over $\mathbb{Q}$.

6. Let $m, n \in \mathbb{Q}^+$, $\alpha = \sqrt{m}$, $\beta = \sqrt{n}$ and $\gamma = \alpha(\beta + 1)$.
   By considering γ^2, show that $\beta \in \mathbb{Q}(\gamma)$ and deduce that $\alpha \in \mathbb{Q}(\gamma)$.
   Hence show that $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\gamma)$.

Answers

1. (a) The Vieta substitution is $x = z + \dfrac{2}{3z}$.

   (b) $\Delta(\mathrm{f}) = -4p^3 - 27q^2 = -76 < 0$ so f has one real zero.

   (c) The splitting field of f is the field $L \subset \mathbb{C}$ such that f splits over L but not over any proper subfield of L.
   Since $\delta(\mathrm{f}) \notin \mathbb{Q}$, the splitting field is not $\mathbb{Q}(\alpha)$ for any zero α of f.

   (d) f is irreducible by EIC with $p = 2$. If α is a zero of f then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \partial \mathrm{f} = 3 \neq 2^m$ for any $m \in \mathbb{N}$, so α is not constructible.

   (e) Let α, β and γ be the zeros of f. Then $\alpha + \beta + \gamma = 0, \alpha\beta + \beta\gamma + \gamma\alpha = -2$ so $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 4$.

2. The cubic resolvent is $\rho = t^3 + 4t^2 + 76t + 144$.
   $\rho(-2) = -8 + 16 - 152 + 144 = 0$ so take $-k^2 = -2$, i.e. $k = \sqrt{2}$.
   Then $\ell = \dfrac{1}{2}\left(2 - 2 - \dfrac{12}{\sqrt{2}}\right) = -3\sqrt{2}$, $m = \dfrac{1}{2}\left(2 - 2 + \dfrac{12}{\sqrt{2}}\right) = 3\sqrt{2}$
   so $\mathrm{f} = (t^2 + \sqrt{2}\,t - 3\sqrt{2})(t^2 - \sqrt{2}\,t + 3\sqrt{2})$.
   Thus the zeros of f are $\dfrac{-\sqrt{2} \pm \sqrt{2 + 12\sqrt{2}}}{2}$, $\dfrac{\sqrt{2} \pm \mathrm{i}\sqrt{12\sqrt{2} - 2}}{2}$.
   Two of these are real.

3. $\mathrm{Df} = nt^{n-1} - n(n-1)t^{n-2}$. If $\mathrm{Df}(x) = 0$ then $nx^{n-2}(x - (n-1)) = 0$, so $x = 0$ or $x = n - 1$. Now $\mathrm{f}(0) = -n \neq 0$
   and $\mathrm{f}(n-1) = (n-1)^n - n(n-1)^{n-1} - n = (n-1)^{n-1}(n-1-n) - n = -(n-1)^{n-1} - n < 0$.
   Hence f and Df have no common zeros, so f has no repeated zeros.

4. (a) $\mathbb{F}_2$ and $\mathbb{F}_4$ are finite fields. $[\mathbb{F}_4 : \mathbb{F}_2] = 2$, so $\mathbb{F}_4 : \mathbb{F}_2$ is a finite extension.

   (b) π is transcendental over $\mathbb{Q}$, so $\mathbb{Q}(\pi) : \mathbb{Q}$ is not an algebraic extension.

   (c) $\mathbb{Q}(2^{1/3}) : \mathbb{Q}$ is a finite extension as $[\mathbb{Q}(2^{1/3}) : \mathbb{Q}] = 3$, but is not a normal extension since $t^3 - 2$ has one zero in $\mathbb{Q}(2^{1/3})$ but does not split over $\mathbb{Q}(2^{1/3})$.

5. (a) $\alpha^2 + 1 = \sqrt{11}$ so $\alpha^4 + 2\alpha^2 - 10 = 0$.
Let $\mu = t^4 + 2t^2 - 10$. Then μ is monic, and irreducible over $\mathbb{Q}$ by EIC with $p = 2$, so it is the minimal polynomial of α over $\mathbb{Q}$.

(b) $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \partial\mu = 4$. A basis is $\{1, \alpha, \alpha^2, \alpha^3\}$.

(c) If $\mu(x) = 0$ then $(x^2 + 1)^2 = 11$, so $x^2 + 1 = \pm\sqrt{11}$. Thus the zeros of μ are $\pm\alpha, \pm\beta$ where $\beta = \mathrm{i}\sqrt{\sqrt{11} + 1}$.
$\mathbb{Q}(\alpha) \subset \mathbb{R}$ and $\beta \notin \mathbb{R}$, so $\beta \notin \mathbb{Q}(\alpha)$. Thus $\mathbb{Q}(\alpha)$ is not the splitting field of μ over $\mathbb{Q}$.

6. $\gamma^2 = \alpha^2(\beta^2+2\beta+1) = m(n+2\beta+1)$, so $\beta = \dfrac{1}{2}\left(\dfrac{\gamma^2}{m} - n - 1\right)$. This is obtained from γ and rationals by field operations, so $\beta \in \mathbb{Q}(\gamma)$.
Then $\alpha = \dfrac{\gamma}{\beta + 1} \in \mathbb{Q}(\gamma)$.
Thus $\mathbb{Q}(\alpha, \beta) \subset \mathbb{Q}(\gamma)$. Also $\mathbb{Q}(\gamma) \subset \mathbb{Q}(\alpha, \beta)$ since $\gamma = \alpha(\beta + 1)$, so $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\gamma)$.