Complex Contact and Lift Transformations

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Abstract

We study mappings from sets of real variables into complex variables, which extend features of lift and contact transformations between real variables that we explored in a previous paper. In particular the relationship between lifts in $\mathbb{R}^{2n+1}$ and the Cauchy-Riemann equations for functions of n complex variables is discussed. Explicit examples are given to illustrate the anatomy of such transformations, including the occurrence of singularities. Applications to nonlinear partial differential equations arising in fluid mechanics are presented.

Key words: Lift transformation; Contact transformation; Complex variables; Cauchy-Riemann equations; Singularity; Monge-Ampère equation.

1 Introduction

The aim of this paper is to extend, to complex variables, some ideas and properties of lift and contact transformations which we explored for real variables in a previous paper (Sewell and Roulstone 1994, called SR94 subsequently here). Several explicit examples are provided to show how real features, such as cubic curves and surfaces with inflexions, map to complex functions whose real and imaginary parts reveal the presence of singularities associated with cusps and inflexions.

The Legendre transformation has been applied to a wide variety of situations, in mechanics in particular, and a review of some of these can be found in an article by Sewell (1982) in The Rodney Hill 60$^{\text{th}}$ Anniversary Volume. Further applications have been described in a subsequent book (Sewell 1987). One of the features of the Legendre transformation is a description of the duality between functions in terms of the pole-polar relationship. The Legendre transformation also figures in the description of lift and contact transformations for real variables provided by SR94 and Sewell (2002).

The studies in geophysical fluid dynamics by McIntyre and Roulstone (2002) have shown how a class of lift transformations (there phrased in the wider context of contact transformations) between real and complex variables illuminates connections between features such as Hamiltonian structure and the Monge-Ampère equations. Such interconnections are at

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the heart of certain solution strategies (cf. Purser 2002, Cullen 2006).

The work by Roubtsov and Roulstone (2001) and McIntyre and Roulstone (2002) has been developed by Delahaies (2009) and Delahaies and Roulstone (2010), to show how lift and contact transformations with complex variables play an important role in the transformations of Monge-Ampère equations in problems relating to dynamical meteorology. These results motivate the present study and, with applications to fluid mechanics in mind, our presentation is intended to be accessible to a broad readership. However, there are clear links with several specialized areas of differential geometry, including Lagrangian, and Special Lagrangian, submanifolds of n-dimensional complex manifolds. We shall discuss some of these connections in the Summary.

Section 2 is a résumé of basic results described in SR94. The main results of this present paper are contained in Theorem 1 (Section 3) and Theorem 2 (Section 8). Theorem 1 treats the three complex variables X, Y, Z defined, in (11) below, in terms of a pair of real variables x, z. The Cauchy-Riemann equations are derived in (15), and several explicit examples of complex lift transformations involving trios of real and complex variables are given in Sections 4 and 5. In the examples which we give, the substance is found to be the relation between the lift property and a holomorphic property of $Y(X)$. Singularities associated with these examples are illustrated, and in Section 6 we derive an involutive property of the lift transformation from real to complex variables. A Legendre transformation from real to complex variables is presented in Section 7. In Section 8, Theorem 2 treats the $2n+1$ complex variables X_i, Y, Z_i defined, in (61) below, in terms of $2n$ real variables x_i, z_i. Theorem 2 contains a pair of n-tuples of equations (67), and each pair includes Cauchy-Riemann equations. Again, the substance is found to be the relation between the lift property and a holomorphic property of $Y(X_j)$. An example to accompany these results is presented in Section 9. In Section 10 we apply lift transformations with complex variables to nonlinear partial differential equations. A summary is presented in Section 11.

2 Basic concepts

It is helpful to have in mind, throughout this investigation, some basic ideas of real geometry in $\mathbb{R}^3$ which were the subject of the previous paper SR94. These concern the lift of a plane curve in $\mathbb{R}^2$ into a space curve in $\mathbb{R}^3$ and the contact in $\mathbb{R}^2$ of a pair of such plane curves, whose lifts intersect when the height of the lift is defined to be the gradient of the plane curve. Our general objective in the present paper is to provide understanding of how such ideas extend to the geometry of complex variables.

A plane curve in $\mathbb{R}^2$, spanned by cartesian coordinates x and y, can often be written $x = x(u), y = y(u)$ in terms of functions of an intermediate parameter u. It may be *lifted* into a space curve in $\mathbb{R}^3$ by nominating *any* other function $z(u)$. The plane curve may then also be called the *projection* of the space curve.

It is useful in the subject of contact geometry to make the particular definition that the function $z(u)$ is $z = (\mathrm{d}y/\mathrm{d}u)/(\mathrm{d}x/\mathrm{d}u)$, which is $z = \mathrm{d}y/\mathrm{d}x$ at *regular* points where $\mathrm{d}x/\mathrm{d}u \neq 0$. Then the height z of the lifted curve is the gradient of the plane curve there.

This explicit definition of z ensures that if two plane curves have a point where they are in contact in the sense of having a common tangent, their two lifts will be space curves which do intersect at the height z above the contact point.

A *transformation* of $\mathbb{R}^3$ to $\mathbb{R}^3$ defined as a mapping

$$X = X(x, y, z), \quad Y = Y(x, y, z), \quad Z = Z(x, y, z), \tag{1}$$

with functions as indicated on the right, will convert any space curve

$$x = x(u), \quad y = y(u), \quad z = z(u) \tag{2}$$

(whether $z = \mathrm{d}y/\mathrm{d}x$ or not) into another space curve

$$X = X(u), \quad Y = Y(u), \quad Z = Z(u). \tag{3}$$

If *both* (2) and (3) are lifted curves, so that $z = \mathrm{d}y/\mathrm{d}x$ and $Z = \mathrm{d}Y/\mathrm{d}X$ at regular points, we can then call (1) a *lift transformation* connecting them. When it is reversible, the inverse will have the form

$$x = x(X, Y, Z), \quad y = y(X, Y, Z), \quad z = z(X, Y, Z). \tag{4}$$

Thus a lift transformation converts one lifted curve into another. This will happen in particular if (1) and/or (4) have the property

$$\mathrm{d}Y - Z\mathrm{d}X = \beta(\mathrm{d}y - z\mathrm{d}x) \tag{5}$$

with a finite and non-zero number β, because then *both* of

$$\mathrm{d}Y - Z\mathrm{d}X = 0 = \mathrm{d}y - z\mathrm{d}x \tag{6}$$

can be satisfied simultaneously. In SR94 we gave several examples of general lift transformations between *real* variables which have these properties.

When a lifted curve is defined by $y(x)$ with $z = \mathrm{d}y/\mathrm{d}x$, its transform by (1) will have the particular form

$$X(x) = X(x, y(x), z(x)), \quad Y(x) = Y(x, y(x), z(x)), \quad Z(x) = Z(x, y(x), z(x)) \tag{7}$$

with that particular $z(x)$. The purpose now is to explore examples of transformations between *complex* variables which have the same algebraic lift transformation property (5) as we have just described for real variables.

We prefer to use the adjective "lift" when referring to a single curve, and "contact" for a pair of curves. It is not uncommon in the literature to find "contact" used for a single curve, but strictly speaking that is a misnomer.

3 Lift transformations with complex variables

In what follows we use $i = \sqrt{-1}$ to introduce some explicit complex variables, and we explore associated lift properties. With real x, y, z we nominate six real functions

$$\hat{a}(x,y,z),\ \ \hat{b}(x,y,z),\ \ \hat{u}(x,y,z),\ \ \hat{v}(x,y,z),\ \ \hat{g}(x,y,z),\ \ \hat{h}(x,y,z). \tag{8}$$

We use them to redefine the real X, Y, Z in Section 2 as new and *complex* functions

$$X = \hat{a} + i\hat{b},\ \ Y = \hat{u} + i\hat{v},\ \ Z = \hat{g} + i\hat{h}. \tag{9}$$

We next introduce a real plane curve $y = y(x)$ into (8), which creates six new real functions each of the two variables x and z. We write these functions as $a(x,z) = \hat{a}(x, y(x), z)$ and similarly for the other five functions, to achieve

$$a(x,z),\ \ b(x,z),\ \ u(x,z),\ \ v(x,z),\ \ g(x,z),\ \ h(x,z), \tag{10}$$

but without at first supposing that $z = \mathrm{d}y/\mathrm{d}x$. Then (9) becomes

$$X = a(x,z) + ib(x,z),\ \ Y = u(x,z) + iv(x,z),\ \ Z = g(x,z) + ih(x,z). \tag{11}$$

We now introduce the additional assumption that the jacobian

$$\frac{\partial(a,b)}{\partial(x,z)} \neq 0,\ \ \pm\infty \tag{12}$$

for the two functions $a(x,z)$ and $b(x,z)$. Then we can use the implicit function theorem to construct the inverse functions $x(a,b)$ and $z(a,b)$, and use them to rewrite (11) as

$$\tilde{X} = a + ib,\ \ \tilde{Y} = \tilde{u}(a,b) + i\tilde{v}(a,b),\ \ \tilde{Z} = \tilde{g}(a,b) + i\tilde{h}(a,b), \tag{13}$$

thus producing new functions of a and b, having the same values as those in (11), i.e. $\tilde{X} = X,\ \ \tilde{Y} = Y,\ \ \tilde{Z} = Z$ and $\tilde{u} = u,\ \ \tilde{v} = v,\ \ \tilde{g} = g,\ \ \tilde{h} = h$.

Now we can rewrite the invariance of the differential form (5) as

$$\beta(\mathrm{d}y - z\mathrm{d}x) = \mathrm{d}\tilde{Y} - \tilde{Z}\mathrm{d}\tilde{X} = \mathrm{d}\tilde{u} + i\mathrm{d}\tilde{v} - (\tilde{g} + i\tilde{h})(\mathrm{d}a + i\mathrm{d}b) = \mathrm{d}\tilde{u} - \tilde{g}\mathrm{d}a + \tilde{h}\mathrm{d}b + i(\mathrm{d}\tilde{v} - \tilde{h}\mathrm{d}a - \tilde{g}\mathrm{d}b). \tag{14}$$

This approach delivers the following result whenever $\beta \neq 0$.

Theorem 1 *($\tilde{Y}(\tilde{X})$ is holomorphic on the lift of $y(x)$)*
At any point with height $z = \mathrm{d}y/\mathrm{d}x$ on the lift of the real curve $y = y(x)$, we deduce from (14) that the functions $\tilde{Y}(\tilde{X})$ and $\tilde{Z}(\tilde{X})$ implied by (13) satisfy $\mathrm{d}\tilde{Y} = \tilde{Z}\mathrm{d}\tilde{X}$ and therefore

$$\tilde{g} = \frac{\partial\tilde{u}}{\partial a} = \frac{\partial\tilde{v}}{\partial b},\ \ \tilde{h} = -\frac{\partial\tilde{u}}{\partial b} = \frac{\partial\tilde{v}}{\partial a}. \tag{15}$$

The Cauchy-Riemann equations included here (on the right of each pair) are the defining requirement for the function $\tilde{Y}(\tilde{X})$ to be holomorphic. (If β is real, the properties of $\tilde{v}$ deduced in (15) follow at once from (14), without reference to any lift hypothesis.) We say that (13) facilitates the construction of a holomorphic complex lift transformation from the real lift $z = \mathrm{d}y/\mathrm{d}x$ to the complex lift $\tilde{Z} = \mathrm{d}\tilde{Y}/\mathrm{d}\tilde{X}$.

Then any plane curve $y(x)$ induces, via the lift $z = \mathrm{d}y/\mathrm{d}x$, a curve in the (a, b)-plane (i.e. in the complex $\tilde{X}$-plane), because x can be eliminated between $\hat{a}(x, y(x), z(x))$ and $\hat{b}(x, y(x), z(x))$ in $(9)_1$ (i.e. between $a(x, z(x))$ and $b(x, z(x))$ in $(11)_1$), thus delivering a curve $\hat{b}(\hat{a})$ (and therefore $b(a)$). The first two equations of (13) define, implicitly, a transformation from the $\tilde{X}$ to the $\tilde{Y}$ plane, so that the real curve $b(a)$ has an image $\tilde{v}(\tilde{u})$ in the $\tilde{Y}$-plane. We shall give some explicit examples of such transformations in Section 5.

4 Families of lift transformations

In SR94 our focus was on a class of real functions of real variables, and that was the ostensible context in which the theorems were proved. They arose from our interest in the semi-geostrophic equations of meteorology. The families of lift transformations there were found by seeking solutions of $\alpha + z\beta = 0$ with $\gamma = 0$ among families of quadratics $X(x, y, z)$ and $Y(x, y, z)$, and of quotients $Z(x, y, z)$ of linear forms, all for real variables, where

$$\alpha = \frac{\partial Y}{\partial x} - Z\frac{\partial X}{\partial x}, \quad \beta = \frac{\partial Y}{\partial y} - Z\frac{\partial X}{\partial y}, \quad \gamma = \frac{\partial Y}{\partial z} - Z\frac{\partial X}{\partial z}. \tag{16}$$

As Roubtsov and Roulstone (2001) showed, there is a natural generalization of this class of real functions to functions of complex variables.

With this in mind, we shall explore the following example of (43) of SR94:

$$X = Px+z, \quad Y = \frac{1}{2}(cP-1)Px^2+\frac{1}{2}cz^2+(cP-1)zx+rPx+y+rz, \quad Z = (cP-1)x+cz+r. \tag{17}$$

It is straightforward to verify that every such transformation has the differential invariant property

$$\mathrm{d}Y - Z\mathrm{d}X = \mathrm{d}y - z\mathrm{d}x. \tag{18}$$

This is true irrespective of whether the trios (x, y, z) and (X, Y, Z) are real or complex, and any or all of the constant coefficients P, c, r may likewise be real or complex. In Section 5 we use (17) as a basis for studying properties of lift transformations from real to complex variables. In Section 5.4 we discuss an example from SR94 for which (18) is satisfied, but (12) is not (and hence this example does not correspond to a complex lift transformation as defined in Section 3).

Another, and different, family of transformations which have the property (18), and the consequences just described, is

$$X = x + Qz, \quad Y = y + \frac{1}{2}Qz^2, \quad Z = z \tag{19}$$

in which the variables and the arbitrary constant Q may be real or complex. This family does not belong to the family in (17). We explore an example of it in Section 5.3, which arises from a problem in geophysical fluid mechanics.

We notice that the special case of (17) in which $c = P = 1$ and $r = 0$, or of (19) with $Q = 1$, namely

$$X = x + z, \ \ Y = y + \frac{1}{2}z^2, \ \ Z = z, \tag{20}$$

is the geostrophic momentum transformation of semi-geostrophic theory in meteorology, which has been exploited by Hoskins (1975).

5 Dualities between plane curves I

In this Section we give three examples of lift transformations between plane curves in $\mathbb{R}^2$ and their image in $\mathbb{C}$. We also give a counter-example in which a lift transformation, i.e. one satisfying (5), does not satisfy (12).

5.1 Example 1

To illustrate this extension of the application of (17) to complex variables, the structure of (17) invites the choice $P = i$ so that we have the particular case

$$X = z+ix, \ \ Y = -\frac{1}{2}(c+i)x^2+\frac{1}{2}cz^2+(ic-1)zx+rix+y+rz, \ \ Z = (ic-1)x+cz+r, \tag{21}$$

in which c and r could still be real or complex.

To obtain an explicit example of (11) we will assume that c and r are real, so that the real functions in (11) are

$$a = z, \ \ b = x, \ \ u = \frac{1}{2}c(z^2-x^2)-zx+y(x)+rz, \ \ v = -\frac{1}{2}x^2+czx+rx, \ \ g = -x+cz+r, \ \ h = cx. \tag{22}$$

Then $\partial(a,b)/\partial(x,z) = -1$, thus satisfying (12) so that we can construct an example of (13) with

$$\tilde{u} = \frac{1}{2}c(a^2 - b^2) - ab + y(b) + ra, \ \ \tilde{v} = -\frac{1}{2}b^2 + cab + rb, \tag{23}$$

which are the components of $\tilde{Y}$, and

$$\tilde{g} = -b + ca + r, \ \ \tilde{h} = cb, \tag{24}$$

which are the components of $\tilde{Z}$. Then (15) in Theorem 1 become

$$-b + ca + r = ca - b + r = -b + ca + r, \ \ cb = cb + a - \frac{\mathrm{d}y}{\mathrm{d}b} = cb. \tag{25}$$

Evidently these four equations are examples of (15) which display, in this case, three identities and just one non-trivial remaining Cauchy-Riemann equation $a = \mathrm{d}y/\mathrm{d}b$, which expresses the lift property that a is the height of the lift of $y(b)$. This is what Theorem 1 tells us in this example.

5.2 Example 2: duality of a cubic

A particular set of lift transformations contained within the family (17) is obtained by choosing $c = r = 0$ there, which gives

$$X = Px + z,\ \ Y = -\frac{1}{2}Px^2 - zx + y,\ \ Z = -x. \tag{26}$$

(i) If we choose $P = i$ and $y(x) = \frac{1}{3}x^3$ in (26) we obtain the mapping

$$X = z + ix,\ \ Y = \frac{1}{3}x^3 - xz - \frac{i}{2}x^2,\ \ Z = -x. \tag{27}$$

This is an example of (11) in which

$$a = z,\ \ b = x,\ \ u = \frac{1}{3}x^3 - xz,\ \ v = -\frac{1}{2}x^2,\ \ g = -x,\ \ h = 0 \tag{28}$$

for which the jacobian in (12) is $\partial(a, b)/\partial(x, z) = -1$ so that we can construct an example of (13), namely

$$\tilde{X} = a + ib,\ \ \tilde{Y} = \frac{1}{3}b^3 - ab - \frac{i}{2}b^2,\ \ \tilde{Z} = -b. \tag{29}$$

This provides another example of Theorem 1. That is, at any point with height $z = \mathrm{d}y/\mathrm{d}x = x^2$ on the lift of $y = \frac{1}{3}x^3$, (15) apply in the form

$$-b = -b = -b,\ \ 0 = -b^2 + a = 0. \tag{30}$$

Therefore the Cauchy-Riemann equations are satisfied on the parabola $a = b^2$ which can be written from (28) as $z = x^2$, which is the height of the lift of $y = \frac{1}{3}x^3$.

Furthermore we can see from (13) and (29) that on this lift, $a = b^2$ implies

$$\tilde{u} = \frac{1}{3}b^3 - ab = -\frac{2}{3}b^3,\ \ \tilde{v} = -\frac{1}{2}b^2 \tag{31}$$

so that

$$\tilde{u} = \mp\frac{2}{3}(-2\tilde{v})^{\frac{3}{2}} \tag{32}$$

there. This curve in $(\tilde{u}, \tilde{v})$ space has, as shown in Figure 1 on the left, an upward pointing cusp (in the direction of the positive $\tilde{v}$ axis) at the origin, and elsewhere has gradient

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{u}} = \frac{\frac{\mathrm{d}\tilde{v}}{\mathrm{d}b}}{\frac{\mathrm{d}\tilde{u}}{\mathrm{d}b}} = \frac{-b}{-2b^2} = \frac{1}{2b} = \pm\frac{1}{2(-2\tilde{v})^{\frac{1}{2}}}. \tag{33}$$

(ii) The special choice $P = 0$ in (26) gives the real transformation

$$X = z,\ \ Y = y - zx,\ \ Z = -x \tag{34}$$

whose invariant $\mathrm{d}Y - Z\mathrm{d}X = \mathrm{d}y - z\mathrm{d}x$ is zero on the lifts $z = \mathrm{d}y/\mathrm{d}x$ and $Z = \mathrm{d}Y/\mathrm{d}X$ of $y(x)$ and $Y(X)$. There we also have the Legendre transformation properties

$$X = \frac{\mathrm{d}y}{\mathrm{d}x},\ \ x = \frac{\mathrm{d}(-Y)}{\mathrm{d}X},\ \ Xx = y + (-Y). \tag{35}$$

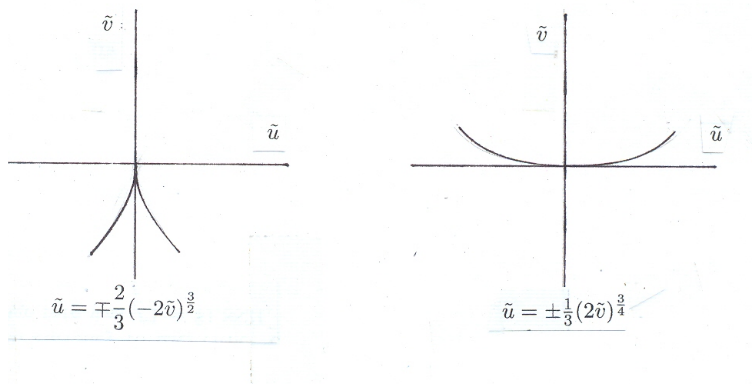

Figure 1: Curves in $(\tilde{u}, \tilde{v})$ space

5.3 Example 3: a canonical example from Delahaies & Roulstone (2010)

In Section 4 we cited two different general families of lift transformations, namely (17) and (19). We have just given two illustrations of (17). Turning now to (19), we note this is a canonical example from Delahaies & Roulstone (2010), in the sense that it is the basis of the generalization of the geostrophic momentum transformation, (20), to complex variables. This complex version (in higher dimensions — cf. Section 8) was shown by Delahaies & Roulstone to be part of the hyper-Kähler structure of the semi-geostrophic equations.

Choosing $Q = i$ gives

$$X = x + iz, \quad Y = y + \frac{i}{2}z^2, \quad Z = z \tag{36}$$

which has the property $\mathrm{d}Y - Z\mathrm{d}X = \mathrm{d}y - z\mathrm{d}x$ so that $\beta = 1$.

This is a different illustration of (9) from those previously considered, and it can be seen to be a consequence of choosing $\hat{a} = x$, $\hat{b} = z$, $\hat{u} = y$, $\hat{v} = \frac{1}{2}z^2$, $\hat{g} = z$, $\hat{h} = 0$.

Then any real plane curve $y = y(x)$ provides an example of (11) with

$$a = x, \quad b = z, \quad u = y(x), \quad v = \frac{1}{2}z^2, \quad g = z, \quad h = 0. \tag{37}$$

This provides an example of (13) with

$$\tilde{X} = a + ib, \quad \tilde{Y} = y(a) + \frac{i}{2}b^2, \quad \tilde{Z} = b \tag{38}$$

so that $\tilde{u} = y(a)$, $\tilde{v} = \frac{1}{2}b^2$, $\tilde{g} = b$, $\tilde{h} = 0$ and (15) become

$$b = \frac{\mathrm{d}y}{\mathrm{d}a} = b, \quad 0 = -0 = 0. \tag{39}$$

Again we have a (different) trio of identities, and one lift statement.

A particular example is provided by the cubic $y = \frac{1}{3}x^3$ whose lift is $z = x^2$, for which $\tilde{X} = a + ia^2$ and $\tilde{Y} = \frac{1}{3}a^3 + \frac{i}{2}a^4$, so that $\tilde{u} = \frac{1}{3}a^3$, $\tilde{v} = \frac{1}{2}a^4$ imply $\tilde{u} = \pm\frac{1}{3}(2\tilde{v})^{\frac{3}{4}}$. This is the description of a curve in the upper half of the ($\tilde{u}$,$\tilde{v}$) plane which is symmetrical about the vertical axis, and has a horizontal tangent at the origin, as shown in Figure 1 on the right.

5.4 Counter-example

The complex transformation

$$X = \frac{1}{2}(x+z)^2 + i(x+z), \;\; Y = zx - y + \frac{1}{2}x^2 + i(x+z), \;\; Z = \frac{x+i}{x+z+i} \tag{40}$$

(derived from the family (21) of SR94 with the particular choice of coefficients $Q = c = 0,\; G = g = 1,\; R = r = i$) has the property

$$\mathrm{d}Y - Z\mathrm{d}X = -(\mathrm{d}y - z\mathrm{d}x), \tag{41}$$

so that $\beta = -1$ in the complex version of (5). Evidently we must require that

$$x + z + i \neq 0, \tag{42}$$

which will certainly be satisfied when x and z are real.

We may pose the following question: does (40) describe a complex lift transformation in the sense that a real lift satisfying $\mathrm{d}y = z\mathrm{d}x$ implies, via (40) and (41) with (42), a complex lift satisfying $\mathrm{d}Y = Z\mathrm{d}X$?

Associated with this question of whether (40) describes a complex lift transformation is a second question: is the function $Y(X)$ implied by the first two parts of (40) a holomorphic function on some real plane curve $y = y(x)$? To answer this we write the first two parts of (40), using the notation in (11), as

$$X = a(x,z) + ib(x,z), \;\; Y = u(x,z) + iv(x,z) \tag{43}$$

in which

$$a = \frac{1}{2}(x+z)^2, \;\; b = x + z, \;\; u = zx - y(x) + \frac{1}{2}x^2, \;\; v = x + z. \tag{44}$$

This shows that the jacobian $\partial(a,b)/\partial(x,z) = 0$ so that (as is otherwise obvious from the fact that $a = \frac{1}{2}b^2$), a and b are *not* independent variables. Therefore the Cauchy-Riemann equations cannot be satisfied, and (43) with (44) do *not* imply the existence of a holomorphic function $Y(X)$ on the lift of $y(x)$.

6 Dualities between plane curves II

Here we describe a procedure that postulates a real curve $\tilde{v}(\tilde{u})$ in the complex $\tilde{Y}$-plane of $(13)_2$ and establishes a correspondence between it and a real function $y(x)$, introduced after (9).

We specify a relation between the real and imaginary parts of $\tilde{Y}$ in (13), of the form

$$\tilde{v}(a,b) = \tilde{f}(\tilde{u}(a,b)), \tag{45}$$

where $\tilde{f}(\tilde{u})$ is a given function. Then assuming (12), so that we have functions $x(a,b)$ and $y(a,b)$ available, (45) takes the form

$$v(x,z) = f(u(x,z)), \tag{46}$$

in terms of a new function f, where $v(x,z) = \tilde{v}(a(x,z), b(x,z)),\;\; u(x,z) = \tilde{u}(a(x,z), b(x,z))$, and $f(u) = \tilde{f}(\tilde{u})$ in value.

Hence we have a relation

$$\hat{v}(x,y,z) = \hat{f}(\hat{u}(x,y,z)), \tag{47}$$

by using the reverse transformation from (10) to (8).

This is for any trio of values x, y, z and, in particular, we have not invoked the lift property. But if we now set $z = \mathrm{d}y/\mathrm{d}x$, (47) becomes an ordinary differential equation for $y(x)$. Hence, we have a duality between (45) and a real function $y(x)$.

For an illustration we can use Example 3 of Section 5.3. We start with the curve

$$\tilde{u} = \pm\frac{1}{3}(2\tilde{v})^{\frac{3}{4}}, \tag{48}$$

in the complex $\tilde{Y} = \tilde{u} + i\tilde{v}$ plane. From (38) *et seq.*, we see it has the property

$$y(a) = \pm\frac{1}{3}b^{\frac{3}{2}}, \tag{49}$$

so then $u(x) = y(x) = \pm\frac{1}{3}z^{\frac{3}{2}}$.

We now use the Cauchy-Riemann equation, (39), which gives the lift property $z = \mathrm{d}y/\mathrm{d}x$ (using (37)), and hence we have the nonlinear ODE

$$y(x) = \pm\frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{\frac{3}{2}}. \tag{50}$$

Equation (50) can be integrated

$$y = \frac{1}{3}(c \pm x)^3, \tag{51}$$

where c is the constant of integration. This is a one-parameter family of cubics in the real (x,y)-plane. A one-parameter family of curves is to be anticipated, because we are inverting the lift $z = \mathrm{d}y/\mathrm{d}x$, i.e. finding y given z.

7 Legendre transformations associated with complex lift transformations

Each mapping (17), and the several special cases of it which we subsequently considered (such as (21) and (26)), have the invariant property (18).

When we add the additional assumptions that

$$y = y(x) \text{ and } z = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{52}$$

so that $z(x)$ is the lift of $y(x)$, we have $\mathrm{d}y - z\mathrm{d}x = 0$ and therefore that there must be a function

$$Y = Y(X) \text{ with } Z = \frac{\mathrm{d}Y}{\mathrm{d}X} \text{ because } \mathrm{d}Y = Z\mathrm{d}X. \tag{53}$$

This fact leads to the construction of an associated Legendre transformation, which we can illustrate via the derivation of such $Y(X)$ for the example (21) with $c = r = 0$ and the additional assumptions $y = y(x)$ and $z = \mathrm{d}y/\mathrm{d}x$. Thus we begin with

$$X = z(x) + ix, \;\; Y = y(x) - xz(x) - \frac{i}{2}x^2, \;\; Z = -x. \tag{54}$$

With x as parameter, (54) is a parametrically defined example of $Y(X)$ with $Z = \mathrm{d}Y/\mathrm{d}X$ and therefore $Z(X)$ as (53) requires.

To construct the inverse function $X(Z)$ of $Z(X)$ we need a Legendre dual function

$$H(Z) = ZX(Z) - Y[X(Z)] \tag{55}$$

which will have the property

$$\frac{\mathrm{d}H}{\mathrm{d}Z} = X + Z\frac{\mathrm{d}X}{\mathrm{d}Z} - \frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}Z} = X. \tag{56}$$

Thus for (54) the “value” of (55) is

$$H[Z] = -x(z + ix) - (y - xz - \frac{i}{2}x^2) = -y - \frac{i}{2}x^2 = -y[-Z] - \frac{i}{2}Z^2 \tag{57}$$

from which we can verify the gradient to be

$$\frac{\mathrm{d}H}{\mathrm{d}Z} = -\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}Z} - iZ = z + ix = X. \tag{58}$$

8 Higher dimensions

Here we generalise Section 3 to many dimensions. We introduce $2n + 1$ real variables $(x_1, ..., x_n; y; z_1, ..., z_n)$. We use them with $i = \sqrt{-1}$ and $4n + 2$ real functions

$$\hat{a}_k(x_i, y, z_i), \;\; \hat{b}_k(x_i, y, z_i), \;\; \hat{u}(x_i, y, z_i), \;\; \hat{v}(x_i, y, z_i), \;\; \hat{g}_k(x_i, y, z_i), \;\; \hat{h}_k(x_i, y, z_i) \tag{59}$$

to construct $2n + 1$ complex functions

$$X_k = \hat{a}_k + i\hat{b}_k, \;\; Y = \hat{u} + i\hat{v}, \;\; Z_k = \hat{g}_k + i\hat{h}_k, \tag{60}$$

thus generalising (9). Now we introduce a real function $y(x_i)$ which allows us to rewrite $\hat{a}_k(x_i, y(x_i), z_i) = a_k(x_i, z_i)$ and similarly for all such $4n + 2$ functions, to achieve

$$X_k(x_i, z_i) = a_k + ib_k, \;\; Y(x_i, z_i) = u + iv, \;\; Z_k(x_i, z_i) = g_k + ih_k. \tag{61}$$

No lift assumption of the type $z_i = \frac{\partial y}{\partial x_i}$ has yet been made. Now we suppose that, for the functions $X_k(x_i, z_i)$ in (61), the jacobian

$$\frac{\partial(a_k, b_k)}{\partial(x_j, z_j)} \neq 0, \pm\infty \tag{62}$$

so that we can use the implicit function theorem to construct the inverse functions $x_i(a_k, b_k)$ and $z_i(a_k, b_k)$ and use them to rewrite (61) as

$$\tilde{X}_k = a_k + ib_k, \quad \tilde{Y} = \tilde{u}(a_j, b_j) + i\tilde{v}(a_j, b_j), \quad \tilde{Z}_k = \tilde{g}_k(a_j, b_j) + i\tilde{h}_k(a_j, b_j) \tag{63}$$

in terms of new functions $\tilde{u}(a_j, b_j)$, $\tilde{v}(a_j, b_j)$, $\tilde{g}_k(a_j, b_j), \tilde{h}_k(a_j, b_j)$, having the same values as the previous u, v, g_k, h_k, thus generalising (13).

The mapping (60) is of the form

$$X_k = X_k(x_i, y, z_i), \quad Y = Y(x_i, y, z_i), \quad Z_k = Z_k(x_i, y, z_i), \tag{64}$$

and when it has the property

$$\mathrm{d}Y - Z_k \mathrm{d}X_k = \beta(\mathrm{d}y - z_k \mathrm{d}x_k) \tag{65}$$

we can rewrite this as

$$\begin{aligned} \beta(\mathrm{d}y - z_k\mathrm{d}x_k) = \mathrm{d}\tilde{Y} - \tilde{Z}_k\mathrm{d}\tilde{X}_k &= \mathrm{d}\tilde{u} + i\mathrm{d}\tilde{v} - (\tilde{g}_k + i\tilde{h}_k)(\mathrm{d}a_k + i\mathrm{d}b_k) \\ &= \mathrm{d}\tilde{u} - \tilde{g}_k\mathrm{d}a_k + \tilde{h}_k\mathrm{d}b_k + i(\mathrm{d}\tilde{v} - \tilde{h}_k\mathrm{d}a_k - \tilde{g}_k\mathrm{d}b_k). \end{aligned} \tag{66}$$

Now we do introduce the lift property $z_i = \frac{\partial y}{\partial x_i}$, thus strengthening (64) to make both sides of (65) zero, to prove the following result.

Theorem 2 *($\tilde{Y}(\tilde{X}_k)$ is holomorphic on the lift of $y(x_i)$)*

At any point with height $z_k = \frac{\partial y}{\partial x_k}$ on the lift of the real function $y(x_k)$, the function $\tilde{Y}(\tilde{X}_k)$ implied by (64) satisfies $\mathrm{d}\tilde{Y} = \tilde{Z}_k\mathrm{d}\tilde{X}_k$ and therefore

$$\tilde{g}_k = \frac{\partial \tilde{u}}{\partial a_k} = \frac{\partial \tilde{v}}{\partial b_k}, \quad \tilde{h}_k = -\frac{\partial \tilde{u}}{\partial b_k} = \frac{\partial \tilde{v}}{\partial a_k}. \tag{67}$$

As in Theorem 1, the second equations in each of this pair are the (generalised) Cauchy-Riemann equations associated with $\tilde{u}(a_k, b_k)$ and $\tilde{v}(a_k, b_k)$, and these are *necessary* for $\tilde{Y}(\tilde{X}_k)$ to be holomorphic on the lift of $y(x_i)$.

In summary we can say that we have constructed a holomorphic complex lift transformation from the real lift $z_i = \frac{\partial y}{\partial x_i}$ to the complex lift $\tilde{Z}_k = \frac{\partial \tilde{Y}}{\partial \tilde{X}_k}$.

9 An example in higher dimensions

We now extend the family described in Section 4 to higher dimensions. If we put $c = 0$ in (17) we obtain

$$X = Px + z, \;\; Y = y - xz + r(Px + z) - \frac{1}{2}Px^2, \;\; Z = r - x \tag{68}$$

for given real or complex constants P and r. This is a family of transformations from the real or complex trio of variables x, y, z to the real or complex trio X, Y, Z.

We now use the two sets each of $2n + 1$ real or complex variables x_i, y, z_i and X_i, Y, Z_i introduced in Section 8. We replace the real or complex constant scalars in (68) by the following real or complex constant matrices. The P_{ij} are n^2 components of a symmetric square matrix, and the r_i are the n components of a row or column matrix. We consider the mapping defined by the following $2n + 1$ definitions, which are generalisations of (68):

$$X_i = P_{ij}x_j + z_i, \tag{69}$$

$$Y = y - x_i z_i + r_i(P_{ij}x_j + z_i) - \frac{1}{2}P_{ij}x_i x_j, \tag{70}$$

$$Z_i = r_i - x_i. \tag{71}$$

It is straightforward to verify that the generalisation of the differential invariant property (18) still applies, namely

$$\mathrm{d}Y - Z_i \mathrm{d}X_i = \mathrm{d}y - z_i \mathrm{d}x_i. \tag{72}$$

This is the generalisation of (5) as in (65), in the case $\beta = 1$ there, as in (18). We can therefore say that (69) to (71) is a (real or complex) lift transformation in the sense that each of $\mathrm{d}Y = Z_i \mathrm{d}X_i$ and $\mathrm{d}y = z_i \mathrm{d}x_i$ implies the other.

For a particular example we now choose $P_{ij} = ip\delta_{ij}$ with real $p \neq 0$, and also real r_i. We also introduce a real function $y(x_i)$, so that (69)–(71) become

$$X_i = z_i + ipx_i, \tag{73}$$

$$Y = y(x_i) - x_i z_i + r_i(z_i + ipx_i) - \frac{1}{2}ipx_i x_i, \tag{74}$$

$$Z_i = r_i - x_i. \tag{75}$$

The real and imaginary parts in (73)–(75) are now all explicit, and we can identify the individual members of (64). In particular we see that

$$a_k = z_k, \;\; b_k = px_k \tag{76}$$

have the inverses

$$x_k = \frac{b_k}{p}, \;\; z_k = a_k. \tag{77}$$

Then (73)-(75) can be rewritten as

$$\tilde{X}_k = a_k + ib_k, \;\; \tilde{Y} = \tilde{u} + i\tilde{v}, \;\; \tilde{Z}_k = \tilde{g}_k + i\tilde{h}_k. \tag{78}$$

In these equations

$$\tilde{u} = y\left(\frac{b_i}{p}\right) - \frac{a_i b_i}{p} + r_i a_i, \quad \tilde{v} = r_i b_i - \frac{1}{2p} b_i b_i \tag{79}$$

and

$$\tilde{g}_k = r_k - \frac{b_k}{p}, \quad \tilde{h}_k = 0. \tag{80}$$

Therefore the equations (67) in Theorem 2, including the Cauchy-Riemann equations, become

$$r_k - \frac{b_k}{p} = r_k - \frac{b_k}{p} = r_k - \frac{b_k}{p}, \quad 0 = -\frac{1}{p}\frac{\partial y}{\partial x_i} + \frac{a_i}{p} = 0. \tag{81}$$

The first pair are trivially satisfied, and the second pair express the lift property.

For an even more specific example we can choose $p = 1$ and $r_i = 0$ in (73) - (75) to give the mapping

$$X_i = z_i + i x_i, \quad Y = y(x_i) - x_i z_i - \frac{i}{2} x_i x_i, \quad Z_i = -x_i, \tag{82}$$

which satisfies $\mathrm{d}Y - Z_i \mathrm{d}X_i = \mathrm{d}y - z_i \mathrm{d}x_i$. The associated Cauchy-Riemann equations (81) are satisfied with $a_i = z_i, \;\; b_i = x_i$ when $z_i = \frac{\partial y}{\partial x_i}$, i.e. on the lift of $y(x_i)$.

For $n = 2$ the cubic example $y(x_i) = \frac{1}{3}({x_1}^3 + {x_2}^3)$ has the lift $(z_1, z_2) = ({x_1}^2, {x_2}^2)$ and

$$Y(x_1, x_2) = -\frac{2}{3}({x_1}^3 + {x_2}^3) - \frac{i}{2}({x_1}^2 + {x_2}^2), \tag{83}$$

so that in (61) we can write

$$u = -\frac{2}{3}({x_1}^3 + {x_2}^3), \quad v = -\frac{1}{2}({x_1}^2 + {x_2}^2). \tag{84}$$

For each fixed x_1 or x_2 this describes functions $u(v)$ and $v(u)$. For example, on the plane $x_1 = 0$ we have $3u + 2 = -2{x_2}^3$ and $2v + 1 = -{x_2}^2$, and eliminating x_2 gives the relation

$$u = -\frac{2}{3}\left[1 \pm (-2v - 1)^{\frac{3}{2}}\right]. \tag{85}$$

10 An application to partial differential equations

The application of lift and contact transformations to nonlinear partial differential equations is a subject with a long and rich history, dating back to the seminal work of Lie, Carathéodory, and others. In this Section we show how the complex lift transformations described earlier in this paper can be applied to partial differential equations.

We consider differential equations involving two independent real variables x_1 and x_2 and one dependent real variable $y(x_1, x_2)$. We define the lift of $y(x_1, x_2)$ by $z_1 = \partial y/\partial x_1$ with $z_2 = \partial y/\partial x_2$. An example of a constant coefficient equation of Monge-Ampère type for y is the jacobian equation

$$\frac{\partial(z_1, z_2)}{\partial(x_1, x_2)} \equiv \frac{\partial^2 y}{\partial {x_1}^2}\frac{\partial^2 y}{\partial {x_2}^2} - \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2 = 1. \tag{86}$$

These equations play an important role in semi-geostrophic theory (Hoskins 1975).

We now consider a transformation of (86) under the following complex lift transformation

$$X_1 = x_1 + iz_1,\quad X_2 = x_2 + iz_2,\quad Y = y + i\frac{1}{2}({z_1}^2 + {z_2}^2),\quad Z_1 = z_1,\quad Z_2 = z_2. \tag{87}$$

The inverse of (87) is $x_1 = X_1 - iZ_1$ etc., and on the lift of y, where $Z_1 = \partial Y/\partial X_1$ and $Z_2 = \partial Y/\partial X_2$, we find that (86) transforms via

$$\frac{\partial(Z_1, Z_2)}{\partial(X_1, X_2)} = \frac{\partial((X_1 - iZ_1), (X_2 - iZ_2))}{\partial(X_1, X_2)} \tag{88}$$

to another nonlinear, but now complex, equation of Monge-Ampère type, namely

$$1 - i\left(\frac{\partial^2 Y}{\partial {X_1}^2} + \frac{\partial^2 Y}{\partial {X_2}^2}\right) - 2\left(\frac{\partial^2 Y}{\partial {X_1}^2}\frac{\partial^2 Y}{\partial {X_2}^2} - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2\right) = 0. \tag{89}$$

It is straightforward to verify that the solution $y = \frac{1}{2}({x_1}^2 + {x_2}^2)$ of (86) transforms under (87) to

$$Y = \frac{1}{2}(1+i)\left[\left(\frac{X_1}{1+i}\right)^2 + \left(\frac{X_2}{1+i}\right)^2\right] \tag{90}$$

and that (90) is a solution of (88).

We may start with a real linear equation, for example Laplace's equation

$$\frac{\partial^2 y}{\partial {x_1}^2} + \frac{\partial^2 y}{\partial {x_2}^2} = 0, \tag{91}$$

which on the lift of y may be written in jacobian form

$$\frac{\partial(z_1, x_2)}{\partial(x_1, x_2)} = -\frac{\partial(x_1, z_2)}{\partial(x_1, x_2)}. \tag{92}$$

Under the transformation (87) this becomes a complex nonlinear equation of Monge-Ampère type, which can be written in the form

$$\frac{\partial(Z_1, X_2 - iZ_2)}{\partial(X_1, X_2)} = -\frac{\partial(X_1 - iZ_2, Z_2)}{\partial(X_1, X_2)}. \tag{93}$$

Equation (93) becomes, on the lift of Y,

$$\frac{\partial^2 Y}{\partial {X_1}^2} + \frac{\partial^2 Y}{\partial {X_2}^2} - 2i\left(\frac{\partial^2 Y}{\partial {X_1}^2}\frac{\partial^2 Y}{\partial {X_2}^2} - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2\right) = 0. \tag{94}$$

If we take a solution of (92), such as $y = \frac{1}{2}({x_1}^2 - {x_2}^2)$, and transform it under (87) (on the lift of y, where $z_1 = x_1, z_2 = -x_2$, and hence $x_1 = X_1/(1+i)$, $x_2 = X_2/(1-i)$), we obtain

$$Y = \frac{1+i}{2}\left(\frac{X_1}{1+i}\right)^2 + \frac{1-i}{2}\left(\frac{X_2}{1-i}\right)^2, \tag{95}$$

which can readily be verified to be a solution of (94).

11 Summary

We have extended the results of SR94 to mappings between real and complex variables, and have demonstrated applications of such mappings to transformations of nonlinear partial differential equations. The application of real lift transformations to the study of symmetries of differential equations is well established. See Hydon (2000) for further details, and for a brief discussion of the role of the lift of a plane curve versus contact between a pair of curves (Hydon, *op. cit.* p. 56).

One of our key results is the connection between the lift of a real function and a holomorphic complex dual function. Our generalization of the lift property to mappings between real and complex variables is not entirely straightforward, as illustrated by the counter-example given in §5.4.

Singularities arising in lift transformations with real variables have been the subject of extensive investigation (for example, see Sewell 1987 for many examples associated with the Legendre transformation), and we have shown how singularities may arise in mappings between real and complex variables. In this regard, our emphasis is on the local behaviour of real submanifolds of complex manifolds. We remark that the methods presented in Section 6 can be applied in higher dimensions, which then involve the solution of (sometimes nonlinear) partial differential equations.

Complex variable structures have been associated with elliptic Monge-Ampère equations in two independent variables (Lychagin et al. 1993, Kushner et al. 2007), and can be extended to equations in three independent variables (Banos 2002). Monge-Ampère equations in n independent variables play a role in the theory of Special Lagrangian submanifolds in $\mathbb{C}^n$ as graphs; see Joyce 2002, Section 7.2, for further discussion. The subject of complex contact manifolds is well established (Kobayashi 1959; Boothby 1961, 1962; Ishihari and Konishi 1980, 1982), but the lift and contact transformations arising in this paper are novel in that they involve mappings between real and complex variables. We also note that the differential one-form $\mathrm{d}Y - Z\mathrm{d}X$ need not be holomorphic: that is, Z is not required to be a holomorphic function of X, and indeed it is not so in the examples we have provided.

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