Potential vorticities in semi-geostrophic theory

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Abstract

We devise a family of kinematically possible motions in shallow water versions of semi-geostrophic theory, with variable Coriolis parameter, for which two distinct measures of potential vorticity take the same value. We describe conditions for which this value is conserved following the actual motion of a particle. We relate the results to the existing literature, in particular for a constant Coriolis parameter.

1 Introduction

Shallow water theory can serve as a simplified basis for the study of so-called semi-geostrophic theories in geophysical fluid dynamics. We examine shallow water theory and two semi-geostrophic models of that type, with particular reference to a definition of potential vorticity associated with each, and allowing for the Coriolis parameter f to be spatially varying wherever possible. We describe a class of vector fields under which the expressions for two of the potential vorticities are the same. We give hypotheses under which that value is conserved following the particle.

The plan of the paper is as follows. In §§2 and 3 we recall the shallow water equations, and the associated quantity (q in (6)), now called potential vorticity, whose conservation was proved by Rossby (1936, equation (75); 1940, equation (9)). We describe a variational principle for these equations, prompted by Salmon's (1983) work. We state the semi-geostrophic approximation to the shallow water equations for constant f, and we recall (in (13)) the measure of potential vorticity, analogous to Hoskins' (1975, §3(*iii*)) definition, which is conserved by motions which satisfy that approximation. In §4 we describe a generalization, for variable f, of the geostrophic coordinate transformation. This

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leads naturally to the third model, which is a pair of pseudo-hamiltonian equations, also with variable f, studied by Salmon (1985, equations (3.22)), and their associated potential vorticity (Q in (25)).

We are then in a position to exhibit, in §5, a class of vector fields for which $q = Q$. We indicate, via an initial value problem, the reasonable extra assumptions which are required for these vector fields to be kinematically possible. In §6 we establish hypotheses which are sufficient to ensure that both q and Q are conserved following the motion of the particle. This applies, in particular, to the semi-geostrophic equations. In the case of constant f, in §7, we make some further connections with the previous literature, particularly Salmon's (1988, equation (5.18)) constraints.

We remark that some of these results, particularly those associated with equation (40), were discovered, in the first instance, using the properties of covariant skew-symmetric tensors, often referred to in the mathematical physics literature as forms, but we have rephrased the proofs in the more elementary style described here. (The calculation about the conservation of q on p. 21 of Rossby (1936) and on pp. 70-71 of Rossby (1940) can be understood in terms of these quantities.)

2 Kinematics

The motion of a typical particle in shallow water theory can be described by expressing the current cartesian horizontal coordinates

$$x = x(a, b, t), \quad y = y(a, b, t) \tag{1}$$

as functions, on the right, of the particle labels a, b and the time t. Throughout this paper we shall use the same symbol to denote a function and its generic values, as in this example (1). If $t = 0$ is the reference time, the functions in (1) have the properties $x(a, b, 0) = a,\ y(a, b, 0) = b$.

The incompressibility hypothesis requires the current depth h to be a function $h(a, b, t)$ with the property

$$\frac{h(a, b, 0)}{h(a, b, t)} = \frac{\partial(x, y)}{\partial(a, b)}\ , \tag{2}$$

where the jacobian on the right is that of the mapping (1). The time derivative of (2) following the particle gives the differential equation of continuity. We regard (2) as describing the explicit solution $h(a,b,t)$ of that continuity equation in terms of (1), and of $h(a,b,0)$. To simplify the algebra, we shall assume that the latter function of a and b only is actually an absolute constant h_0 (say), i.e.

$$h(a,b,0) = h_0. \tag{3}$$

We note that this does constrain the free surface to be parallel to the bed at $t = 0$. In compensation one can take the view that events to be studied take place at much later times, so that (3) is not a serious physical restriction. (We do *not* write $h_0 = 1$, because this would make it harder to check subsequent equations for dimensional consistency.)

The inverse

$$a = a(x,y,t), \quad b = b(x,y,t) \tag{4}$$

of the mapping (1), when inserted into (2), expresses $h(a,b,t)$ as another function

$$h(x,y,t) = \frac{\partial(a,b)}{\partial(x,y)} h_0. \tag{5}$$

We define a *kinematically possible motion* to be any mapping (1) with the properties (4) and (5), regardless of whether the dynamical equations in §3 are satisfied or not. An example will be given in (35).

Another important kinematical concept is the so-called potential vorticity defined by

$$q = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right). \tag{6}$$

This name was introduced by Rossby (1940, p. 72) for a datum value of the vorticity. The superposed dot signifies time differentiation following the particle, i.e. partial differentiation with respect to t when a and b are held constant. In (6) $\dot{x}$ and $\dot{y}$ are each being regarded as functions of x, y, t obtained by inserting (4) into the derivatives of (1) following the particle.

Common choices of the Coriolis parameter are $f =$ *constant* or βy, as approximations to $2\Omega \sin\phi$, depending on purpose. Here β and Ω are constants, and ϕ is latitude. In other words, we shall regard f as a constant or as a given function $f(y)$, not depending on x.

3 Dynamics

The equations of horizontal momentum balance for flows over a bed which is rotating with position dependent Coriolis parameter $f(y)$ are

$$\ddot{x} + g\frac{\partial h}{\partial x} - \dot{y}f = 0, \quad \ddot{y} + g\frac{\partial h}{\partial y} + \dot{x}f = 0\ . \tag{7}$$

Here g is a given constant, representing the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation. The derivatives of (5) appear in (7). The basic problem is therefore to solve (7) with (5) for (1), which will then deliver $h(a, b, t)$ from (2).

A result of Salmon (1983, equation (2.20)) can be adapted to show under what circumstances (7) are the natural conditions of a variational principle. Let $p(x, y)$ and $r(x, y)$ be any two functions which satisfy the partial differential equation

$$\frac{\partial p}{\partial x} + \frac{\partial r}{\partial y} = f(y). \tag{8}$$

The motion (1) can be used to define a functional

$$F[x, y] = \int\int_{t_1}^{t_2} \left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - r\dot{x} + p\dot{y} - \frac{1}{2}gh\right] \mathrm{d}A_0 \mathrm{d}t \tag{9}$$

where the integral is over the area A_0 of label space occupied by the fluid particles under consideration, and over an arbitrary time interval $t_1 \leq t \leq t_2$. The first variation of (9) for small variations $\delta x(a, b, t)$ and $\delta y(a, b, t)$, taking due account of (2) and (8), is

$$\begin{aligned} \delta F \quad &= \quad \left[\int((\dot{x} - r)\delta x + (\dot{y} + p)\delta y)\mathrm{d}A_0\right]_{t_1}^{t_2} + \int\int_{t_1}^{t_2} \frac{1}{2}gh^2(l\delta x + m\delta y)\mathrm{d}S\mathrm{d}t \\ &\quad - \int\int_{t_1}^{t_2} \left[(\ddot{x} + g\frac{\partial h}{\partial x} - \dot{y}f)\delta x + (\ddot{y} + g\frac{\partial h}{\partial y} + \dot{x}f)\delta y\right] \mathrm{d}A_0 \mathrm{d}t \end{aligned} \tag{10}$$

where S, l, m denote the current boundary of the fluid and its outward unit normal components. Thus whenever the first two terms are zero in (10), we see that (7) implies $\delta F = 0$ and stationary F. It is necessary that, for example, either $h = 0$ or $\delta x = \delta y = 0$ on S.

The semi-geostrophic approximation to equations (7), in the case when f is a constant, is the replacement of the true acceleration by the time derivative of another vector

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial h}{\partial x} \tag{11}$$

following the particle. The vector (11) is a notional velocity, called the geostrophic velocity for the reason mentioned after (14). The semi-geostrophic approximation therefore seeks to find motions (1) satisfying

$$\dot{u}_g + g\frac{\partial h}{\partial x} - \dot{y}f = 0, \ \ \dot{v}_g + g\frac{\partial h}{\partial y} + \dot{x}f = 0 \tag{12}$$

with (5) and constant f. Associated with these equations, the expression

$$\frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right] \tag{13}$$

is conserved. This is the shallow water version of Hoskins' (1975) potential vorticity. When f is variable, no measure of potential vorticity associated with (12) is known to be conserved.

4 Transformation to geostrophic coordinates

We recall from §2 that f is being regarded as a given function of y in the first instance. Then, for such $f(y)$, we define the transformation of coordinates

$$X = x + \frac{g}{f^2}\frac{\partial h}{\partial x}, \ \ Y = y + \frac{g}{f^2}\frac{\partial h}{\partial y} \tag{14}$$

where (5) is used. (These are called geostrophic coordinates because, when f is a constant, $\dot{X} = u_g$, $\dot{Y} = v_g$ from (11) and (12).) Equations (14) are a transformation

$$X = X(x, y, t), \ \ Y = Y(x, y, t) \tag{15}$$

which, at each t, has an inverse

$$x = x(X, Y, t), \ \ y = y(X, Y, t) \tag{16}$$

certainly if

$$\frac{\partial(X, Y)}{\partial(x, y)} \neq 0. \tag{17}$$

Then (16) can be used to write $f(y) = f(X, Y, t)$, and the latter will *not* be a *given* function if the motion is unknown in advance. The explicit presence of t in this last expression becomes important in Theorem 2 below.

Solution of the ensuing dynamical equations are facilitated when (14) is a gradient transformation of the type

$$X = \frac{\partial P}{\partial x}, \ \ Y = \frac{\partial P}{\partial y}. \tag{18}$$

For this, it is necessary that $\partial X/\partial y = \partial Y/\partial x$, and therefore

$$\frac{\mathrm{d}f}{\mathrm{d}y}\frac{\partial h}{\partial x} = 0. \tag{19}$$

This is always satisfied for constant f, in which case

$$P(x,y,t) = \frac{1}{2}(x^2+y^2) + \frac{g}{f^2}h(x,y,t). \tag{20}$$

For variable f, however, $\partial h/\partial x = 0$ is required, so that the motion (1) can only be such that $h(y,t)$ in (5), i.e. the depth cannot change in the x-direction. Then

$$P(x,y,t) = \frac{1}{2}(x^2+y^2) + \int \frac{g}{f^2}\frac{\partial h}{\partial y}\,\mathrm{d}y. \tag{21}$$

In either case (18) becomes a Legendre transformation with inverse (16) of the form

$$x = \frac{\partial R}{\partial X}, \quad y = \frac{\partial R}{\partial Y}, \quad \text{with} \;\; R(X,Y,t) = Xx + Yy - P. \tag{22}$$

Such Legendre transformations have been studied systematically by Chynoweth and Sewell (1989, 1991) for constant f.

Salmon (1985, equations (3.22)) studied certain *generalized* semi-geostrophic equations with pseudo-hamiltonian form in X, Y space, namely

$$\dot{X} = -\frac{1}{f}\frac{\partial H}{\partial Y}, \quad \dot{Y} = \frac{1}{f}\frac{\partial H}{\partial X} \tag{23}$$

where

$$H(X,Y,t) = \frac{1}{2}(u_g^2 + v_g^2) + gh. \tag{24}$$

Associated with these equations is a *generalized* potential vorticity defined by

$$Q = \frac{f}{h}\frac{\partial(X,Y)}{\partial(x,y)}. \tag{25}$$

Salmon's version (1985, equations (3.12)) of (14) has $f(X,Y)$ written in. Thus (17) is implicitly assumed, but the t dependence in $f(X,Y,t)$ is omitted. When f is a constant, (25) is equivalent to (13).

The definition (25) can be rewritten, using (18), as

$$Q = \frac{f}{h}\begin{vmatrix} \dfrac{\partial^2 P}{\partial x^2} & \dfrac{\partial^2 P}{\partial x \partial y} \\ \dfrac{\partial^2 P}{\partial y \partial x} & \dfrac{\partial^2 P}{\partial y^2} \end{vmatrix}. \tag{26}$$

It will be a particular consequence of Theorem 2 that, for constant f, $\dot{Q} = 0$. In that case, (26) is the starting point for a Monge-Ampère type of equation in which $P(x,y,t)$ is the unknown, to be found in conjunction with suitable boundary conditions.

5 Identification of potential vorticities

We are now in a position to establish the main results of this paper, as follows. The reader is asked to observe that we use u and v to denote the components of a vector, but we do not assume them to be the components of a velocity unless (33) are satisfied.

Theorem 1 *There is a class of vector fields $u(x,y,t), v(x,y,t)$, described in the following proof, and permitting f to be a function $f(y)$ or a constant, for which*

$$\frac{1}{h}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=Q. \tag{27}$$

Proof

We use the variables X, Y defined by the geostrophic transformation (14) to construct another transformation

$$\tilde{X}=\tilde{X}(X,Y,t),\quad \tilde{Y}=\tilde{Y}(X,Y,t) \tag{28}$$

to new coordinates $\tilde{X},\tilde{Y}$. The functions on the right of (28) are required to be any which satisfy the partial differential equation

$$\frac{\partial(\tilde{X},\tilde{Y})}{\partial(X,Y)}=f(X,Y,t). \tag{29}$$

This is one equation, from which to determine two functions, so it will have many solutions. For example, in the special case when f is a non-zero constant, (28) may be any canonical transformation (cf. Sewell and Roulstone, 1993), with t absent.

The geostrophic transformation (15) can be inserted into the transformation (28) to define a pair of functions $\tilde{X}(x,y,t),\tilde{Y}(x,y,t)$. Let $\lambda(x,y,t)$ be any function, and construct the family of vector fields $u(x,y,t),\ v(x,y,t)$ defined by

$$\left.\begin{aligned} u &= r+\frac{\partial\lambda}{\partial x}+\frac{1}{2}\left[\tilde{X}\frac{\partial\tilde{Y}}{\partial x}-\tilde{Y}\frac{\partial\tilde{X}}{\partial x}\right], \\ v &= -p+\frac{\partial\lambda}{\partial y}+\frac{1}{2}\left[\tilde{X}\frac{\partial\tilde{Y}}{\partial y}-\tilde{Y}\frac{\partial\tilde{X}}{\partial y}\right], \end{aligned}\right\} \tag{30}$$

where p and r are the functions which satisfy (8). By differentiation we see that (30) satisfy

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-f+\frac{\partial(\tilde{X},\tilde{Y})}{\partial(x,y)}. \tag{31}$$

This is why Schubert and Magnusdottir (1994, §2a) call $\tilde{X}$ and $\tilde{Y}$ *vorticity coordinates*. The chain rule for jacobians gives

$$\frac{\partial(\tilde{X},\tilde{Y})}{\partial(x,y)} = f\frac{\partial(X,Y)}{\partial(x,y)} = hQ \tag{32}$$

using (29) and (25).

By (31) and (32) we obtain (27).

□

We observe that the manipulation in the proof of Theorem 1 can also be carried through when X and Y are *not* necessarily the geostrophic coordinates defined by (14), and with $f(X,Y,t)$ interpreted as *any* function, not necessarily having the values of the Coriolis parameter.

It is an initial value problem in the theory of coupled pairs of ordinary differential equations

$$\dot{x} = u(x,y,t), \quad \dot{y} = v(x,y,t) \tag{33}$$

to decide whether the given functions on the right allow a solution (1) to exist satisfying $x(a,b,0) = a$, $y(a,b,0) = b$, for any constants of integration a and b. It is a particular consequence of a standard result (see, for example, Theorem 1.3.1 of Sánchez, 1968) that, if $u(x,y,t)$ and $v(x,y,t)$ *and* their spatial gradients are defined and continuous in some region $\mathcal{D}$ (say) of x,y,t-space, then there exists a unique solution (1) of (33) satisfying the initial conditions for any choice of $a,b,0$ in $\mathcal{D}$, and valid in some neighbourhood of such a point. (More sophisticated results of this nature are proved by Coddington and Levinson (1955).)

For example, the pair

$$\dot{x} = \frac{x^2}{y}, \quad \dot{y} = x \tag{34}$$

(which is equivalent to the single nonlinear second order equation $y\ddot{y} = \dot{y}^2$) has, in either half-space $y > 0$ or $y < 0$, the unique general solution

$$x = a\, e^{at/b}, \quad y = b\, e^{at/b} \tag{35}$$

when $b \neq 0$, with the required initial values. This is also an example of a kinematically possible motion in the sense defined after (5). It has an inverse

$$a = x\, e^{-xt/y}, \quad b = y\, e^{-xt/y} \tag{36}$$

and associated fluid depth

$$h = h_0\, e^{2at/b} = h_0\, e^{2xt/y} \tag{37}$$

from (2) and (5). It happens that (35) does not satisfy (7), not surprisingly, so that some *other* kinematically possible motion is required to be dynamically possible as well, corresponding to another choice of functions on the right of (33).

The counter example

$$\dot{x} = \frac{x}{t} + t\, e^{2t}, \quad \dot{y} = x, \tag{38}$$

which has the general solution

$$x = \frac{1}{2} t\, e^{2t} + ct, \quad y = \frac{1}{4} t\, e^{2t} - \frac{1}{8}\left(e^{2t} - 1\right) + \frac{1}{2} ct^2 + b \tag{39}$$

with arbitrary b and c, has $y(c, b, 0) = b$ but $x(c, b, 0) = 0$. This illustrates that it is not always possible to impose $x(c, b, 0) = c$ for *any* c, because the function on the right of $(38)_1$ does not satisfy the hypotheses for $u(x, y, t)$ quoted after (33).

With this background, we shall now assume that any pair of functions permitted on the right of (33) will satisfy the sufficient conditions quoted there, and that this resulting unique solution will define a kinematically possible velocity field. Then (27) with (33) gives $q = Q$ from (6) and Theorem 1, and differentiating following a particle gives $\dot{q} = \dot{Q}$. Summarizing thus far, the two potential vorticities (6) and (25) are then the same for any vector field (and, by our assumption, for any kinematically possible field) in the family (30). In other words, (30) with (33) satisfies

$$\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} = \left(\frac{\partial(X, Y)}{\partial(x, y)} - 1\right) f. \tag{40}$$

6 Rate of change of potential vorticity

It follows that either both potential vorticities are conserved, or neither is, when (30) is a kinematically possible motion.

Theorem 2 *Any solution $X = X(t)$, $Y = Y(t)$ of (23), for* any *hamiltonian function $H(X, Y, t)$ (not only (24)) and for variable $f(X, Y, t)$ $(= f(y)$ via (15)), has the property*

$$\dot{Q} = \frac{1}{h_0} \frac{\partial(X, Y)}{\partial(a, b)} \frac{\partial f}{\partial t}. \tag{41}$$

Here $\partial f/\partial t$ is the partial t derivative of $f(X,Y,t)$, and the jacobian is that of the transformation

$$X = X(a,b,t), \quad Y = Y(a,b,t) \tag{42}$$

obtained by substituting (1) into (15).

Proof

Combining (5) and (25) gives

$$Q = \frac{f}{h_0}\frac{\partial(X,Y)}{\partial(a,b)}. \tag{43}$$

Differentiating following the particle gives

$$\dot{Q} = \frac{\dot{f}}{h_0}\frac{\partial(X,Y)}{\partial(a,b)} + \frac{f}{h_0}\frac{\partial(\dot{X},Y)}{\partial(a,b)} + \frac{f}{h_0}\frac{\partial(X,\dot{Y})}{\partial(a,b)}. \tag{44}$$

The chain rule, with the pseudo-Hamilton equations (23), implies

$$\dot{f} = \frac{\partial f}{\partial X}\dot{X} + \frac{\partial f}{\partial Y}\dot{Y} + \frac{\partial f}{\partial t} = \frac{1}{f}\frac{\partial(H,f)}{\partial(X,Y)} + \frac{\partial f}{\partial t} \tag{45}$$

and

$$\frac{\partial(\dot{X},Y)}{\partial(a,b)} + \frac{\partial(X,\dot{Y})}{\partial(a,b)} = \frac{1}{f^2}\frac{\partial(X,Y)}{\partial(a,b)}\frac{\partial(f,H)}{\partial(X,Y)}. \tag{46}$$

The result (41) follows.

□

When f and Q are constants, (43) shows that (42) is canonical.

In passing we remark that, with this definition of $\partial/\partial t$ after (41), it is not necessary to introduce a second symbol to denote time, as some authors occasionally do.

Theorem 3 *Suppose that*

(a) the velocity field (33) has the form (30), and

(b) it satisfies the pseudo-hamiltonian equations (23), for any *hamiltonian (not only (24)), and*

(c) $\partial f/\partial t = 0$, so that t is absent from f when expressed as $f(y(X,Y,t))$ in terms of X and Y via (16).

The latter requirement is satisfied a fortiori when f is a constant.

Then

$$\dot{Q} = \dot{q} = 0. \tag{47}$$

Proof

This is a consequence of Theorems 1 and 2.

□

7 Some properties of f-plane semi-geostrophic theory

In this Section we suppose that $f =$ *constant* throughout.

If, in (30), we make the particular choices

$$p = \frac{1}{2}fx, \quad r = \frac{1}{2}fy, \quad \tilde{X} = |f|^{\frac{1}{2}}X, \quad \tilde{Y} = |f|^{\frac{1}{2}}Y, \quad \lambda = \chi - \frac{f}{2}(xY - yX) \tag{48}$$

which satisfy (8) and (29), where $\chi(x, y, t)$ is an arbitrary function, then

$$\left.\begin{aligned} u &= \frac{\partial \chi}{\partial x} + \frac{1}{2}f\left[(X-x)\frac{\partial Y}{\partial x} - (Y-y)\left(\frac{\partial X}{\partial x} + 1\right)\right], \\ v &= \frac{\partial \chi}{\partial y} + \frac{1}{2}f\left[(X-x)\left(\frac{\partial Y}{\partial y} + 1\right) - (Y-y)\frac{\partial X}{\partial y}\right]. \end{aligned}\right\} \tag{49}$$

Here X and Y may be, but are not necessarily, geostrophic coordinates (as is also the case in Theorem 1). It follows from (25) and (27) that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f = f\frac{\partial(X, Y)}{\partial(x, y)} \tag{50}$$

in this particular case.

Schubert and Magnusdottir (1994) prove that (50) can be rearranged as a zero curl condition which implies (49). We have shown that (50) is also satisfied by a wider class of vector fields (30).

A brief and elementary calculation using (11) and (14) allows us to point out that, when $\chi =$ constant, the vector field on the right of (49) can be rewritten as

$$\left.\begin{aligned} u &= u_g + \frac{1}{2f}\left(u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)v_g \,, \\ v &= v_g - \frac{1}{2f}\left(u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)u_g \,. \end{aligned}\right\} \tag{51}$$

Therefore (51) also satisfies (50).

These particular examples (49) and (51) are available as vector fields whether or not (33) are imposed. When these fields do satisfy the hypotheses which the initial value theorem requires, and the consequent mapping (1) is invertible, we can interpret such u and v as kinematically possible velocity components.

The interest of (51) is that its right hand side is identical with a form proposed by Salmon (1988, equation (5.18)) to serve as what he calls a constraint imposed upon *dynamically* possible motions. That is, that (51) is required to be a kinematically possible velocity field which *also* satisfies (7). He claims that such constraints "define the slow manifold for the semi-geostrophic equations of Salmon (1985, §3)", i.e. for (23) with (24).

When the vector field (51) is substituted into (27) we obtain (13). Therefore, whenever (51) is a *dynamically* possible motion that satisfies (7), the resulting Rossby expression q will be equal to the analogue (13) of Hoskins' formula.

Inserting the geostrophic coordinates (14) into the right hand sides of (49) to get (51), as we have described, is different from the discussion that leads to the semi-geostrophic approximation given in Schubert and Magnusdottir (1994, §2*c*).

8 Conclusions

Shallow water theory over a rotating bed is a convenient simplified vehicle for studying certain properties of the basic equations of meteorology and oceanography. It has been used by a number of authors to help the meteorologist to focus upon important theoretical concepts. In this paper we have adopted this model.

We have found it helpful at the outset to make a clear distinction between kinematically possible and dynamically possible motions. The former are a much wider class. We have compared the two distinct scalar quantities q in (6) and Q in (25), each of which has been investigated, by Rossby (1936, 1940) and Salmon (1985) respectively, and in each of which the Coriolis parameter can vary with physical position in an assigned way (or be a constant). Under light assumptions, we have exhibited, in §5, a family of kinematically possible motions for which $q = Q$, and therefore for which the time derivative following *such* motions satisfy $\dot{q} = \dot{Q}$.

In §6 we exhibit restrictions on the Coriolis parameter which are sufficient to ensure that, when such motions are also dynamically possible in the sense of satisfying the pseudo-hamiltonian equations (23) for any hamiltonian, then $\dot{q} = \dot{Q} = 0$, i.e. both measures of potential vorticity are conserved.

When the Coriolis parameter is a constant, and the geostrophic coordinates (14) are the variables in the pseudo-hamiltonian equations (23), Q becomes the potential vorticity known to be conserved in the dynamical solutions of semi-geostrophic theory, and we make some further connections with that theory in §7.

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