Examples of Quaternionic and Kähler Structures in Hamiltonian Models of Nearly Geostrophic Flow

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Abstract

We study 2-forms on phase spaces of hamiltonian models of nearly geostrophic flows. A quaternionic structure is identified, and the complex part of a symplectic representation of this structure corresponds to an elliptic Monge-Ampère equation. The real part is an invariant Kähler structure.

1 Introduction

Atmospheric cyclones and anticyclones, and ocean eddies, can be idealized as the stratified, rotating coherent structures that correspond to circular vortices in ordinary two-dimensional Euler flow. Their interaction and evolution, which play a major role in weather developments and in the behaviour of ocean eddies, have been much studied using approximations to Newton's second law of motion. These approximate models seek to describe flows in which there is a dominant balance between the Coriolis, buoyancy and pressure-gradient forces. Such approximations to Newton's second law are commonly referred to as *balanced models*. The balance between the Coriolis and pressure gradient forces is referred to as *geostrophic* balance. Such balance conditions can be used to define *slow manifolds* within the full phase space of the unapproximated dynamics (Salmon 1988; Allen and Holm 1996; McIntyre and Roulstone 1996).

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Many of these balanced models have one particular feature in common, that is, they have one degree of freedom. In practice, this is realized by the reduction of Newton's laws of motion to a single partial differential equation for a scalar variable. For example, the motion of air on a surface of constant potential temperature in the atmosphere might, under certain physical conditions, be approximated by the equation $\varpi_t + \mathbf{u}.\nabla\varpi = 0$, where the vorticity $\varpi = -\Delta\psi$ and the velocity $\mathbf{u} = (-\psi_y, \psi_x)$.

In this paper we demonstrate, by means of explicit examples, how the reduction of the degrees of freedom can be viewed as a contactification of the symplectic structure of the exact, unapproximated dynamics, followed by the restriction to the graph of a single scalar variable. In section 2 we show how this works for the semi-geostrophic equations, a balanced model whose mathematical structure has been studied thoroughly (Cullen and Purser 1984; Salmon 1985, 1988; Purser and Cullen 1987; Roulstone and Norbury 1994; Roulstone and Sewell 1996a,b). We show how the Monge-Ampère operator of that model can be obtained from the underlying geometry. In section 3 we construct a quaternionic description of balanced models. The quaternionic structure possesses three pairs of self-dual and anti-self-dual two forms, one pair of which encodes a Monge-Ampère equation. The quaternionic structure provides a means of describing, in a unified way, how a Monge-Ampère operator is common to, (i) the model referred to as L_1 dynamics by Salmon (1985, 1988) and McIntyre and Roulstone (1996), (ii) the semi-geostrophic model and (iii), a new model discussed by McIntyre and Roulstone (*op. cit.*, §9). A symplectic representation of the quaternion possesses a natural Kähler structure.

2 A geometric approach to semi-geostrophic theory

2.1 Contactification of a symplectic structure

Consider fluid flow on a region $\mathcal{D} \in \mathbb{R}^2$. The cotangent bundle $\mathcal{V}^* = T^*\mathbb{R}^2$ has coordinates $\{x, y, p, q\}$ which span the phase space of a fluid obeying Newton's second law (e.g. the shallow water equations (Salmon 1983)).

Let $(\mathcal{V}, \Omega)$ be a symplectic space with $\Omega \in \bigwedge^2(\mathcal{V}^*)$ and in coordinates we write

$$\Omega = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q.$$

The transformation

$$f : \{x, y, p, q\} \mapsto \{x + p, y + q, p, q\} \equiv \{X, Y, P, Q\}, \tag{1}$$

has the property $f^*(\Omega) = \Omega$ and is therefore canonical. The function $S = \frac{1}{2}(p^2 + q^2)$ is a generating function for f in the usual sense:

$$P - p = \frac{\partial S}{\partial x} = 0, \qquad X - x = \frac{\partial S}{\partial p} = p,$$
$$Q - q = \frac{\partial S}{\partial y} = 0, \qquad Y - y = \frac{\partial S}{\partial q} = q.$$

Consider the contactification of $\mathcal{V}^*$ (Arnol'd 1989, Appendix 4L)

$$\{x, y, p, q\} \mapsto \{x, y, \phi, p, q\} \equiv \mathcal{C},$$

where ϕ would be identified with a suitably scaled geopotential function in most applications. Then there exists a natural lift $f_\mathcal{C}$, of f, to the contact bundle

$$f_\mathcal{C} : \{x, y, \phi, p, q\} \mapsto \{x + p, y + q, \phi + \tfrac{1}{2}(p^2 + q^2), p, q\} \equiv \{X, Y, \Phi, P, Q\}. \tag{2}$$

The map $f_\mathcal{C}$ preserves the Cartan form

$$\theta = \mathrm{d}\phi - p\mathrm{d}x - q\mathrm{d}y \tag{3}$$

i.e. $f_\mathcal{C}^*(\theta) = \theta$, and is therefore a contact transformation.

The graph of ϕ is a legendrian submanifold in $J^1\mathcal{D}$, the manifold of 1-jets of smooth functions on $\mathcal{D}$ with coordinates

$$\{x, y, \phi, p = \phi_x, q = \phi_y\}. \tag{4}$$

Let us recall briefly the correspondence between the Monge-Ampère operators (equations) and the contact geometry of $J^1\mathcal{D}$ (e.g. Lychagin 1979). A Monge-Ampère equation is a relation of the form

$$A + B\phi_{xx} + 2C\phi_{xy} + D\phi_{yy} + E(\phi_{xx}\phi_{yy} - \phi_{xy}^2) = 0. \tag{5}$$

The coefficients A, B, C, D, E depend on x, y, ϕ, p, q where x and y are independent variables and $\phi = \phi(x, y)$ is an unknown function. We suppose also that

$$A^2 + B^2 + C^2 + D^2 + E^2 \neq 0. \tag{6}$$

It is known (Lychagin 1979) that the left-hand side of (5) is given by the differential 2-form

$$\omega = A\mathrm{d}x \wedge \mathrm{d}y + B\mathrm{d}p \wedge \mathrm{d}y + C(\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}q \wedge \mathrm{d}y) + D\mathrm{d}x \wedge \mathrm{d}q + E\mathrm{d}p \wedge \mathrm{d}q. \tag{7}$$

That is, evaluating ω on the graph of ϕ yields a Monge-Ampère equation on the second jet bundle $J^2\mathcal{D}$ with coordinates $\{x, y, \phi, p, q, r, s, t\}$, where

$$p = \phi_x, \quad q = \phi_y, \quad r = \phi_{xx}, \quad s = \phi_{yy} \quad \text{and} \quad t = \phi_{xy}. \tag{8}$$

The left-hand sides of equations of type (5) are in one-to-one correspondence with the differential 2-forms of type (7).

Consider the exterior differential of the Cartan form θ

$$\Omega = \mathrm{d}\theta = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q. \tag{9}$$

The differential 2-form (7) is *effective*, which means that

$$i_{\frac{\partial}{\partial\phi}}\omega = 0 \quad \text{and} \quad \omega \wedge \Omega = 0. \tag{10}$$

2.2 Semi-geostrophic theory

The motion of a typical particle in $\mathcal{D}$ can be described by expressing the current horizontal coordinates

$$x = x(a, b, t), \quad y = y(a, b, t) \tag{11}$$

as functions, on the right, of the particle labels a, b and the time t. If $t = 0$ is the reference time, the functions in (11) have the properties $x(a, b, 0) = a,\ y(a, b, 0) = b$. Care must be taken when interpreting a partial derivative, e.g. $\partial/\partial x$, as this means varying x *on a particle*, holding y, p and q constant.

The transformation (2), when restricted to the graph of ϕ, is the *geostrophic momentum transformation* (Sewell and Roulstone 1994, Theorem 11), and the semi-geostrophic equations can be written in the hamiltonian form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\partial \Phi}{\partial Y}, \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\partial \Phi}{\partial X}, \tag{12}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + \dot{X}\frac{\partial}{\partial X} + \dot{Y}\frac{\partial}{\partial Y} \tag{13}$$

and the canonical coordinates X, Y, are given in this notation by

$$X = x + p, \quad Y = y + q, \quad X, Y \in J^1\mathcal{D}. \tag{14}$$

The function ϕ represents some suitably scaled geopotential (or the depth in shallow water theory). In meteorological terminology, the vector field $(-q, p) \equiv (u_{\mathrm{G}}, v_{\mathrm{G}})$ is known as the geostrophic wind. The mapping $\{x, y\} \mapsto \{X, Y\}$ is a Legendre transformation

$$X = \psi_x, \quad Y = \psi_y, \quad \text{where} \quad \psi = \tfrac{1}{2}(x^2 + y^2) + \phi \tag{15}$$

and

$$x = \Psi_X, \quad y = \Psi_Y, \quad \text{where} \quad \Psi = \tfrac{1}{2}(X^2 + Y^2) - \Phi = \mathbf{X}.\mathbf{x} - \psi. \tag{16}$$

The singularities of this map can be interpreted as atmospheric fronts (Chynoweth *et al.* 1988; Chynoweth and Sewell 1989, 1991).

Using (12), and a continuity equation of the form (Roulstone and Sewell 1996a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(x, y)}{\partial(a, b)}\right) = 0, \tag{17}$$

one can show (using (8))

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + r + s + rs - t^2\right) = 0. \tag{18}$$

This is an expression for the conservation of potential vorticity.

The Monge-Ampère operator in (18) can be obtained from Ω, in (9), via a transformation from $\mathcal{V}^*$ to $J^2\mathcal{D}$. Explicitly,

$$F_{J^2\mathcal{D}} \equiv \{x, y, p, q\} \mapsto \{-q + \tfrac{1}{2}(pt - qr - y), p - \tfrac{1}{2}(qt - ps - x), x, y\}, \tag{19}$$

for then

$$F^*_{J^2\mathcal{D}}(\Omega) = [1 + r + s + rs - t^2]\mathrm{d}x \wedge \mathrm{d}y. \tag{20}$$

The transformation $\{p, q\} \mapsto \{-q + \frac{1}{2}(pt - qr), p - \frac{1}{2}(qt - ps)\}$ is exactly the transformation of Roulstone and Sewell (1996a, Eqn. (49)). The two form $\mathrm{d}X \wedge \mathrm{d}Y$, expressed on $J^2\mathcal{D}$, is simply

$$\mathrm{d}X \wedge \mathrm{d}Y = [1 + r + s + rs - t^2]\mathrm{d}x \wedge \mathrm{d}y = F^*_{J^2\mathcal{D}}(\Omega). \tag{21}$$

In the next section we show how this 2-form arises from a symplectic representation of a quaternionic structure.

3 A quaternionic structure for balanced models

Consider the quaternion

$$\ell = x + icq + jcp + ky \tag{22}$$

where $c \in \mathbb{R}$ is a constant, to be specified later, and

$$\left.\begin{array}{rcccl} i^2 &=& j^2 &=& k^2 = -1, \\ ij &=& -ji &=& k, \\ jk &=& -kj &=& i, \\ ki &=& -ik &=& j, \end{array}\right\} \tag{23}$$

is the standard quaternion algebra. Using the representation (22), the 2-form $\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}$ can be written as a triplet of self-dual 2-forms.

Using (23), we write ℓ in a symplectic representation

$$\ell = x + icq + k(y + icp) \tag{24}$$

and define

$$z \equiv x + icq, \quad w \equiv y - icp. \tag{25}$$

Then, using (24), we construct the 2-form

$$\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell} = \mathrm{d}z \wedge \mathrm{d}\bar{z} - \mathrm{d}w \wedge \mathrm{d}\bar{w} - k(\mathrm{d}z \wedge \mathrm{d}w + \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{w}), \tag{26}$$

whose real part

$$\Omega_{\mathbb{C}} \equiv \mathrm{d}z \wedge \mathrm{d}\bar{z} - \mathrm{d}w \wedge \mathrm{d}\bar{w} \tag{27}$$

is a Kähler structure with respect to the symplectic representation (24). The complex part of $\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}$ corresponds to a Monge-Ampère equation on $J^2\mathcal{D}$

$$\mathrm{d}z \wedge \mathrm{d}w + \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{w} = 2[1 + c^2(rs - t^2)]\mathrm{d}x \wedge \mathrm{d}y. \tag{28}$$

Consider the $\mathbb{C}^2$ spanned by $\{z, w\}$. Then the transformation (cf. (1))

$$g : \{z, w\} \mapsto \{z + ap, w + aq\} \equiv \{Z, W\} \quad a \in \mathbb{R} \text{ a constant,} \tag{29}$$

preserves $\Omega_{\mathbb{C}}$, i.e. $g^*(\Omega_{\mathbb{C}}) = \Omega_{\mathbb{C}}$. Furthermore

$$g^*(\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}) = \Omega_{\mathbb{C}} - 2k[1 + a(r+s) + (a^2 - c^2)(rs - t^2)]\mathrm{d}x \wedge \mathrm{d}y \tag{30}$$

of which the complex part is a Monge-Ampère operator.

To understand the appearance of the representation (24) we recall some facts about the connections between *elliptic* Monge-Ampère equations and families of complex structures on their solutions (Lychagin and Rubtsov 1983, Lychagin *et al.* 1993). Let us consider an elliptic Monge-Ampère equation with constant coefficients (28). The corresponding 2-form (7) defines a family of operators $A_\mu : T_\mu \mathcal{V}^* \mapsto T_\mu \mathcal{V}^*, \mu \in \mathcal{V}^*$ by the rule

$$\Omega(A_\mu X, Y) = \omega(X, Y) \quad X, Y \in T_\mu \mathcal{V}^*. \tag{31}$$

The operators A_μ satisfy the characteristic equation which in our case reads $A_\mu^2 + 1 = 0$ and the field $\mu \mapsto A_\mu$ defines an almost complex structure on $\mathcal{V}^*$. Direct verification (see Lychagin *et al.* 1993) shows that the closedness of (7) implies vanishing of the Nijenhuis tensor of A_μ and hence, by Newlander-Nirenberg theorem, the integrability of the complex structure. This is exactly the *new* complex structure compatible with the representation (24).

Conversely, given any almost complex structure A on $\mathcal{V}^*$ there is a unique 2-form ω on $\mathcal{V}^*$ such that $\Omega + i_A\omega$ is (2,0)-form on $\mathcal{V}^*$ with the respect to A and it provides the ellipticity of the Monge-Ampère operator, corresponding to ω. The legendrian submanifolds of (28) are A-*(pseudo) holomorphic* curves.

We remark that the quaternionic representation gives a simple explanation and a proof of the isomorphism between the grassmanian of Cartan's planes for an elliptic Monge-Ampère equation and $\mathbb{CP}^1$ (Lychagin 1982). The point of the Cartan grassmanian is a tangent plane to a solution of the Monge-Ampère equation and, hence, carries an A-complex structure, but the quaternionic representation permits us to consider this structure as a point of a S^2 ($\mathbb{CP}^1$) parametrizing all complex structures on the underlying quaternionic space.

Using (30), we can now see how the quaternionic structure arises in L_1 dynamics, and in other models discussed by McIntyre and Roulstone (1996, §9). In that paper, L_1 corresponds to the choice $a = c = 1$. The 2-form (20) is obtained by setting $a = 1, c = 0$, so the complex part of (30) becomes

$$-2k[1 + r + s + rs - t^2]\mathrm{d}x \wedge \mathrm{d}y, \tag{32}$$

which, apart from the $-2k$ which arises from the multiplication of the quaternion, is the canonical 2-form of semi-geostrophic theory. The choice $a = 1, c = \sqrt{3}$ gives a new balance model (see McIntyre and Roulstone *op. cit.* for further details). In all three examples, we identify (29) with the notation of McIntyre and Roulstone *op. cit.* as simply $Z = X$ and $W = Y$, except that McIntyre and Roulstone choose to include $\sqrt{-1}$ in c.

4 Summary

The geometric description of the semi-geostrophic equations has been crucial in the construction of finite-element numerical models (Cullen and Purser 1984, Purser and Cullen 1987). A description in terms of Legendre transformations is summarized in Chynoweth and Sewell (1991), and a description in terms of the polar factorization of maps is given in Brenier (1996).

We have shown how the geostrophic coordinates X, Y, of semi-geostrophic theory and the complex coordinates for L_1 dynamics discovered by McIntyre and Roulstone (1996) can be viewed as part of an underlying quaternionic description of the slow manifold defined by geostrophic balance. This slow manifold possesses a natural Kähler structure with respect to a symplectic representation of the quaternion.

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