COM 1021 Mathematical Methods for Computing I
Autumn 2009

COURSEWORK I - SOLUTIONS

Functions

1. (a) Find the **range** of the function in each case:
   (i) $y \in [-8, \infty)$ (ii) $y \in [-46, 2]$. [4]

   (b) Find the **domain** of the function in each case:
   (i) $x \in \mathbb{R}$ (ii) $x \leq -1, x \geq 2$ (ii) $x \in \mathbb{R} \setminus \{-1, 2\}$. [5]

2. If $f(x) = 4x - 3$ and $g(x) = 5 - 2x$;

   (a) Find
   (i) $f^{-1}(x) = \frac{x+3}{4}$ (ii) $g^{-1}(x) = \frac{5-x}{2}$
   (iii) $f(g(x)) = 17 - 8x$ (iv) $g(g(x-1)) = 4x - 9$

   (b) Find
   (i) $g(f(x)) - f(g(x)) = 17 - 8x - (11 - 8x) = 6$
   (ii) $f^{-1}(g^{-1}(x)) - g^{-1}(f^{-1}(x)) = \frac{11-x}{8} - \frac{17-x}{8} = -\frac{3}{4}$.

   (c) Find $f(f(f(x)))$, $f^{(4)}(x)$ and $f^{(5)}(x)$. Can you find a general expression for $f^{(n)}(x)$?
   $f^{(3)}(x) = 64x - 63$, $f^{(4)}(x) = 256x - 255$, $f^{(5)}(x) = 1024x - 1023$,
   $f^{(n)}(x) = 4^n(x-1) + 1$

   (d) Suppose $F(x) = f(x^2)$ and $G(x) = g(x^2)$. Solve $F(G(x)) = 0$, expressing your answers in exact (i.e. surd) form. [27]
   $F(G(x)) = 4(5 - 2x^2)^2 - 3 \Rightarrow (5 - 2x^2)^2 = \frac{3}{4} \Rightarrow 5 - 2x^2 = \pm\frac{\sqrt{3}}{2} \Rightarrow x = \pm\frac{1}{2}\sqrt{10 \pm \sqrt{3}}$

3. Find the inverse of

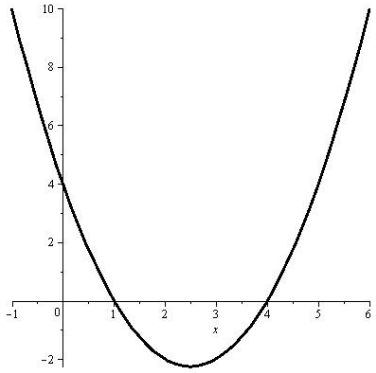
   (i) $y = \ln(3x + 2)$, $y^{-1} = \frac{1}{5}(e^x - 2)$

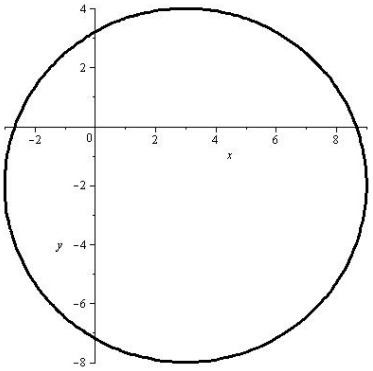
   (ii) $y = \dfrac{1 + 3x}{3 - x}$, $y^{-1} = \dfrac{3x - 1}{3(x + 1)}$ [4]

Graphs

4. Sketch the graphs of the following, clearly indicating the coordinates of the points at which the curve crosses the x and y axes and any local minima or maxima.

   (a) $2y + 3x - 4 = 0$ (b) $y = x^2 - 5x + 4$ (c) $x^2 - 6\,x + y^2 + 4\,y - 23 = 0$ (10)





For the circle, intersection with x axis $x = 3 \pm 4\sqrt{2}$, with y axis $y = -2 \pm 3\sqrt{3}$. Maxima and minima at $x = 3, y = 4$ max $y = -8$ min.

Equations and inequalities

5. The pressure P experienced by a diver under water is related to the diver's depth d by the equation

$$P = kd + c.$$

At the surface the pressure is 1 atmosphere and at a depth of 200 metres the pressure is 21 atmospheres.

(a) Find c and k. $c = 1, \quad k = \frac{1}{10}$

(b) Find the pressure at 117 metres. $P(117) = 12.7$

(c) Find the depth at which the pressure is 39 atmospheres.$d(39) = 380$ [3]

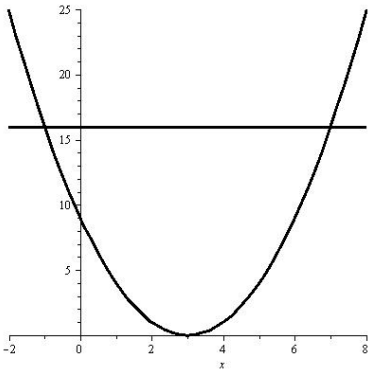
6. Find all the solutions of the following equations

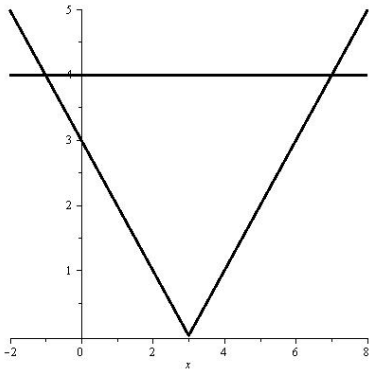
(a) $3x^2 - 4x - 5 = 0, \qquad x = \frac{2}{3} \pm \frac{\sqrt{19}}{3} = -0.786, 2.120$

(b) $6x^2 + x - 2 = 0, \qquad x = \frac{1}{2}, \quad -\frac{2}{3}$

(c) $x^3 - 5x^2 + 6x = 0 \qquad x = -1, 0, 6.$ [7]

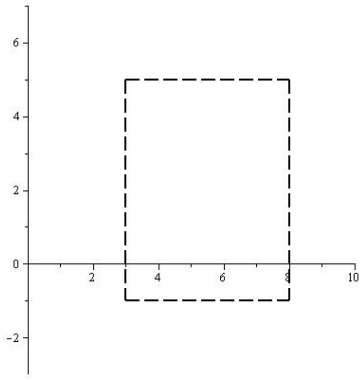
7. (a) Solve the following inequalities and sketch the appropriate areas on a graph.
(i) $(x-3)^2 \leq 16 \quad -1 \leq x \leq 7$(ii) $|x-3| > 16. \quad x < -1, \quad x > 7$ [6]



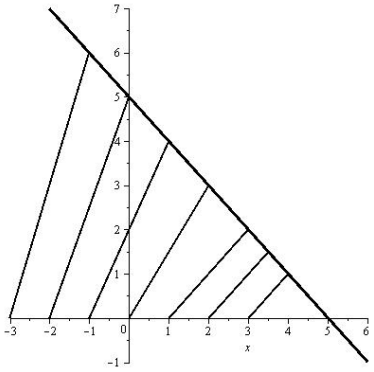


(b) Sketch the areas on the graph represented by

(i) $3 < x < 8, -1 < y < 5$ (ii) $x + y \leq 5$. [6]



Area inside rectangle excluding boundary



Area below the line including the line

Trigonometry

8. (a) Find the distance between the points $(-2, -7)$ and $(5, -4)$. $d = \sqrt{(-2-5)^2 + (-7-(-4))^2} = \sqrt{58} = 7.616$

(b) In a right angled triangle the hypotenuse is of length 12 and one of the angles is 55^o. Find the other angle and the lengths of the other two sides. $55^o, 12 \sin 55^o (9.83) 12 \cos 55^o (6.88)$

(c) (*) Use Maple to find **all** of the solutions of
$18 \sin x + x - 1 = 0$. Hint: plot a graph of the function first. [10]
Use the command seq(fsolve$(18 * \sin(x) + x - 1, x = -13 + n.. - 12 + n), n = 0..31)$ to generate all the 11 solutions;
$-11.77710108, -10.08846558, -5.890362316, -3.387842905, 0.05265462609, 3.267924210,$
$6.001612008, 9.944841945, 11.91486844, 16.77644178, 17.66615190$

Linear algebra

9. Matrices A and B are defined as follows $A = \begin{pmatrix} 1 & 5 & -4 \\ 4 & -1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 6 & 1 & 5 \\ -2 & 0 & 3 \\ 1 & 1 & -2 \end{pmatrix}$

Find $A - B$, AB and BA. [8]

$$A - B = \begin{pmatrix} -5 & 4 & -9 \\ 6 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} -8 & -3 & 28 \\ 29 & 7 & 11 \\ -2 & -2 & 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 10 & 29 & -31 \\ -2 & -10 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

10. Find the inverse of each of the following matrices:

$$C = \begin{pmatrix} 3 & 8 \\ 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad E = \begin{pmatrix} 7 & 7 \\ 3 & -1 \end{pmatrix}. \qquad [6]$$

$$C^{-1} = \begin{pmatrix} 1/3 & -8/3 \\ 0 & 1 \end{pmatrix} \quad D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 1/28 & 1/4 \\ \frac{3}{28} & -1/4 \end{pmatrix}$$

11. Solve the following system of simultaneous equations using Gaussian elimination [8]

$$\begin{aligned} x_1 + x_2 &= 7 \\ x_2 - x_3 + x_4 &= 5 \\ x_1 - x_2 + x_3 + x_4 &= 6 \\ x_2 + x_4 &= 10 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 7 \\ 0 & 1 & -1 & 1 & 5 \\ 1 & -1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 1 & 10 \end{array}\right)$$

$$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 7 & \\ 0 & 1 & -1 & 1 & 5 & \\ 0 & -2 & 1 & 1 & -1 & r_3 := r_3 - r_1 \\ 0 & 1 & 0 & 1 & 10 & \end{array}\right)$$

$$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & 7 & \\ 0 & 1 & -1 & 1 & 5 & \\ 0 & 0 & -1 & 3 & 9 & r_3 := r_3 + 2r_2 \\ 0 & 10 & 0 & 1 & 5 & r_4 := r_4 - r_2 \end{array}\right)$$

Then $x_3 = 5, \quad x_4 = \frac{14}{3}, \quad x_2 = \frac{16}{3}, \quad x_1 = \frac{5}{3}$.

12. (*) Find the inverse of the matrix M, where

$$M = \begin{pmatrix} 2 & 5 & 9 \\ 8 & 11 & 7 \\ 4 & 6 & 13 \end{pmatrix}.$$

$$M^{-1} = \begin{pmatrix} -\frac{101}{142} & \frac{11}{142} & \frac{32}{71} \\ \frac{38}{71} & \frac{5}{71} & -\frac{29}{71} \\ -\frac{2}{71} & -\frac{4}{71} & \frac{9}{71} \end{pmatrix}$$

(Hint: use Maple) Hence, solve the following system of simultaneous equations: [8]

$$\begin{aligned} 2x + 5y + 9z &= 4 \\ 8x + 11y + 7z &= 7 \\ 4x + 6y + 13z &= 1 \end{aligned}$$

$$\begin{pmatrix} -\frac{101}{142} & \frac{11}{142} & \frac{32}{71} \\ \frac{38}{71} & \frac{5}{71} & -\frac{29}{71} \\ -\frac{2}{71} & -\frac{4}{71} & \frac{9}{71} \end{pmatrix} \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{263}{142} \\ \frac{158}{71} \\ -\frac{27}{71} \end{pmatrix}$$