Module MAT1015: Techniques in Calculus

PART A

Course Structure

The first three weeks of this course will consist of revision and practice of A-level work, using the examples and exercises in Part A of these notes. Most of this work should be familiar. Ask your personal tutor about anything that you do not fully understand.

In Weeks 1,2 and 3 there will be a short class test The three sections in Part A of these notes will be tested as follows:

week commencing 5 October (week 1)	Algebraic methods
week commencing 12 October (week 2)	Series, functions, etc.
week commencing 17 October (week 3)	Calculus (Differentiation and Integration)

You must score at least 8/12 in each of these tests. If you do not achieve this, you will have to do a re-test. You must pass these tests (eventually) to pass the module. In the tests you may use a calculator, but **not** notes, books or lists of formulae. You should bring some paper for your working.

In Week 4, lectures will start on the main content of the course. Unless you have done Further Maths A-level, much of this will probably be new to you.

Three courseworks will be set, these will be corrected but you will not be assessed on the work. You will be able to discuss this work in the weekly meeting with your personal tutor. Some of the lecture sessions will be full-class tutorials, where you can ask for guidance with any of your work for this module.

There will be two tests on the new material, later in the semester (approximately weeks 11 and 15). See Part C for practice tests with solutions.

Module web-page

Go to the Maths Department home page (http://www.maths.surrey.ac.uk/teaching/) and click on 'Level 1' (under 'For Current Students'). Alongside 'MAT1015 Techniques in Calculus' is a link to the information page where you will find announcements, dates of tests, etc.

Books

In the University Library, the section coded **51** contains books on 'general mathematics'. More books on Calculus can be found in the section coded **517**.

A good textbook is 'Guide to Mathematical Methods' by J. Gilbert and C. Jordan (Palgrave Mathematical Guides). Parts of this should also be useful for other modules. A excellent but much larger book is 'Calculus a complete course' by R. Adams, (which covers most of the calculus you will need for the entire degree course) while a simpler and cheaper book, giving a good basic introduction to the material, is 'Calculus Demystified' by Steven G. Krantz.

Other relevant books which may be found in the library and/or the bookshop are those by Anton; Thomas; Mustoe & Barry; McGregor, Nimmo & Stothers. 'Engineering Mathematics' by K. Stroud also contains many excellent worked examples and exercises.

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1. Algebraic Methods

Algebra is the language of mathematical reasoning. It is important to be fluent in this language and to know its conventions, so that you can read and write mathematics in a way which is correct and easily understood.

You should know most of the following methods already. Study the examples and make sure you understand them. There are exercises at the end of each main section, starting on pages 8, 14 and 19. You should do as many of them as you need to. If you find them easy, try the 'Longer Questions' on page 20. Answers to the exercises are at the end of Part A.

Simplification

1. $2a(3b-4)-3b(a-2) \equiv 6ab-8a-3ab+6b \equiv 3ab-8a+6b$
2. $\dfrac{1+x}{2}+\dfrac{2-x}{3} \equiv \dfrac{3(1+x)+2(2-x)}{6} \equiv \dfrac{x+7}{6}$
3. $\dfrac{2}{x}\times\dfrac{3x}{y} \equiv \dfrac{6x}{xy} \equiv \dfrac{6}{y}$
4. $\dfrac{3}{2x}\div\dfrac{6}{x^2} \equiv \dfrac{3}{2x}\times\dfrac{x^2}{6} \equiv \dfrac{x}{4}$
5. $\dfrac{2}{x+3}-\dfrac{1}{x-4} \equiv \dfrac{2(x-4)-(x+3)}{(x+3)(x-4)} \equiv \dfrac{x-11}{(x+3)(x-4)}$
6. $\dfrac{1}{(x+1)(x-1)}-\dfrac{1}{x+1} \equiv \dfrac{1-(x-1)}{(x+1)(x-1)} \equiv \dfrac{2-x}{x^2-1}$

Note that the above have all been written with '$\equiv$' signs, to show that they are *identities*, i.e. they are true for *all* values of x, y, etc. There is nothing wrong with using '$=$' signs, but it is important to know the difference between an identity and an *equation*, which is true only for certain values of the symbols.

Examples. Identity: $\sin 2x = 2\sin x\cos x$ Equation: $2x = 4$ (which implies that $x = 2$).

Expansion

1. $(x-2y)(2x-3y) \equiv 2x^2-3xy-4xy+6y^2 \equiv 2x^2-7xy+6y^2$
2. $(x+y)^2 \equiv (x+y)(x+y) \equiv x^2+2xy+y^2$

   You should be able to square a bracket this way without multiplying out. Just add the squares of the terms and twice their product, as in the next examples:

3. $(2p-3q)^2 \equiv (2p)^2+2(2p)(-3q)+(-3q)^2 \equiv 4p^2-12pq+9q^2$
4. $\left(\dfrac{2}{x}+\dfrac{x}{2}\right)^2 \equiv \dfrac{4}{x^2}+2+\dfrac{x^2}{4}$
5. $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2xz$
6. $(x+y)^3 \equiv (x+y)(x^2+2xy+y^2) \equiv x^3+3x^2y+3xy^2+y^3$ [Useful to learn]
7. $(3a-2b)^3 \equiv (3a)^3+3(3a)^2(-2b)+3(3a)(-2b)^2+(-2b)^3$ [using the last result]
   $\equiv 27a^3-54a^2b+36ab^2-8b^3$

Factorisation

To *factorise* an expression means to write it as the *product* of its factors. Look for any *common factors* first, then see if one of the standard methods can be used: grouping in pairs; trinomials (quadratic-type expressions); difference of two squares; and two which you may not have met before but which should be learnt, the sum and difference of two cubes. Always check that your answer cannot be factorised any further.

1. $12x^2y + 18xy^3 \equiv 6xy(2x + 3y^2)$
2. $4ax - 2ay - 6bx + 3by \equiv 2a(2x - y) - 3b(2x - y) \equiv (2a - 3b)(2x - y)$
3. $x^2 - 11x + 10 \equiv (x - 1)(x - 10)$
4. $2x^2 + 7xy - 15y^2 \equiv (2x - 3y)(x + 5y)$
5. $x^2 + 2xy + y^2 \equiv (x + y)(x + y) \equiv (x + y)^2$
6. $x^2 - y^2 \equiv (x + y)(x - y)$ [Difference of Squares — this should be learnt.]
7. $32m^2 - 72n^2 \equiv 8(4m^2 - 9n^2) \equiv 8(2m + 3n)(2m - 3n)$, using the previous result
8. $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$ [This and the next one should be learnt.]
9. $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$
10. $8p^3 - 27q^3 \equiv (2p - 3q)(4p^2 + 6pq + 9q^2)$, using the previous result
11. $x^4 - y^4 \equiv (x^2 + y^2)(x^2 - y^2) \equiv (x^2 + y^2)(x + y)(x - y)$
12. $x^4 - 8x^2 - 9 \equiv (x^2 - 9)(x^2 + 1) \equiv (x - 3)(x + 3)(x^2 + 1)$
13. $x^3 - 3x^2 + 4x - 2$: notice that this is 0 when $x = 1$, so $x - 1$ must be a factor. Divide by $x - 1$ to get $(x - 1)(x^2 - 2x + 2)$, which *in this case* does not factorise further.
14. $(x - 1)(x^2 + 2) + (2x - 2)(x^2 + 1) \equiv (x - 1)[(x^2 + 2) + 2(x^2 + 1)] \equiv (x - 1)(3x^2 + 4)$
    [Spot the common factor $(x - 1)$ at the start and so avoid multiplying out.]

Completing the Square

This method is useful for solving quadratic equations (it's how we get the quadratic formula) and also for finding the maximum or minimum of a quadratic without differentiating.

1. $x^2 - 8x + 12 \equiv (x - 4)^2 - 4$

   Hence $x^2 - 8x + 12 = 0$ when $(x - 4)^2 = 4$, i.e. $x - 4 = \pm 2$ so $x = 2$ or $x = 6$

   $x^2 - 8x + 12$ has minimum value -4, when $(x - 4)^2 = 0$, i.e. when $x = 4$
2. $5 - 4x - x^2 \equiv -(x^2 + 4x - 5) \equiv -((x + 2)^2 - 9) \equiv 9 - (x + 2)^2$,

   so $5 - 4x - x^2$ has maximum value 9, when $x = -2$
3. $2x^2 - 8x + 13 \equiv 2(x^2 - 4x) + 13 \equiv 2((x - 2)^2 - 4) + 13 \equiv 2(x - 2)^2 + 5$
4. $x^4 + 2x^2 + 11 \equiv (x^2 + 1)^2 + 10$

Algebraic Long Division; Factor and Remainder Theorems

1. $\dfrac{x^2+4}{x+1} \equiv \dfrac{(x+1)(x-1)+5}{x+1} \equiv x-1+\dfrac{5}{x+1}$
2. $(6x^3-3x^2+5x+4) \div (2x+1) \equiv 3x^2-3x+4$, by Algebraic Long Division.
   See a text book or ask for help if you can't remember how this is done.
3. $(x^3-2x^2+3x+4) \div (x+2) \equiv x^2-4x+11-\dfrac{18}{x+2}$
4. $x+3$ is a factor of $\mathrm{f}(x) \equiv x^4+2x^3-27$ by the Factor Theorem, since $\mathrm{f}(-3)=0$
5. The remainder when $\mathrm{g}(x) \equiv x^3+x^2+1$ is divided by $x+2$ is equal to $\mathrm{g}(-2)=-3$

If you didn't already know the method in the last example, learn it now. It is the **Remainder Theorem**: to find the numerical remainder when a polynomial is divided by $(ax+b)$, substitute $x=-b/a$ in the expression. It's much quicker than doing the division, if the remainder is all you need to find.

Partial Fractions

A numerical fraction can be split up into a sum of two fractions in infinitely many ways (e.g. $\frac{1}{2}=\frac{1}{4}+\frac{1}{4}=\frac{1}{8}+\frac{3}{8}=\ldots$), but with algebraic fractions this can be done in just one way.

At this stage there will only be *linear* factors in the denominator. For each factor $(ax+b)$ in the denominator, there is a partial fraction $\dfrac{p}{ax+b}$.

If the degree of the numerator is *not less* than that of the denominator, *divide* first.

1. To express $\dfrac{x}{x^2-x-12}$ in partial fractions, let it be $\dfrac{p}{x-4}+\dfrac{q}{x+3}$.

   Then $p(x+3)+q(x-4) \equiv x$, so $p=\dfrac{4}{7}, q=\dfrac{3}{7}$.

   Answer: $\dfrac{x}{x^2-x-12} \equiv \dfrac{4}{7(x-4)}+\dfrac{3}{7(x+3)}$.

2. $\dfrac{x^2+3}{(x-1)(x-2)(x+3)} \equiv \dfrac{p}{x-1}+\dfrac{q}{x-2}+\dfrac{r}{x+3}$

   $\equiv \dfrac{p(x-2)(x+3)+q(x-1)(x+3)+r(x-1)(x-2)}{(x-1)(x-2)(x+3)}$

   so $p(x-2)(x+3)+q(x-1)(x+3)+r(x-1)(x-2) \equiv x^2+3$.

   Put $x=1$: $-4p=4$. Put $x=2$: $5q=7$. Put $x=-3$: $20r=12$.

   Hence $p=-1, q=7/5, r=3/5$, giving $\dfrac{-1}{x-1}+\dfrac{7}{5(x-2)}+\dfrac{3}{5(x+3)}$.

3. $\dfrac{x^3+3}{x^2-1} \equiv x+\dfrac{x+3}{x^2-1} \equiv x+\dfrac{p}{x+1}+\dfrac{q}{x-1}$, so $p(x-1)+q(x+1) \equiv x+3$.

   Then $p=-1, q=2$, so $\dfrac{x^3+3}{x^2-1} \equiv x-\dfrac{1}{x+1}+\dfrac{2}{x-1}$.

Linear Equations

1. $3(x+4) - 2(3-2x) = 4x - 9 \quad \Rightarrow \quad 7x + 6 = 4x - 9 \quad \Rightarrow \quad 3x = -15 \quad \Rightarrow \quad x = -5$
2. $\dfrac{x+2}{3} = \dfrac{2x-1}{5} \quad \Rightarrow \quad 5(x+2) = 3(2x-1) \quad \Rightarrow \quad x = 13$
3. $5x + 3y = 7, 3x - y = 1$. Multiply the second equation by 3 to get $9x - 3y = 3$. Add to the first equation : $14x = 10$ so $x = \dfrac{5}{7}$. Substitute back to get $y = \dfrac{8}{7}$.

   The solutions of these *simultaneous* linear equations give the x and y coordinates of the point where two straight lines intersect.

Quadratic and higher-order Equations

The values of x which make an equation true are called the *solutions* or *roots* of the equation. The values of a function $\mathrm{f}(x)$ for which $\mathrm{f}(x) = 0$ can also be called the *zeros* of f.

1. $x^2 - 18x + 45 = 0$. By factorisation, $(x-3)(x-15) = 0$. The roots are $x = 3, x = 15$.
2. $x^2 - 18x + 70 = 0$. By Completing the Square: $(x-9)^2 - 11 = 0$, so $x = 9 \pm \sqrt{11}$.
3. $3x^2 - 16x + 11 = 0$. By the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{16 \pm \sqrt{124}}{6} = \dfrac{8 \pm \sqrt{31}}{3}$.
4. $(x+2)(x^2+4x+5) = 2(x+2) \Rightarrow (x+2)(x^2+4x+5-2) = 0 \Rightarrow (x+2)(x+1)(x+3) = 0 \Rightarrow x = -1, x = -2, x = -3$. [Do not divide each side of the original equation by $(x+2)$, as you would not get the solution $x = -2$.]
5. *Non-linear* simultaneous equations: make one unknown the subject of the simpler equation and substitute in the other.

   e.g. $xy = 3, x^2 + y = 8$. Write $y = 8 - x^2$, so $x(8 - x^2) = 3$, i.e. $x^3 - 8x + 3 = 0$. Notice a root $x = -3$; divide by $x + 3$ to get $(x+3)(x^2 - 3x + 1) = 0$.

   Hence $x = -3$ or $x = \dfrac{3 \pm \sqrt{5}}{2}$. Solutions are approx. $(-3, -1), (0.38, 7.85), (2.62, 1.15)$. These are the points of intersection of the two graphs.

The formula for solving $ax^2 + bx + c = 0$ shows that this quadratic equation has two real roots if $b^2 - 4ac > 0$, one (repeated) real root if $b^2 - 4ac = 0$ and no real roots if $b^2 - 4ac < 0$.

Example Find a condition for $9x^2 - 8kx + 4 = 0$ to have two distinct real roots.

We need $64k^2 > 4(9)(4)$, so $4k^2 > 9$, so $k < -\frac{3}{2}$ or $k > \frac{3}{2}$.

If $ax^2+bx+c = 0$, $a \neq 0$, has roots $x = \alpha, x = \beta$ then the equation must be $(x-\alpha)(x-\beta) = 0$, i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Comparing this with the original equation, which is $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$, we see that $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$. You may not have met these formulae before, so learn them now. They are very easy to remember and use.

Example Find the sum and the product of the roots of $5x^2 - 90x - 2000 = 0$.

The sum is $-\dfrac{b}{a} = -\dfrac{-90}{5} = 18$ and the product is $\dfrac{c}{a} = \dfrac{-2000}{5} = -400$.

Inequalities

If an inequality is multiplied or divided through by a *negative* quantity, then the inequality sign must be *reversed*. It is therefore *not* safe to multiply both sides by an algebraic quantity unless it is known to be positive.

1. $2x - 7 < 8x + 5 \Rightarrow -6x < 12 \Rightarrow x > -2$
2. $|x - 3| > 5 \Rightarrow x - 3 > 5$ or $x - 3 < -5 \Rightarrow x < -2$ or $x > 8$
3. $x^2 - 8x + 12 \leq 0 \Rightarrow (x-2)(x-6) \leq 0 \Rightarrow 2 \leq x \leq 6$. A graph will help. Sketch $y = x^2 - 8x + 12$ and read off the values of x where it lies *below or on* the x-axis.
4. $x^2 - 8x + 12 > 0 \Rightarrow (x-2)(x-6) > 0 \Rightarrow x < 2$ or $x > 6$ (where the graph is *above* the x-axis). Do not write nonsense like $2 > x > 6$, since 2 is not > 6 !!
5. $\dfrac{x}{2} \geq \dfrac{2}{x} \Rightarrow x > 0$ and $x^2 \geq 4$, or $x < 0$ and $x^2 \leq 4$. Solution is $x \geq 2$ or $-2 \leq x < 0$.

   Alternatively, sketch the graphs of $y = \dfrac{x}{2}$ and $y = \dfrac{2}{x}$ and find the set of values of x for which the first graph lies *on or above* the second graph.

Transforming Formulae

1. If $2ay + 3bx = c$ then $3bx = c - 2ay$ so $x = \dfrac{c - 2ay}{3b}$
2. If $\dfrac{x^2}{a} = p - q$ then $x^2 = a(p-q)$ so $x = \pm\sqrt{a(p-q)}$.
3. If $2ax + 3by = cx + dy$ then $2ax - cx = dy - 3by$, so $x(2a - c) = y(d - 3b)$, giving $x = \dfrac{y(d-3b)}{2a - c}$
4. If $(x-a)(x-b) = ab$ then $x^2 - (a+b)x = 0$ so $x(x - [a+b]) = 0$ so $x = 0$ or $x = a + b$

Exercises for Section 1

1. Write each of the following expressions in its simplest form without brackets:

   (a) $5\{x - 4[y + 3(z - 6)]\}$ (b) $\dfrac{1}{4}[(x+y)^2 - (x-y)^2]$

   (c) $(x-y)(x^2 + xy + y^2)$ (d) $x(y - 2z) - y(z - 2x) - z(x - 2y)$

2. Express as single algebraic fractions in their simplest forms :

   (a) $\dfrac{x^7y^5}{y^8x^3}$ (b) $\dfrac{6x}{12x^2 + 18xy}$

   (c) $\dfrac{8y^2}{y+2} \div \dfrac{2y^3}{y^2 + 3y + 2}$ (d) $\dfrac{x+2}{6} + \dfrac{3x-4}{9}$

   (e) $\dfrac{x+1}{x} + \dfrac{x}{x+1}$ (f) $\dfrac{2x}{x^2 - 1} - \dfrac{3}{x+1}$

3. Complete the square for each of the following expressions. Hence find the maximum or minimum value of each, and the value of x at which this occurs.

   (a) $x^2 - 8x - 7$, (b) $11 + 2x - 4x^2$, (c) $x^4 + 4x^2 + 7$.

4. Factorise the following as far as possible:
   (a) $x^2 + 5x + 6$ (b) $x^4 - x^2 - 12$ (c) $4ax - 6ay - 12x^2 + 18xy$
   (d) $4x^2 + 8x + 3$ (e) $x^3 - 2x^2 - 5x + 6$ (f) $121p^2 - 169q^2$
   (g) $x^3 + 64y^3$ (h) $x^4 - 16x^2$ (i) $(4x - 3y)^2 - (2x + 3y)^2$
   (j) $(x^2 - 4)(x + 3) - (x + 2)(x^2 + x - 6)$ (Do not multiply the brackets out!)

5. Perform the following divisions:
   (a) $(x^2 - 9x + 20) \div (x - 4)$ (b) $(x^2 + x + 1) \div (x + 1)$
   (c) $(2x^3 - 7x^2 + 8x - 8) \div (2x - 3)$ (d) $(x^3 + 2x^2 + x + 5) \div (x^2 + x + 1)$

6. Express the following in partial fractions:
   (a) $\dfrac{5x + 1}{x^2 + x - 2}$ (b) $\dfrac{x(x + 31)}{(x + 1)(x - 4)(2x - 1)}$ (c) $\dfrac{x^3 - 1}{(x - 2)(x - 3)}$

7. Answer the following **without** doing any algebraic division.
   (a) Show that $2x - 1$ is a factor of $2x^4 - x^3 - 8x + 4$.
   (b) Find the remainder when $x^3 - 7x^2 + 9$ is divided by $x + 1$.
   (c) $\mathrm{f}(x) = x^3 + ax^2 + bx - 1$. When $\mathrm{f}(x)$ is divided by $x + 2$, the remainder is 3. When $\mathrm{f}(x)$ is divided by $x - 3$, the remainder is 8. Find the values of a and b.

8. Solve the following equations by the method indicated.
   (a) $\dfrac{x - 2}{3} + \dfrac{1 - 2x}{5} = \dfrac{7}{10}$, by multiplying through by 30.
   (b) $2x + 5y = -9, -4x - 3y = 4$ by any method you know for simultaneous equations.
   (c) $x^2 - 8x - 13 = 0$, by completing the square.
   (d) $x^3 - 7x + 6 = 0$, by dividing by an obvious factor.
   (e) $xy = 7, x + y^2 = 50$, by forming a cubic equation in y.

9. (a) Find the range of values of k for which $kx^2 - 2kx + 1 = 0$ has no real roots.
   (b) Without solving the equation, find in terms of p the sum and the product of the roots of the equation $2px^2 - 4p^2x = 6p^3$, where $p \neq 0$.

10. Solve the following inequalities for x
    (a) $5x + 2 \geq 2x + 1$ (b) $(4x - 2)(3 - x) < 0$ (c) $x^2 + 2x \geq 15$
    (d) $\dfrac{1}{x + 1} < 2$ (e) $|x - 3| < 4$

11. In each of the following, express x in terms of the other letters.
    (a) $ax + b = cx + d$ (b) $\sqrt{p(x + p)} = x - p$ (c) $x = kx(1 - x)$
    (d) $x^2 - 4xy + 4y^2 = z^2$ [Hint : factorise the left-hand side.]

12. Rewrite the following expressions using the equations given and simplify fully:
    (a) $x^2 + y^2$ where $x = \dfrac{at}{\sqrt{1 + t^2}}, y = \dfrac{a}{\sqrt{1 + t^2}}$
    (b) $\dfrac{s^2 - 1}{s^2 - t^2}$ where $s = \sqrt{1 - x^2}, t = \sqrt{1 + x^2}$
    (c) $\dfrac{(p^2 + q^2)}{(p^2 - q^2)}$ where $p = \sqrt{q^2 - 1}$ (d) $x^2 + \dfrac{36}{x^2}$ where $x + \dfrac{6}{x} = 5$

2. Series, Functions, etc.

Sequences and Series

A **sequence** or **progression** is an ordered list of numbers. A **series** is formed by adding the successive terms of the sequence. For example : 1, 2, 4, 8, ... is a sequence, but 1 + 2 + 4 + 8 + ... is a series.

Sigma notation: $\sum_{k=a}^{b} \mathrm{f}(k)$ means the sum of all terms of the form f(k) as k takes integer values from a to b inclusive.

- The Arithmetic Progression (A.P.) with first term a and common difference d is $a, a+d, a+2d, ...$ The nth term is $T_n = a + (n-1)d$.

  Sum of first n terms is $S_n = \frac{n}{2}(2a + (n-1)d)$.

  Example : $\sum_{k=0}^{12}(4k-5) = (-5) + (-1) + ... + 43 = \frac{13}{2}(-10 + 12(4)) = 247$

- The Geometric Progression (G.P.) with first term a and common ratio r is $a, ar, ar^2, ...$ The nth term is $T_n = ar^{n-1}$.

  Sum of first n terms is $S_n = \frac{a(1-r^n)}{1-r}$.

  If $|r| < 1$, the series converges to the *sum to infinity* $S_\infty = \frac{a}{1-r}$

  Example : $\sum_{t=0}^{\infty} 3(2^{-t}) = 3 + \frac{3}{2} + \frac{3}{4} + ... = \frac{3}{1-1/2} = 6$

The Binomial Theorem

The expansions of $(x+y)^2$ and $(x+y)^3$ can be generalised to give the **Binomial Theorem**. Let n be a positive integer. Then

$$(x+y)^n \equiv x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n.$$

The coefficient of $x^{n-r}y^r$ is $\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$. Note that $\binom{n}{r} \equiv \binom{n}{n-r}$ and $0! = 1$.

The *binomial coefficients* $\binom{n}{r}$ can also be worked out from **Pascal's Triangle**,

whose rows are 1, 1 1, 1 2 1, 1 3 3 1, 1 4 6 4 1, 1 5 10 10 5 1, etc.

1. $(2x-3y)^5$
   $\equiv (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5$
   $\equiv 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

2. The term in a^5b^4 in the expansion of $\left(3a - \frac{1}{2}b\right)^9$ is

   $$\binom{9}{4}(3a)^5\left(\frac{-b}{2}\right)^4 = \frac{9!}{4!5!}\frac{243a^5b^4}{16} = \frac{126 \times 243a^5b^4}{16} = \frac{15309a^5b^4}{8}$$

Binomial Series

If n is a non-negative integer, the binomial expansion terminates; otherwise, it does not. In the case where it does not terminate, it takes the form of the infinite series

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

which is convergent provided that $|x| < 1$. The first term in the bracket *must* be 1.

When the second term in the bracket is small enough, the first few terms of the series give a good approximation to the function being expanded.

1. $(1-2x)^{-2} \equiv 1 - 2(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 + \frac{(-2)(-3)(-4)}{6}(-2x)^3 + ...$
   $\equiv 1 + 4x + 12x^2 + 32x^3 + ...$ Valid for $|-2x| < 1$, i.e. $-\frac{1}{2} < x < \frac{1}{2}$.
2. $(4+12x)^{1/2} \equiv 4^{1/2}(1+3x)^{1/2} = 2\left(1 + \frac{3}{2}x - \frac{9}{8}x^2 + ...\right) \equiv 2 + 3x - \frac{9}{4}x^2 + ...$
   Valid for $|3x| < 1$, i.e. $-\frac{1}{3} < x < \frac{1}{3}$.
3. $\frac{1}{1-x} \equiv (1-x)^{-1} \equiv 1 + x + x^2 + x^3 + ...$
4. $\frac{1}{1+x} \equiv (1+x)^{-1} \equiv 1 - x + x^2 - x^3 + ...$
   It's worth knowing these last two, which are valid for $|x| < 1$. You can easily check them by multiplying out: $(1-x)(1+x+x^2+...) \equiv 1$. (Why?)

Indices and Surds

For $x \neq 0$ we define $x^0 \equiv 1,\ x^{-1} \equiv \frac{1}{x},\ x^{-2} \equiv \frac{1}{x^2}$, etc.

Roots are given by fractional powers, so $x^{1/2} \equiv \sqrt{x}, x^{1/3} \equiv \sqrt[3]{x}, 2x^{-1/2} \equiv \frac{2}{\sqrt{x}}, x^{3/4} \equiv (\sqrt[4]{x})^3$.

Note that $\sqrt{x}$ or $x^{\frac{1}{2}}$ means the **positive** real square root of x. Thus $\sqrt{x^2} = x$ if $x \geq 0$, but $-x$ if $x < 0$. Never write things like $\sqrt{4} = \pm 2$.

A number written as an unevaluated root, like $\sqrt{2}$, is said to be in *surd form*. This is exact, whereas any decimal approximation is not.

Laws of indices: $x^a \times x^b \equiv x^{a+b},\ x^a \div x^b \equiv x^{a-b}$.

If $x > 0$ then $(x^a)^b \equiv x^{ab}$. Note that this may fail if $x < 0$, e.g. $[(-2)^2]^{1/2} = 4^{1/2} = 2 \neq (-2)^1$ so $(x^a)^b \neq x^{ab}$ here!

1. $27^{2/3} = 3^2 = 9$, $\qquad 16^{-3/4} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{8}$
2. $x^5 \times x^{-2} \equiv x^3$, $\qquad x^2 \div x^{-3} \equiv x^{2-(-3)} \equiv x^5$
3. $(x^4)^{-2} \equiv x^{-8}$, or $\frac{1}{x^8}$ $(x \neq 0)$ $\qquad (x^2)^{\frac{3}{4}} \equiv x^{\frac{3}{2}}$, or $\sqrt{x^3}$ $(x > 0)$
4. $\frac{x+1}{\sqrt{x^2-1}} \equiv \frac{x+1}{\sqrt{x+1}\sqrt{x-1}} \equiv \frac{\sqrt{x+1}}{\sqrt{x-1}}$, provided $x > 1$. [What is it if $x < -1$?]

5. $\sqrt{4x^2+4x+1} \equiv 2x+1$ if $x \geq -\dfrac{1}{2}$, but $-(2x+1)$ if $x < -\dfrac{1}{2}$.

6. $\dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ (Rationalising the denominator.)

7. $\dfrac{2+3\sqrt{2}}{3-4\sqrt{2}} = \dfrac{(2+3\sqrt{2})(3+4\sqrt{2})}{(3-4\sqrt{2})(3+4\sqrt{2})} = \dfrac{-30-17\sqrt{2}}{23}$

Exponential and Logarithmic Functions

An expression of the form a^x is called an **exponential** function of x. A special case, known as *the* exponential function, is $\exp(x)$ or e^x, where $\exp(1)$ or e is about 2.718.

If $x = a^y$, there is no simple way of making y the subject of this formula so for $a > 0$ we define the **logarithm** of x, to the **base** a, by $\log_a x = y$ where $a^y = x$.

In other words, the logarithm of a number is the power to which the base must be raised to give that number. In the case $a = \mathrm{e}$, we write $\ln x$ for $\log_{\mathrm{e}} x$ and call this the **natural logarithm** of x. Note that $\ln(\mathrm{e}^x) = \mathrm{e}^{\ln x} = x$.

Negative numbers and zero do not have real logarithms. Because logarithms are powers of a positive number, the laws of indices apply. Hence we have these **laws of logarithms**:

- $\log_a xy = \log_a x + \log_a y$, (Note that $\log_a(x+y)$ cannot be expanded.)
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$,
- $\log_a x^n = n\log_a x$,
- $\log_b x = \log_a x \div \log_a b$ (Change of Base rule).

Equations in which the unknown appears as a power can often be solved by taking logarithms, to a suitable base, of both sides.

1. $\log_3 9 = 2$; $\log_a 1 = 0$; $\log_4 2 = \dfrac{1}{2}$; $\log_2 \dfrac{1}{8} = -3$
2. $\ln \mathrm{e}^4 = 4$; $\exp(\ln 3) = 3$; $\log_3 5 = \ln 5 \div \ln 3 \approx 1.465$
3. If $\log_3 x = 4$ then $x = 3^4 = 81$. If $\ln(x+1) = 2$ then $x = \mathrm{e}^2 - 1$.
4. If $2\log_3 x + \log_3(x-2) = 2$ then $\log_3[x^2(x-2)] = 2$ so $x^3 - 2x^2 = 9$ Hence $x = 3$.
5. If $3^{x+1} = 2^{x-2}$ then, taking natural logs of both sides, $(x+1)\ln 3 = (x-2)\ln 2$, so $(\ln 2 - \ln 3)x = \ln 3 + 2\ln 2$, so $x \approx -6.13$.
6. If $5^{2x} - 5^{x+1} + 4 = 0$, we have a quadratic in 5^x. It is $(5^x)^2 - 5(5^x) + 4 = 0$, so $(5^x - 1)(5^x - 4) = 0$. Thus $5^x = 1$ or $5^x = 4$, so $x = 0$ or $x = \log_5 4 \approx 0.861$.

Trigonometric Functions

You should know the definitions of sine, cosine and tangent as ratios of sides in a right-angled triangle. These can be extended using their periodic properties, and illustrated using the four quadrants of a unit circle. Hence the trigonometric functions are sometimes called the **circular functions**. In the context of calculus, all angles are assumed to be measured in *radians*. There are 2π radians in a complete rotation, so 1 radian $= \frac{180}{\pi}^\circ$ and $1^\circ = \frac{\pi}{180}$ radians. We often express angles in terms of π.

You should learn, or be able to work out quickly, conversions such as:

Degrees	0	30	45	60	90	120	135	150	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

and also the following exact trigonometric ratios:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Really we should not write $\tan\frac{\pi}{2} = \infty$, but $\tan x \to \infty$ as $x \to \frac{\pi}{2}$.

Note that $\tan x \equiv \dfrac{\sin x}{\cos x}$.

The secant, cosecant and cotangent functions are defined by :

$$\sec x \equiv \frac{1}{\cos x},\ \text{cosec}\ x \equiv \frac{1}{\sin x},\ \cot x \equiv \frac{1}{\tan x}.$$

It follows from $\sin^2 x + \cos^2 x \equiv 1$ (the trigonometric form of Pythagoras' Theorem) that

$$\tan^2 x + 1 \equiv \sec^2 x \quad \text{and} \quad 1 + \cot^2 x \equiv \text{cosec}^2 x.$$

You should know all the trigonometric identities given on the formula card. From them we can deduce these *factor formulae*:

$$\sin x \pm \sin y \equiv 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right), \quad (\mp \text{ means } - \text{ when } \pm \text{ is } +\text{, and vice-versa})$$

$$\cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\ \cos x - \cos y \equiv -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

1. Prove that $\sin x \cos y \equiv \frac{1}{2}(\sin(x+y) + \sin(x-y))$.
   $\sin(x+y) \equiv \sin x \cos y + \cos x \sin y$ and $\sin(x-y) \equiv \sin x \cos y - \cos x \sin y$.
   Adding these and dividing by 2 gives the required result.

2. If $\sin x = \dfrac{3}{5}$ and $\sin y = \dfrac{5}{13}$, where $0 < x < \dfrac{\pi}{2}, 0 < y < \dfrac{\pi}{2}$, find $\tan(x+y)$.

   Using the $3, 4, 5$ and $5, 12, 13$ triangles we see that $\tan x = \dfrac{3}{4}, \tan y = \dfrac{5}{12}$, so

   $$\tan(x+y) = \frac{3/4 + 5/12}{1 - 3/4 \times 5/12} = \frac{56}{33}$$

3. Find $\cos\dfrac{\pi}{12}$ in surd form

   $$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Trigonometric Equations

Using the periodic properties of the trig. functions, we can find all solutions of a trig. equation in a given interval. If no interval is specified then generally there are infinitely many solutions.

A formula for these solutions in terms of a variable integer n is called the **general solution**.

1. Solve the equation $\sec x = 2$ (a) for $0 \leq x < 2\pi$, (b) generally.

   (a) If $\sec x = 2$ then $\cos x = \dfrac{1}{2}$.

   Cosines are positive in the first and fourth quadrants, so $x = \dfrac{\pi}{3}$ or $x = \dfrac{5\pi}{3}$.

   (b) The generalisation of this is that x is either of the above angles plus or minus an integer multiple of 2π. More simply, $x = 2n\pi \pm \dfrac{\pi}{3}, n \in \mathbb{Z}$. ($\mathbb{Z}$ = set of integers.)

2. Find the general solution of the equation $\cos^2 x - \sin x - 1 = 0$.

   We can re-write this as $1 - \sin^2 x - \sin x - 1 = 0$, so $\sin x(1 + \sin x) = 0$, giving $\sin x = 0$ or $\sin x = -1$. Hence $x = n\pi$ or $x = 2n\pi - \dfrac{\pi}{2}, n \in \mathbb{Z}$.

3. Solve the equation $\cos 3x + \cos x = \dfrac{1}{2}$.

   By the factor formulae (see above), we get $2\cos 2x \cos x = \dfrac{1}{2}$, so $4\cos x(2\cos^2 x - 1) = 1$, so $8\cos^3 x - 4\cos x - 1 = 0$. Note that $\cos x = -\dfrac{1}{2}$ is a solution, and factorise as $(2\cos x + 1)(4\cos^2 x - 2\cos x - 1) = 0$. Thus $\cos x = -\dfrac{1}{2}$ or $\dfrac{2 \pm \sqrt{20}}{8}$; hence find x.

Exercises for Section 2

1. Evaluate the following summations using the formula for an arithmetic or geometric progression.

   (a) $\sum_{r=0}^{10} (2 - 4r)$ (b) $\sum_{k=0}^{10} 8(3^k)$ (c) $\sum_{t=0}^{\infty} 0.5^t$

2. Without using a calculator, evaluate or simplify the following:

   (a) $\dbinom{6}{3}$ (b) $\dbinom{40}{38}$ (c) $\dfrac{12!}{10!}$ (d) $\dfrac{(n+1)!}{(n-1)!}$ (e) $n! + (n+2)! - (n+1)!$

3. Use Pascal's triangle or the binomial theorem to expand the following fully:

   (a) $(x - 4)^4$ (b) $(2x + 3y)^6$

4. Find the coefficients of x^r in the following expansions.

   (a) $(1 + 2x)^{14}$ where $r = 5$ (b) $(5 - 2x)^{10}$ where $r = 8$

   (c) $\left(1 - \dfrac{x}{7}\right)^5$ where $r = 3$

5. Obtain the series expansion of each of the following, as far as the fourth non-zero term. State the range of validity in each case.

   (a) $(2 + x)^{-1}$ (b) $(1 + 4x)^{-3}$ (c) $\dfrac{x}{(1 + x)^2}$ (d) $(8 - x)^{2/3}$

6. Without using a calculator, express each of the following in its simplest form:

   (a) $16^{-3/2}$ (b) $\log_2 \frac{1}{8}$ (c) $\log_9 3$

   (d) $(x^3)^{-2} \times (x^{-1})^4$, where $x > 0$ (e) $\sqrt{1 - 2x + x^2}$

   (f) $\sqrt{a} + \frac{b}{\sqrt{a}}$ (g) $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ (h) $(x+y)^{\frac{1}{2}} - x(x+y)^{-\frac{1}{2}}$

7. Solve the following equations for x:

   (a) $3^{-x} = 1$ (b) $\mathrm{e}^{2x} = \mathrm{e}$ (c) $\log_4 x = 5$

   (d) $\log_x 36 = 2$ (e) $4^{x+2}5^{x+1} = 32000$ (f) $4^x + \frac{1}{4^x} = 2$

   (g) $\mathrm{e}^{2x} - 5\mathrm{e}^x + 6 = 0$ (h) $2^{2x} + 8 = 9(2^x)$ (i) $\ln(\sqrt{x}) = \ln x + \ln 3$

8. Expand the following in terms of $\sin a, \sin b, \cos a$ and $\cos b$:

   (a) $\sin(a + 2b)$ (b) $\tan 2a$ (c) $\cos(a+b) - \cos(a-b)$ (d) $\cos 3b$

9. (a) Express $150°$ in radians. (b) Express $\frac{3\pi}{5}$ radians in degrees.

   (c) Write down the exact values of $\cos\frac{3\pi}{4}$ and $\tan\frac{7\pi}{6}$.

   (d) An arc of a circle of radius r subtends an angle θ radians at the centre.

   Write down the formulae for the perimeter and area of the sector formed.

   (e) In triangle ABC, $AB = 10$ cm, $BC = 8$ cm and angle $ABC = \frac{2\pi}{3}$.

   What is the exact area of the triangle?

10. Find in surd form (a) $\sin\frac{7\pi}{12}$, (b) $\tan\frac{\pi}{8}$.

    (c) Without using a calculator, prove that $\cos 80° + \cos 40° = \cos 20°$.

11. Solve the following trigonometric equations

    (a) $4\cos^2 x = 1$, giving the general solution. (b) $2\tan^2 x + \sec x = 1, -\pi \le x < \pi$

    (c) $\sin 2x = \tan x$, giving the general solution. (d) $\sin 3x - \sin x = \cos 2x, 0 \le x < \pi$

3. Calculus

Later in the course we shall go into the theory of differentiation and integration in more detail. At this stage, you should just be able to use the following methods.

Differentiation

If $y = \mathrm{f}(x)$, the **derived function**, **derivative** or **differential coefficient** of y with respect to x is $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$. This gives the **rate of change** of y with respect to x.

Graphically, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the *gradient* of the graph of $y = \mathrm{f}(x)$ at the point (x, y), i.e. the slope of the **tangent** to the graph at this point. The **normal**, which is perpendicular to the tangent, has gradient $-\frac{1}{\mathrm{f}'(x)}$.

(Recall that the product of the gradients of perpendicular lines is –1.)

At a turning point or **stationary point**, $f'(x) = 0$. The nature of such a point can be determined from the sign of the **second derivative** $\dfrac{d^2y}{dx^2} = f''(x)$: this is positive at a minimum and negative at a maximum. If at a point $x = x_0$, $f''(x_0) = 0$, and f'' changes sign as x passes through x_0, then f has a point of inflection at $x = x_0$.

There may also exist points of inflection which are not stationary, where $f''(x) = 0$ but $f'(x) \neq 0$. The tangent to a curve at any point of inflection *crosses* the curve there.

We have the following standard rules. Let u and v be functions of x.

- Sum / Difference Rule : $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$.
- Product Rule: To differentiate one function multiplied by another, $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.
- Quotient Rule: To differentiate one function divided by another, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
- Function of a Function (Composite Function) or Chain Rule:
  To differentiate one function *of* another function, $\dfrac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$.
  This may be easier to use in the following form : if y is a function of u and u is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.
- Standard Derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$(ax+b)^n$	$na(ax+b)^{n-1}$	e^{ax+b}	ae^{ax+b}
$\ln(ax+b)$	$\frac{a}{ax+b}$	$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$	$\tan(ax+b)$	$a\sec^2(ax+b)$
$\cot(ax+b)$	$-a\ \mathrm{cosec}^2(ax+b)$	$\sec(ax+b)$	$a\sec(ax+b)\tan(ax+b)$
$\mathrm{cosec}(ax+b)$	$-a\ \mathrm{cosec}(ax+b)\cot(ax+b)$		

  You should know these results when $a = 1, b = 0$ especially well.

Examples

1. Find the gradient of $y = \tan(2x + \pi)$ at the point $P(\pi, 0)$.
   $\dfrac{dy}{dx} = 2\sec^2(2x+\pi) = 2\sec^2 3\pi$ at P. Now $\cos 3\pi = -1$, so the gradient is 2.
2. Find the point on $y = \sqrt{2-3x}$ where the gradient is -3.
   $\dfrac{d}{dx}(2-3x)^{1/2} = \dfrac{1}{2}(2-3x)^{-1/2} \cdot (-3) = \dfrac{-3}{2\sqrt{2-3x}}$.
   When this equals -3, $\sqrt{2-3x} = \dfrac{1}{2}$, so $2 - 3x = \dfrac{1}{4}$, so $x = \dfrac{7}{12}$. The point is $\left(\dfrac{7}{12}, \dfrac{1}{2}\right)$.
3. Find $f''(x)$ when $f(x) = \ln(2x+1)$.
   $f'(x) = \dfrac{2}{2x+1} = 2(2x+1)^{-1}$, so $f''(x) = -2(2x+1)^{-2} \cdot (2) = \dfrac{-4}{(2x+1)^2}$

4. Find the turning points on the graph of $y = x^3 - 3x^2$. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x = 0$ at turning points. $3x(x-2) = 0$, so $x = 0$ or $x = 2$. The points are $(0,0)$ and $(2,-4)$.

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{3x}\sin 2x) = 2\mathrm{e}^{3x}\cos 2x + 3\mathrm{e}^{3x}\sin 2x$ [Using product rule]

6. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{x^2}\right) = \dfrac{(x^2)(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{1 - 2\ln x}{x^3}$ [Using quotient rule]

7. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin^3(x+3)) = 3\sin^2(x+3)\cos(x+3)$ [Using composite function rule]

8. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec(\ln x)) = \dfrac{\sec(\ln x)\tan(\ln x)}{x}$ [Using composite function rule]

Integration

Integration is the reverse of differentiation. Given a formula for the gradient or rate of change of a function, we can integrate to find the function itself.

An **indefinite** integral, written $\displaystyle\int \mathrm{f}(x)\ \mathrm{d}x$, must include an arbitrary constant.

A **definite** integral, written $\int_a^b \mathrm{f}(x)\ \mathrm{d}x$, means [$\int \mathrm{f}(x)\ \mathrm{d}x$ evaluated at $x = b$] − [$\int \mathrm{f}(x)\ \mathrm{d}x$ evaluated at $x = a$]. This has a fixed numerical value, and represents the area under the graph of $y = \mathrm{f}(x)$, above the x-axis, between $x = a$ and $x = b$. Any area below the x-axis is negative, so if the graph crosses the x-axis between $x = a$ and $x = b$ the definite integral does not give the total area of the regions formed.

When the area under a graph is rotated completely about the x-axis, the volume of the solid generated is $\displaystyle\pi\int_a^b [\mathrm{f}(x)]^2\ \mathrm{d}x$.

Standard Integrals [All + c.] Again, the special cases $a = 1, b = 0$ should be familiar.

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\mathrm{d}x$	
$(ax+b)^n \quad (n \neq -1)$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$	
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$	
e^{ax+b}	$\frac{1}{a}\mathrm{e}^{ax+b}$	
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)$	
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)$	
$\tan(ax+b)$	$\frac{1}{a}\ln\lvert\sec(ax+b)\rvert$	
$\cot(ax+b)$	$\frac{1}{a}\ln\lvert\sin(ax+b)\rvert$	
$\sec(ax+b)$	$\frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert$	
$\mathrm{cosec}(ax+b)$	$\frac{1}{a}\ln\lvert\mathrm{cosec}(ax+b)-\cot(ax+b)\rvert$	
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)$	(Inverse of differentiation result)
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	(proved by partial fractions)
$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$	$\ln\lvert\mathrm{f}(x)\rvert$	(proved by substituting $u = \mathrm{f}(x)$)

Integration by Parts

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \quad \text{or equivalently} \quad \int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Other methods which are often useful are algebraic substitutions, partial fractions and trigonometric identities. At this stage, you will be given suitable substitutions when needed.

Examples

1. Find the area bounded by the curve $y = \mathrm{e}^{2x-1}$, the x-axis and the lines $x = 0$ and $x = 3$.

   Area $= \displaystyle\int_0^3 \mathrm{e}^{2x-1} \, \mathrm{d}x = \left[\frac{1}{2}\mathrm{e}^{2x-1}\right]_0^3 = \frac{1}{2}\left(\mathrm{e}^5 - \mathrm{e}^{-1}\right).$

2. Find the volume formed when the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated once about the x-axis.

   Volume $= \pi \displaystyle\int_0^\pi \sin^2 x \, \mathrm{d}x = \pi \int \frac{1 - \cos 2x}{2} \, \mathrm{d}x = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi^2}{2}.$

3. Find $\displaystyle\int x\mathrm{e}^{2x} \, \mathrm{d}x$. Integrate by parts : let $u = x$, $\mathrm{d}v = \mathrm{e}^{2x} \, \mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$, $v = \dfrac{1}{2}\mathrm{e}^{2x}$

   so we get $\dfrac{1}{2}x\mathrm{e}^{2x} - \displaystyle\int \frac{1}{2}\mathrm{e}^{2x} \, \mathrm{d}x = \frac{1}{4}\mathrm{e}^{2x}(2x - 1) + c.$

4. Using partial fractions : $\displaystyle\int_6^7 \frac{x}{x^2 - x - 12} \, \mathrm{d}x = \int_6^7 \frac{4}{7(x-4)} + \frac{3}{7(x+3)} \, \mathrm{d}x$

   $= \left[\dfrac{4}{7}\ln|x - 4| + \dfrac{3}{7}\ln|x + 3|\right]_6^7 = \dfrac{1}{7}(4\ln 3 + 3\ln 10 - 4\ln 2 - 3\ln 9) = \dfrac{1}{7}\ln\dfrac{125}{18}.$

   Check that you understand how the logs have been simplified.

5. Find $\displaystyle\int x(x^2 + 3)^{-1/2} \, \mathrm{d}x.$

   If you really understand the composite function rule for differentiation, you can spot the answer to this at a glance. Since x^2+3 differentiates to $2x$ and $u^{1/2}$ differentiates to $\dfrac{1}{2}u^{-1/2}$, the answer must be $(x^2+3)^{1/2}+c$. If you don't see this, make the substitution $u = x^2 + 3$, so $\mathrm{d}u = 2x \, \mathrm{d}x$, etc.

6. $\displaystyle\int x\sqrt{2x+3} \, \mathrm{d}x.$ [This is defined only if $x \geq -\dfrac{3}{2}$.]

   Substitute $u = 2x + 3$, so $\mathrm{d}u = 2 \, \mathrm{d}x$. Then $\displaystyle\int \frac{u-3}{2}u^{1/2}\frac{1}{2} \, \mathrm{d}u = \frac{1}{4}\int u^{3/2} - 3u^{1/2} \, \mathrm{d}u$

   $= \dfrac{1}{4}\left(\dfrac{2}{5}u^{5/2} - 3 \times \dfrac{2}{3}u^{3/2} + c\right) = \dfrac{1}{10}(2x+3)^{5/2} - \dfrac{1}{2}(2x+3)^{3/2} + c$

7. Find $\displaystyle\int \cos^5 x \, \mathrm{d}x.$

   $\cos^5 x = \cos^4 x \cos x = (1 - \sin^2 x)^2 \cos x$, so substitute $u = \sin x, \mathrm{d}u = \cos x \, \mathrm{d}x$ to get

   $\displaystyle\int (1 - u^2)^2 \, \mathrm{d}u = \int (1 - 2u^2 + u^4) \, \mathrm{d}u = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c.$

8. $\displaystyle\int \frac{x^2}{x^3+4}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2}{x^3+4}\,\mathrm{d}x = \frac{1}{3}\ln|x^3+4| + c.$

(Recognising an integrand of the form $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$. Alternatively, substitute $u = x^3 + 4$.)

Exercises for Section 3

1. Differentiate each of the following with respect to x:

   (a) e^{4x+3} (b) $x^3\cos\dfrac{x}{4}$ (c) $x^4\ln 2x$

   (d) $\tan^2\left(\dfrac{x}{2}\right)$ (e) $\log_{10} x$ (f) $\ln(\cos(x^2))$

2. (a) Find the gradient of the curve $y = \mathrm{e}^x\sin x$ at the point where $x = 0$.

   (b) Find the coordinates of the point on the graph of $y = \dfrac{1}{x^2}$ where the gradient is 2.

   (c) Find the coordinates of the three turning points on the graph of $y = 3x^4 - 4x^3 - 24x^2 + 48x$, and identify each as a maximum or minimum.

   Also find the x-coordinates of the points of inflection.

   Illustrate these features on a sketch of the graph.

   (d) Find equations of the tangent and normal to the curve $y = \sqrt{x+1}$ at $(3, 2)$.

3. Integrate each of the following with respect to x:

   (a) $\dfrac{5}{x^6}$ (b) $6x(x^2-2)^{1/2}$ (c) $-9\mathrm{e}^{-3x} + \dfrac{1}{2x}$

   (d) $\dfrac{x}{x+2}$ (e) $\dfrac{3-4x}{9-x^2}$ (f) $\sec\left(\dfrac{3x+2}{4}\right)$

4. Evaluate the following definite integrals:

   (a) $\displaystyle\int_0^{4\pi} \sin(3x-4\pi)\,\mathrm{d}x$ (b) $\displaystyle\int_1^2 (x+1)^4\,\mathrm{d}x$

   (c) $\displaystyle\int_0^3 \frac{2x+1}{x^2+x+2}\,\mathrm{d}x$ (d) $\displaystyle\int_1^{\mathrm{e}} \frac{1}{x(1+\ln x)}\,\mathrm{d}x$

5. Find (a) the area bounded by $y = \mathrm{e}^{2x+3}$, the x and y axes and the line $x = -2$,

   (b) the volume formed when the area under $y = \sec x$ between $x = 0$ and $x = \dfrac{\pi}{4}$ is rotated once about the x-axis.

6. Evaluate the following using the given substitution (or otherwise):

   (a) $\displaystyle\int x\sqrt{x^2-1}\,\mathrm{d}x \quad (u = x^2-1)$ (b) $\displaystyle\int_0^{\sqrt{\pi}} 3x\sin(x^2)\,\mathrm{d}x \quad (u = x^2)$

   (c) $\displaystyle\int \cos^3 x\,\mathrm{d}x \quad (u = \sin x)$ (d) $\displaystyle\int_0^{\pi/2} \cos^4 x\sin x\,\mathrm{d}x \quad (u = \cos x)$

7. Use integration by parts to find:

   (a) $\displaystyle\int x\sec^2 x\,\mathrm{d}x$ (b) $\displaystyle\int x^2\ln x\,\mathrm{d}x$ (c) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\,\mathrm{cosec}^2 x\,\mathrm{d}x$

8. Using the Partial Fractions which you may already have found in an earlier exercise, integrate each of the following with respect to x:

   (a) $\dfrac{5x+1}{x^2+x-2}$ (b) $\dfrac{x^2+31x}{(x+1)(x-4)(2x-1)}$ (c) $\dfrac{x^3-1}{(x-2)(x-3)}$

   (d) If the area under the curve $y = \dfrac{x+1}{3x^2+2x}$ between $x = 1$ and $x = 3$ is $\dfrac{1}{6}\ln k$, find the exact value of k.

Longer Questions

1. Find the set of real values of k, $k \neq -1$, for which the roots of the equation $x^2 + 4x - 1 + k(x^2 + 2x + 1)$ are real and distinct.

2. Find constants a and b such that when $x^5 - ax^3 - bx^2$ is divided by $x^2 - x - 2$, the remainder is $x + 2$.

3. Express $6x^2 - 25x - \dfrac{25}{x} + \dfrac{6}{x^2}$ in terms of y, where $y = x + \dfrac{1}{x}$.
   Hence solve the equation $6x^4 - 25x^3 + 38x^2 - 25x + 6 = 0$.

4. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, find expressions in terms of a, b, c, d for $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

5. The arithmetic mean and the geometric mean of the positive real numbers x and y are respectively $A = \dfrac{x+y}{2}$ and $G = \sqrt{xy}$. Prove that $A > G$. [Hint : consider $A - G$.]

6. The second and fifth terms of a geometric progression of positive terms are $\sqrt{2}$ and $8\sqrt{2}$ respectively. Find the exact value of the sum of the first nine terms.

7. Given that $\mathrm{f}(x) \equiv \dfrac{x+1}{(x-1)(2x+1)}$, find an expression for $\mathrm{f}(x)$ as a series in ascending powers of x up to and including the term in x^5.

8. Given that $x > 2$, find the constants a, b, c for which

   $$\left(\frac{x+2}{x}\right)^{-1/2} = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \cdots$$

   Use this series with a suitable value of x to find an approximate value for $\left(\dfrac{450}{51}\right)^{1/2}$, to 4 decimal places.

9. If a, b, x are positive and neither a nor b is 1, prove that $\log_b x = \dfrac{\log_a x}{\log_a b}$

10. Sketch on the same diagram the graphs of $y = \dfrac{2}{3}\cos x$ and $y = \tan x$ for $0 \leq x \leq 2\pi$. Calculate the values of x at the points of intersection of the two graphs. Hence state the set of values of x in the given interval for which $2\cos x > 3\tan x$.

11. $ABCD$ is a square field and O is the mid-point of AB. A goat is tethered to O and can graze exactly half the field. If L, M are the furthest points which the goat can reach on AD, BC respectively, and angle $LOM = \alpha$ radians, prove that $\dfrac{\alpha}{2} = 1 - \cos\alpha - \dfrac{1}{2}\sin\alpha$.

12. Starting from the formulae for $\sin(x+y)$ and $\cos(x+y)$, prove the formula for $\tan(x+y)$. Use this result to find, to 2 decimal places, the positive value of x for which $\arctan x + \arctan 2x = \dfrac{\pi}{4}$. [$\arctan x$ is the inverse tangent $\tan^{-1} x$.]

13. Find the values of x, in terms of p, which satisfy the equation $x^3 - 7p^2x + 6p^3 = 0$.

    Hence find the values of t, in radians to 2 decimal places in the interval $0 \le t < 2\pi$, for which $4\sec^3 t - 7\sec t + 3 = 0$.

14. Given that $x = 2 + 2\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$, find the maximum value of x as t varies.

15. Find the turning points of the curve $y = x^3 - x$ and decide their nature. Find also the coordinates of the point of inflection of the curve. Sketch the curve.

    Find the equation of the tangent to the curve at (1, 0) and find also the coordinates of the other point at which this tangent cuts the curve.

16. Find the maximum and minimum turning points of $y = x + \dfrac{1}{x+1}$. Sketch the graph of this function.

17. Find the values of x for which the function $\mathrm{e}^{-x/a}\sin x$, where a is a positive constant, has maximum and minimum values. Show that the sequence of maximum and minimum values of the function forms a geometric progression and state its common ratio.

18. Using the substitution $u = \sin x$, evaluate to two decimal places the integral

$$\int_{\pi/6}^{\pi/2} \frac{4\cos x}{3+\cos^2 x}\,\mathrm{d}x.$$

19. Find the area bounded by the curve $y = \dfrac{x+1}{\sqrt{x+2}}$, the lines $x = 1$ and $x = 2$ and the x-axis. Also find the volume formed when this area is rotated once about the x-axis.

20. Use the substitution $u = \dfrac{\pi}{2} - x$ to show that if

$$I = \int_0^{\pi/2} \frac{2\cos x + 7\sin x}{\cos x + \sin x}\,\mathrm{d}x \qquad \text{and} \qquad J = \int_0^{\pi/2} \frac{2\sin x + 7\cos x}{\cos x + \sin x}\,\mathrm{d}x,$$

    then $I = J$.

    By considering $I + J$, find the value of I.

Answers to Exercises

Please let me know if you find any mistakes.

1. Algebraic Methods

1. (a) $5x - 20y - 60z + 360$
   (b) xy
   (c) $x^3 - y^3$
   (d) $3xy - 3xz + yz$

2. (a) $\frac{x^4}{y^3}$
   (b) $\frac{1}{2x+3y}$
   (c) $\frac{4y+4}{y}$
   (d) $\frac{9x-2}{18}$ or $\frac{x}{2} - \frac{1}{9}$
   (e) $\frac{2x^2+2x+1}{x^2+x}$
   (f) $\frac{3-x}{x^2-1}$

3. (a) $(x-4)^2 - 23$
   Min. -23 when $x = 4$
   (b) $\frac{45}{4} - (2x - \frac{1}{2})^2$
   Max. $\frac{45}{4}$ when $x = \frac{1}{4}$
   (c) $(x^2 + 2)^2 + 3$
   Min. 7 when $x = 0$

4. (a) $(x + 2)(x + 3)$
   (b) $(x - 2)(x + 2)(x^2 + 3)$
   (c) $2(a - 3x)(2x - 3y)$
   (d) $(2x + 1)(2x + 3)$
   (e) $(x - 1)(x + 2)(x - 3)$
   (f) $(11p - 13q)(11p + 13q)$
   (g) $(x + 4y)(x^2 - 4xy + 16y^2)$
   (h) $x^2(x + 4)(x - 4)$
   (i) $12x(x - 3y)$
   (j) 0

5. (a) $x - 5$
   (b) $x + \frac{1}{x+1}$
   (c) $x^2 - 2x + 1 - \frac{5}{2x-3}$
   (d) $x + 1 + \frac{4-x}{x^2+x+1}$

6. (a) $\frac{2}{x-1} + \frac{3}{x+2}$
   (b) $\frac{4}{x-4} - \frac{2}{x+1} - \frac{3}{2x-1}$
   (c) $x + 5 + \frac{26}{x-3} - \frac{7}{x-2}$

7. (a) Show f(1/2) = 0
   (b) f(−1) = 1
   (c) $a = 0, b = -6$

8. (a) $x = -35/2$
   (b) $x = 1/2, y = -2$
   (c) $x = 4 \pm \sqrt{29}$
   (d) $x = 1, x = 2, x = -3$
   (e) $(1, 7), (-0.98, -7.14), (49.98, 0.14)$

9. (a) $0 < k < 1$
   (b) Sum = $2p$, product = $-3p^2$

10. (a) $x \geq -1/3$
    (b) $x < \frac{1}{2}$ or $x > 3$
    (c) $x \leq -5$ or $x \geq 3$
    (d) $x < -1$ or $x > -0.5$
    (e) $-1 < x < 7$

11. (a) $x = \frac{d-b}{a-c}$
    (b) $x = 0, x = 3p$
    (c) $x = 0, x = \frac{k-1}{k}$
    (d) $x = 2y \pm z$

12. (a) a^2
    (b) $\frac{1}{2}$
    (c) $1 - 2q^2$
    (d) 13

2. Series, Functions, etc.

1. (a) −198
   (b) 708584
   (c) 2

2. (a) 20
   (b) 780
   (c) 132
   (d) $n(n + 1)$
   (e) $n!(n^2 + 2n + 2)$

3. (a) $x^4 - 16x^3 + 96x^2 - 256x + 256$
   (b) $64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3 + 4860x^2y^4 + 2916xy^5 + 729y^6$

4. (a) 64064
   (b) 288000
   (c) $-10/343$

5. (a) $\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3, |x|<2$
   (b) $1-12\,x+96\,x^2-640\,x^3, |x|<1/4$
   (c) $x-2\,x^2+3\,x^3-4\,x^4, |x|<1$
   (d) $4-\frac{1}{3}x-\frac{1}{144}x^2-\frac{1}{2592}x^3, |x|<8$

6. (a) $\frac{1}{64}$
   (b) -3
   (c) $\frac{1}{2}$
   (d) x^{-10}
   (e) $x-1$ if $x\geq 1$, $\quad 1-x$ if $x\leq 1$
   (f) $\frac{a+b}{\sqrt{a}}$
   (g) $-3-2\sqrt{2}$
   (h) $\frac{y}{(x+y)^{\frac{1}{2}}}$ or $\frac{y}{\sqrt{x+y}}$

7. (a) 0
   (b) $\frac{1}{2}$
   (c) 1024
   (d) 6
   (e) 2
   (f) 0
   (g) ln 2, ln 3
   (h) 0, 3
   (i) $\frac{1}{9}$

8. (a) $2\sin a\cos^2 b-\sin a+2\cos a\sin b\cos b$
   (b) $\frac{2\sin a\cos a}{2\cos^2 a-1}$
   (c) $-2\sin a\sin b$
   (d) $4\cos^3 b-3\cos b$

9. (a) $\frac{5\pi}{6}$
   (b) $108°$
   (c) $\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$
   (d) $r\theta+2r, \frac{1}{2}r^2\theta$
   (e) $20\sqrt{3}$ cm^2

10. (a) $\frac{\sqrt{6}+\sqrt{2}}{4}$
    (b) $\sqrt{2}-1$
    (c) By $\cos x+\cos y$ formula,
    $\cos 80°+\cos 40°=2\cos 60°\cos 20°$
    $=2(\frac{1}{2})\cos 20°=\cos 20°$

11. (a) $n\pi\pm\pi/3$
    (b) $-2.3, 0, 2.3$
    (c) $n\pi, \quad n\pi\pm\pi/4$
    (d) $\pi/6, \pi/4, 3\pi/4, 5\pi/6$

3. Calculus

1. (a) $4e^{4x+3}$
   (b) $3x^2\cos\frac{x}{4}-\frac{x^3}{4}\sin\frac{x}{4}$
   (c) $x^3(1+4\ln 2x)$
   (d) $\tan\frac{x}{2}\sec^2\frac{x}{2}$
   (e) $\frac{1}{x\ln 10}$
   (f) $-2x\tan(x^2)$

2. (a) 1
   (b) $(-1, 1)$
   (c) $(1, 23)$ max, $(-2, -112)$ and $(2, 16)$ min. Inflexion points when $x=\frac{1}{3}(1\pm\sqrt{13})$
   (d) $4y-x=5, \ y+4x=14$

3. (a) $-\frac{1}{x^5}+c$
   (b) $2(x^2-3)^{3/2}+c$
   (c) $3e^{-3x}+\frac{1}{2}\ln|x|+c$
   (d) $x-2\ln|x+2|+c$
   (e) $\frac{3}{2}\ln|x-3|+\frac{5}{2}\ln|x+3|+c$
   (f) $\frac{4}{3}\ln\left|\sec\left(\frac{3x+2}{4}\right)+\tan\left(\frac{3x+2}{4}\right)\right|+c$

4. (a) 0
   (b) 211/5
   (c) ln 7
   (d) ln 2

5. (a) $\frac{1}{2}(e^3-e^{-1})$
   (b) π

6. (a) $\frac{1}{3}(x^2-1)^{3/2}+c$
   (b) 3
   (c) $\sin x-\frac{1}{3}\sin^3 x+c$
   (d) $\frac{1}{5}$

7. (a) $x\tan x-\ln|\sec x|+c$
   (b) $\frac{x^3}{9}(3\ln x-1)+c$
   (c) π

8. (a) $2\ln|x-1| + 3\ln|x+2| + c$
   (b) $4\ln|x-4| - \frac{3}{2}\ln|2x-1|$
   $- 2\ln|x+1| + c$
   (c) $\frac{1}{2}x^2 + 5x + 26\ln|x-3|$
   $- 7\ln|x-2| + c$
   (d) $\frac{135}{11}$

18. 0.59

19. $4/3, \quad \pi(3/2 + \ln(4/3))$

20. $9\pi/4$

Longer Questions

1. $k > -5/4$

2. $a = 3, b = 1$

3. $6y^2 - 25y - 12 \qquad x = 2/3, 1, 3/2$

4. $-b/a, c/a, -d/a$

5. Show that $A - G = (\sqrt{x} - \sqrt{y})^2/2$, etc.

6. $511\sqrt{2}/2$

7. $-1 - 2\,x^2 + 2\,x^3 - 6\,x^4 + 10\,x^5$

8. $a = 1, b = -1, c = 3/2, d = -5/2.$

   Put $x = 100$ (why?), get 2.9704

9. Let $y = \log_b x$, so $x = b^y$. Take logs to base a of both sides.

10. Intersect at $x = \pi/6, 5\pi/6$.

    $0 \le x < \pi/6, \quad \pi/2 < x < 5\pi/6,$

    $3\pi/2 < x \le 2\pi$

11. Let side of field $= 2a$ and find radius of sector in terms of a ...

12. $\frac{\sqrt{17}-3}{4}$

13. $x = p, 2p, -3p \quad t = 0, 2.30, 3.98$

14. 86/27

15. $(1/\sqrt{3}, -2/3\sqrt{3})$ minimum,

    $(-1/\sqrt{3}, 2/3\sqrt{3})$ maximum.

    Point of inflection (0, 0).

    $y = 2x - 2 \quad (-2, -6)$

16. (0, 1), $(-2, -3)$.

    Curve with asymptotes $x = -1, y = x$

17. Max / min when $\tan x = a$.

    Ratio $= -\exp(\pi/a)$

Module MAT1015: Techniques in Calculus

PART B

Outline Lecture Notes

Complex Numbers

A quadratic equation $x^2 + ax + b = 0$ may or may not have solutions in $\mathbb{R}$. Consider the equation $x^2+4x+5 = 0$. If its roots are α and β then by the theory of quadratic equations, $\alpha + \beta = -4, \alpha\beta = 5$.

Solving the equation gives $x = -2\pm\sqrt{-1}$. We know that $\sqrt{-1}$ does not exist in $\mathbb{R}$. However, if $\alpha = -2+\sqrt{-1}$ and $\beta = -2-\sqrt{-1}$, then $\alpha+\beta = -4$ and $\alpha\beta = (-2+\sqrt{-1})(-2-\sqrt{-1}) = 4 + 2\sqrt{-1} - 2\sqrt{-1} - (\sqrt{-1})^2 = 4 - (-1) = 5$.

Let the symbol i have the property that $\mathrm{i}^2 = -1$. An expression of the form $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$, is called a **complex number**. If $x = 0$ then $z = y\mathrm{i}$ is called a **purely imaginary** number. The set of all complex numbers is denoted by $\mathbb{C}$.

x is called the **real part** of z, $\mathrm{Re}(z)$. y is called the **imaginary part** of z, $\mathrm{Im}(z)$.

Two complex numbers are defined to be **equal** if their real parts are equal and their imaginary parts are equal, e.g. $x + y\mathrm{i} = 3 - 5\mathrm{i} \Leftrightarrow x = 3, y = -5$.

We do arithmetic in $\mathbb{C}$ by treating a complex number as a linear function of i, where $\mathrm{i}^2 = -1$. Thus

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}.$$
$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$$

It can be shown that all the usual rules of arithmetic apply to complex numbers, e.g. if $z_1, z_2, z_3 \in \mathbb{C}$ then $z_1 + z_2 = z_2 + z_1$, $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$, $(z_1z_2)z_3 = z_1(z_2z_3)$.

$\overline{z} = x - y\mathrm{i}$ is called the **conjugate** of $z = x + y\mathrm{i}$. It is sometimes denoted by z^*.
Note that $z\overline{z} = x^2 + y^2$, which is a real number.

Division of complex numbers is carried out by making the denominator real: $\dfrac{w}{z} = \dfrac{w\overline{z}}{z\overline{z}}$.

The complex number $z = x + y\mathrm{i}$ can be represented by the point (x, y) in an **Argand diagram** or **complex plane**. Then addition and subtraction correspond to the same operations with vectors.

A point in two-dimensions can be specified by its **cartesian** coordinates (x, y) or its **polar** coordinates (r, θ). If $P\ (x, y)$ represents $z = x+y\mathrm{i}$, then r and θ determine the **polar form** of z.

The **modulus** of $z = x + y\mathrm{i}$ is $|z| = r = \sqrt{x^2 + y^2}$. Thus $z\overline{z} = |z|^2$.

The **argument** of $z = x + y\mathrm{i}$ is $\arg(z) = \theta$ where $\tan\theta = \dfrac{y}{x}$.

The **principal value** of the argument is in the interval $(-\pi, \pi]$, so we take $0 < \theta \le \pi$ if $y > 0$ and $-\pi < \theta < 0$ if $y < 0$.

In polar form, the complex number with modulus r and argument θ is $r(\cos\theta + \mathrm{i}\sin\theta)$.

Let $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$.

Using the trigonometric addition formulae, $z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$.

Thus z_1z_2 has modulus r_1r_2 and argument $(\theta_1 + \theta_2)$.

It follows that if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ then for $n \in \mathbb{N}, z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$; this is called **De Moivre's Theorem** and in fact it is true for all $n \in \mathbb{R}$.

Dividing z_1 by z_2 gives $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2))$.

so $\dfrac{z_1}{z_2}$ has modulus $\dfrac{r_1}{r_2}$ and argument $(\theta_1 - \theta_2)$.

If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, then $\dfrac{1}{z} = \dfrac{1}{r}(\cos\theta - \mathrm{i}\sin\theta)$ and $\dfrac{1}{z^n} = \dfrac{1}{r^n}(\cos n\theta - \mathrm{i}\sin n\theta)$.

The **Fundamental Theorem of Algebra** states that every polynomial equation with coefficients in $\mathbb{C}$ has all its solutions in $\mathbb{C}$.

It can be shown that complex roots of **real** polynomials occur in conjugate pairs.

Exercises

1. Express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$,

   (a) $(2 - 3\mathrm{i}) - (4 - 5\mathrm{i})$, (b) $(3 + 4\mathrm{i})(2 - 3\mathrm{i})$, (c) $(5 - \mathrm{i})^2$, (d) $\dfrac{6 - 2\mathrm{i}}{3 + 4\mathrm{i}}$.

2. For each of the following, find (i) its modulus, (ii) its argument in radians between $-\pi$ and π, in terms of π or as a decimal.

   (a) $1 + \mathrm{i}$, (b) $3 - 4\mathrm{i}$, (c) $-2 + 5\mathrm{i}$, (d) $-\sqrt{3} - \mathrm{i}$, (e) $-7\mathrm{i}$.

3. Find all the solutions in $\mathbb{C}$ of the equations (a) $4x^2 + 1 = 0$, (b) $x^2 + 2x + 5 = 0$.

4. Prove from the definitions that for complex numbers w and z,

   (a) $\overline{w + z} = \overline{w} + \overline{z}$, (b) $\overline{wz} = \overline{w}\ \overline{z}$.

5. Show that $z \div \overline{z}$ has modulus 1. Express $\arg(z \div \overline{z})$ in terms of $\arg(z)$.

6. Find the quadratic equation which has $2 + 3\mathrm{i}$ as one of its roots.

7. Suppose $(a + b\mathrm{i})^2 = 5 + 12\mathrm{i}$. By expanding the left-hand side and equating the real and imaginary parts, find the possible values of the real numbers a and b. Hence write down the two square roots of $5 + 12\mathrm{i}$.
   Deduce the value of $\tan\phi$, if $\tan 2\phi = 12/5$ and $0 < \phi < \pi$.

8. If $z = \cos\theta + \mathrm{i}\sin\theta$, expand z^4 by the binomial theorem. Hence express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

9. If $z = \cos\theta + \mathrm{i}\sin\theta$, show that $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$.

   Deduce that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

10. Factorise $z^3 - 1$. Hence find all the solutions in $\mathbb{C}$ of $z^3 - 1 = 0$. Express the complex roots in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

Functions

Let X and Y be sets. A **function** $\mathrm{f} : X \to Y$ is a rule which assigns to each x in the **domain** X *exactly one* element $\mathrm{f}(x)$ in the **codomain** Y.

If $Y \subseteq \mathbb{R}$, f is a **real-valued** function. If $Y \subseteq \mathbb{C}$, f is a **complex-valued** function.

In this course, X and Y will be subsets of $\mathbb{R}$ unless otherwise stated.

We write $\mathrm{f} : x \mapsto \mathrm{f}(x)$, where $x \in X$, and read this as f *maps* x to its *image* $\mathrm{f}(x)$.

If no domain is specified, we take f to have its **maximal domain**, e.g. $\mathrm{f} : x \mapsto \sqrt{1-x^2}$ can only be defined for $-1 \leq x \leq 1$.

The subset of Y given by $\{\mathrm{f}(x) \ : \ x \in X\}$, or $\mathrm{f}(X)$, is called the **range** of f.

Thus the range of a function is the set of y-coordinates at all the points on the graph.

If f is quadratic, we can find its range by finding its minimum or maximum point, e.g. by completing the square. $\mathrm{f} : x \mapsto x^2 + 4x - 3 \equiv (x+2)^2 - 7$ has range $\mathrm{f}(x) \geq -7$, or equivalently $[-7, \infty)$. The minimum point on the graph of $\mathrm{f}(x)$ is at $(-2, -7)$.

For more general functions with domain $\mathbb{R}$, the range can often be found by writing $\mathrm{f}(x) = y$ and finding a condition for this to have real roots for x.

A real-valued function f with domain X is defined to be:

- **even** if $\mathrm{f}(-x) = \mathrm{f}(x)$ for all $x \in X$.
  The graph of an even function is symmetric about the y-axis.
- **odd** if $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all $x \in X$.
  The graph of an odd function is the same when rotated by 180° about (0, 0).
- **periodic** if for some $k > 0$, $\mathrm{f}(x+k) = \mathrm{f}(x)$ for all $x \in X$.
  The smallest such positive k is called the *minimal* period.
- **one-to-one** or **injective** if $\mathrm{f}(a) = \mathrm{f}(b) \Rightarrow a = b$,
  i.e. no two elements of X have the same image under f.
- **onto** or **surjective** if the range of f is the whole codomain of f.
- **bijective** if it is both injective and surjective.

A **polynomial** function of x is an expression of the form $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where $a_0, ..., a_n$ are constants. n is the **degree** of the polynomial.

The **identity function** id is defined by

$$\mathrm{id}(x) = x \text{ for all } x.$$

The **absolute value** or **modulus** $\mathrm{abs}(x)$ or $|x|$ is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Note that $\sqrt{x}$ always means the *positive* square root of x, and so $|x| = \sqrt{x^2}$.

The **signum function** sgn is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Thus for $x \neq 0$, $\mathrm{sgn}(x) = \dfrac{x}{|x|}$ and $\dfrac{\sqrt{x^2}}{x} = \mathrm{sgn}(x)$.

The **integer part** of x, denoted by $\lfloor x \rfloor$ or $[x]$, is defined to be the largest integer less than or equal to x. For example, $\lfloor 7 \rfloor = 7$, $\lfloor -2.3 \rfloor = -3$ and $\lfloor \pi \rfloor = 3$.

Composition of Functions

For functions f and g, the **composite function** $\mathrm{g} \circ \mathrm{f}$, or gf, is defined by $(\mathrm{g} \circ \mathrm{f})(x) \equiv \mathrm{g}(\mathrm{f}(x))$. To find this, substitute $\mathrm{f}(x)$ in place of x in $\mathrm{g}(x)$, and simplify.

If f has domain X and g has domain X', the domain of $\mathrm{g} \circ \mathrm{f}$ is $\{x \in X \ : \ \mathrm{f}(x) \in X'\}$.

Similarly $(\mathrm{f} \circ \mathrm{g})(x) = \mathrm{f}(\mathrm{g}(x))$. In general $\mathrm{g} \circ \mathrm{f} \neq \mathrm{g} \circ \mathrm{f}$. However, composition *is* associative, i.e. $(\mathrm{f} \circ \mathrm{g}) \circ \mathrm{h} = \mathrm{f} \circ (\mathrm{g} \circ \mathrm{h})$. Note that $\mathrm{f} \circ \mathrm{id} = \mathrm{id} \circ \mathrm{f} = \mathrm{f}$.

It is useful to be able to identify functions as compositions, e.g. $\sin(\sqrt{x+3}) = (\mathrm{f} \circ \mathrm{g} \circ \mathrm{h})(x)$, where $\mathrm{f}(x) = \sin x$, $\mathrm{g}(x) = \sqrt{x}$, $\mathrm{h}(x) = x + 3$.

Inverse Functions

If f is bijective, with domain X and codomain Y, then there is an **inverse function** f^{-1} with domain Y and range X. The graph of $\mathrm{f}^{-1}(x)$ is the reflection of the graph of $\mathrm{f}(x)$ in the line $y = x$.

f^{-1} is defined by the property $\mathrm{f} \circ \mathrm{f}^{-1} = \mathrm{f}^{-1} \circ \mathrm{f} = \mathrm{id}$, i.e. $\mathrm{f}(\mathrm{f}^{-1}(x)) = \mathrm{f}^{-1}(\mathrm{f}(x))$.

Sometimes the inverse can be found by inspection. Otherwise, make x the subject of $y = \mathrm{f}(x)$ and then swap x and y in the answer. It is normal to use x in defining the inverse function, but y or any other symbol is not wrong if it is used consistently.

A function whose graph is symmetric about $y = x$ is **self-inverse**, i.e. $\mathrm{f} = \mathrm{f}^{-1}$.

If f is not bijective, we may obtain an invertible function by restricting the domain.

Partial Fractions

A **rational function** of x has the form $\dfrac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials.

Many rational functions can be expressed in **partial fractions**. The principles are:

(i) a polynomial of degree n in the denominator, which does not factorise, requires a polynomial of degree $n-1$ in its numerator, e.g.

$$\frac{ax^2+bx+c}{(x+d)(x^2+ex+f)} \equiv \frac{p}{x+d} + \frac{qx+r}{x^2+ex+f}.$$

(ii) If there is a linear factor to the power n in the denominator, there may be partial fractions with all powers of this linear factor up to the nth in their denominators, e.g.

$$\frac{ax^2+bx+c}{(x+d)^2(x^2+ex+f)} \equiv \frac{p}{x+d} + \frac{q}{(x+d)^2} + \frac{rx+s}{x^2+ex+f}.$$

(iii) If degree of numerator $\geq$ degree of denominator, there will be some non-fractional terms in the answer. In this case, do a long division first:

$$\frac{2x^4}{x^3+4x^2+3x+12} \equiv 2x - 8 + \frac{26x^2+96}{(x^2+3)(x+4)} \equiv 2x - 8 + \frac{px+q}{x^2+3} + \frac{r}{x+4}, \text{ etc.}$$

We usually find partial fractions over $\mathbb{Q}$. However, we can also have partial fractions over $\mathbb{R}$ or $\mathbb{C}$, e.g. $\dfrac{1}{x^2-3} \equiv \dfrac{p}{x+\sqrt{3}} + \dfrac{q}{x-\sqrt{3}}$, $\dfrac{1}{x^2+4} \equiv \dfrac{p}{x+2\mathrm{i}} + \dfrac{q}{x-2\mathrm{i}}$.

Exercises

1. State the maximal domains of the functions

   (a) $\mathrm{f} : x \mapsto \sqrt{x^2 - 9}$, (b) $\mathrm{g} : x \mapsto \dfrac{3x+1}{x^2 - 2x - 3}$.

2. Find the range of the function $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ defined by $\mathrm{f}(x) = x^2 - 4x - 3$. Why does f not have an inverse? How can the domain and codomain of f be restricted so that the resulting function *does* have an inverse?

3. State the values of $\mathrm{sgn}(-3)$, $|-\pi|$, $\lfloor\sqrt{2}\rfloor$, $\sqrt{(-1)^2}$.

4. Sketch graphs of (i) $\mathrm{sgn}(1-x)$, (ii) $x - \lfloor x \rfloor$, (iii) $\mathrm{sgn}(\sin x)$.

5. If $\mathrm{f} : (-\infty, 0) \to (1, \infty)$ by $x \mapsto 1 - 5x$ and $\mathrm{g} : [1, \infty) \to [0, \infty)$ by $x \mapsto \sqrt{x-1}$, define (if they exist) the functions f^{-1}, g^{-1}, $\mathrm{f} \circ \mathrm{g}$ and $\mathrm{g} \circ \mathrm{f}$.

6. For each of the following functions with maximal domain, and codomain $\mathbb{R}$, state whether it is even, odd, periodic, one-to-one, onto. Find and simplify the composite functions $\mathrm{f} \circ \mathrm{f}$, $\mathrm{f} \circ \mathrm{g}$, $\mathrm{g} \circ \mathrm{f}$, $\mathrm{g} \circ \mathrm{h}$, with their domains.

   (a) $\mathrm{f} : x \mapsto x^2 + 1$, (b) $\mathrm{g} : x \mapsto \dfrac{2x}{x-2}$, (c) $\mathrm{h} : x \mapsto \tan 2x$,

7. If $\mathrm{f}(x) = \dfrac{2x+1}{x^2+2}$ for $x \in \mathbb{R}$, find the set of values of y for which $\mathrm{f}(x) = y$ has real roots for x. Hence state the range of f.

8. Sketch a graph of $\mathrm{f} : x \mapsto \left\{ \begin{array}{ll} 2x, & -1 \le x < 0 \\ 4x^2, & 0 \le x \le 1 \end{array} \right\}$. Define the inverse function.

9. Show that if $\mathrm{f} \circ \mathrm{g}$ is invertible, then $(\mathrm{f} \circ \mathrm{g})^{-1} = \mathrm{g}^{-1} \circ \mathrm{f}^{-1}$.

10. Express each of the following in partial fractions:

    (a) $\dfrac{4x-3}{(x+1)(x^2+x+1)}$ (b) $\dfrac{2x}{(x-5)^2(x+1)}$

    (c) $\dfrac{x^3+1}{x^2+7x+12}$ (d) $\dfrac{3x+7}{(x+1)^2(x+3)}$

Trigonometric, Exponential and Hyperbolic Functions

Let $P\ (x, y)$ be a point on the circle with centre (0, 0) and radius 1, and let θ be the angle measured anti-clockwise from the positive x-axis to OP. Then we define:

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}.$$

The sine and cosine functions have range $[-1, 1]$, so a function of the form $r\sin(nx + \alpha)$ has range $[-r, r]$. r is the **amplitude**, and the **period** is $\dfrac{2\pi}{n}$.

If $\mathrm{f}(x) = \sin x$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ then f is one-to-one. With this restriction of the domain, the **inverse trig. function** $\arcsin x$ or $\sin^{-1} x$ is defined for $-1 \le x \le 1$.

Similarly, restricting the domain of $\cos x$ to $[0, \pi]$, the inverse function is $\arccos x$ or $\cos^{-1} x$, for $-1 \le x \le 1$. Note that $\arccos x = \dfrac{\pi}{2} - \arcsin x$.

Also, restricting the domain of $\tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the inverse function is $\arctan x$ or $\tan^{-1} x$, for $x \in \mathbb{R}$.

The **exponential function** $\exp(x)$ can be defined as the sum of the convergent series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

$\exp(1)$ is an irrational number about 2.718, which we call e. It can be shown that $\exp(x)$ is this number to the power x, so $\exp(x)$ is usually denoted by e^x.

The natural logarithm $\ln x$ is the inverse of e^x. It is defined only for $x > 0$.

Other exponential functions can be defined on $\mathbb{R}$, e.g. $2^x = \exp(x \ln 2)$.

Any function f, defined on a domain symmetrical about 0, can be expressed as the sum of an odd function and an even function as follows :

$$\mathrm{f}(x) \equiv \frac{1}{2}(\mathrm{f}(x) - \mathrm{f}(-x)) + \frac{1}{2}(\mathrm{f}(x) + \mathrm{f}(-x)).$$

Taking $\mathrm{f}(x) = \mathrm{e}^x$, the odd and even components are the **hyperbolic functions**

$$\sinh x \equiv \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \quad \text{and} \quad \cosh x \equiv \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

We also define $\quad \operatorname{cosech} x$ or $\operatorname{csch} x \equiv \dfrac{1}{\sinh x}, \quad \tanh x \equiv \dfrac{\sinh x}{\cosh x},$

$$\operatorname{sech} x \equiv \frac{1}{\cosh x}, \quad \coth x \equiv \frac{\cosh x}{\sinh x} \equiv \frac{1}{\tanh x}.$$

The hyperbolic functions have many properties similar to those of the trigonometric functions, but they are not periodic.

For example, $\cosh^2 x - \sinh^2 x \equiv 1$; compare this with $\cos^2 x + \sin^2 x \equiv 1$.

Similarly, $\quad 1 - \tanh^2 x \equiv \operatorname{sech}^2 x, \quad \coth^2 x - 1 \equiv \operatorname{cosech}^2 x.$

Osborne's Rule

To convert a trig. identity into a hyperbolic one, replace cos by cosh and sin by sinh *but* whenever $\sin^2$ occurs either explicitly or implicitly (e.g. in $\tan^2$), change the sign.

1. $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ becomes
   $\sinh(A + B) \equiv \sinh A \cosh B + \cosh A \sinh B$;

2. $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ becomes
   $\cosh(A + B) \equiv \cosh A \cosh B + \sinh A \sinh B$;

3. $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ becomes $\tanh(A - B) \equiv \dfrac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$
   (Note the implicit product of sines in $\tan A \tan B$).

To solve hyperbolic equations and establish identities we can use the definitions in terms of exponentials, *or* any of the standard identities.

The following **inverse hyperbolic functions** are defined on the given domains:

$$\begin{aligned} \operatorname{arsinh} x &\equiv \sinh^{-1} x \equiv \ln(x + \sqrt{x^2 + 1}), \quad x \in \mathbb{R}; \\ \operatorname{arcosh} x &\equiv \cosh^{-1} x \equiv \ln(x + \sqrt{x^2 - 1}), \quad x \in [1, \infty); \end{aligned}$$

$$\operatorname{artanh} x \quad \equiv \quad \tanh^{-1} x \quad \equiv \quad \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad x \in (-1,1).$$

Limits

The **limit** of $\mathrm{f}(x)$ as x tends to a, written $\lim_{x\to a} \mathrm{f}(x)$, is a number ℓ such that we can make $\mathrm{f}(x)$ as close as we like to ℓ by taking x very close (but not equal) to a.

Some functions have different limits as $x \to a$ from below (from the left) and from above (from the right). In such a case, $\lim_{x\to a} \mathrm{f}(x)$ does not exist.

$\lim_{x\to\infty} \frac{1}{x} = 0$ means that as x gets larger and larger, $\frac{1}{x}$ tends to 0.

Do NOT write $\frac{1}{0} = \infty$ or $\frac{1}{\infty} = 0$, as ∞ is not a real number.

We *can* write $\lim_{x\to 0+} \frac{1}{x} = \infty$, but this limit does not exist as a real number.

Rules for limits

If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ both have finite limits as $x \to a$, then

$\lim_{x\to a} c\mathrm{f}(x) = c \lim_{x\to a} \mathrm{f}(x)$ for any $c \in \mathbb{R}$, $\qquad \lim_{x\to a} [\mathrm{f}(x) + \mathrm{g}(x)] = \lim_{x\to a} \mathrm{f}(x) + \lim_{x\to a} \mathrm{g}(x)$,

$\lim_{x\to a} [\mathrm{f}(x)\mathrm{g}(x)] = \lim_{x\to a} \mathrm{f}(x). \lim_{x\to a} \mathrm{g}(x)$, $\qquad \lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{\lim_{x\to a} \mathrm{f}(x)}{\lim_{x\to a} \mathrm{g}(x)}$, provided $\lim_{x\to a} \mathrm{g}(x) \neq 0$.

Geometrically it can be shown that for any acute angle x, $\sin x < x < \tan x$. Dividing through by $\sin x$, $\quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$, so $\cos x < \frac{\sin x}{x} < 1$.

Now as $x \to 0, \cos x \to 1$, so also $\frac{\sin x}{x} \to 1$. It can be deduced that:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0, \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1.$$

The exponential function can be defined in terms of a limit as follows:

$$\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n,$$

Letting $n = \frac{1}{p}$, we also have $\mathrm{e}^x = \lim_{p\to 0} (1+px)^{1/p}$.

Now the inverse function of $(1+px)^{1/p}$ is $\frac{x^p - 1}{p}$, so the inverse function of e^x can also be expressed as a limit:

$$\ln x = \lim_{p\to 0} \left(\frac{x^p - 1}{p}\right).$$

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing a (except possibly at $x = a$ itself) and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$

Curve Sketching

To sketch the graph of $y = \mathrm{f}(x)$, consider the following :

1. Intersections with the axes

Put $y = 0$, $x = 0$ to find the intersections with the x and y axes respectively.

2. Vertical asymptotes

When the denominator of a rational function is zero then $y \to \pm\infty$. At such a value of x there is a vertical line which the graph approaches but never meets.

3. Horizontal or oblique asymptotes

Find $\lim_{x\to\pm\infty} \mathrm{f}(x)$ to see what the graph is like when $|x|$ is very large.

If the numerator is of higher degree than the denominator, there may be **oblique asymptotes**. To find these, divide; e.g. $y = \dfrac{x^2 + 2x + 3}{x - 1} = x + 3 + \dfrac{6}{x - 1}$ has asymptotes $x = 1$, $y = x + 3$.

A graph may sometimes cross an oblique asymptote, but not a vertical one.

4. (Obvious) symmetries

If the function is even then there is symmetry about the y-axis. If the function is odd then there is 180° rotational symmetry about the origin.

If the function is self-inverse then there is symmetry about $y = x$.

5. Stationary points

Turning points can be found (if necessary) by the usual calculus methods: solve $\mathrm{f}'(x) = 0$. The second derivative $\mathrm{f}''(x)$ is positive at a minimum, negative at a maximum and 0 at a point of inflexion.

Sketching the graph can often help with the solution of inequalities. To find where $\mathrm{f}(x) > 0$, sketch $\mathrm{f}(x)$ and see where the graph lies above the x-axis.

To solve $\mathrm{f}(x) > \mathrm{g}(x)$, either sketch both graphs and see where $\mathrm{f}(x)$ is above $\mathrm{g}(x)$, or sketch $\mathrm{f}(x) - \mathrm{g}(x)$ and find the values of x where this is positive.

Related graphs can be obtained in various standard ways. For example, to obtain $y = |\mathrm{f}(x)|$, simply reflect in the x-axis those parts of the graph that lie below it.

To get $y = \dfrac{1}{\mathrm{f}(x)}$ note that $\dfrac{1}{\mathrm{f}(x)}$ has asymptotes where $\mathrm{f}(x) = 0$, and vice-versa.

Other standard transformations are :

$y = \mathrm{f}(x) + a$: translate $y = \mathrm{f}(x)$ by a units parallel to the y-axis (upwards).

$y = \mathrm{f}(x + a)$: translate $y = \mathrm{f}(x)$ by $-a$ units parallel to the x-axis (a to the left).

$y = \mathrm{f}(-x)$: reflect $y = \mathrm{f}(x)$ in the y-axis.

$y = -\mathrm{f}(x)$: reflect $y = \mathrm{f}(x)$ in the x-axis.

$y = k\mathrm{f}(x)$: stretch $y = \mathrm{f}(x)$ by a factor of k parallel to the y-axis.

$y = \mathrm{f}(kx)$: 'stretch' $y = \mathrm{f}(x)$ by a factor of $\dfrac{1}{k}$ parallel to the x-axis.

Implicit Equations

Up to now we have looked at functions defined *explicitly* in the form $y = \mathrm{f}(x)$. However, sometimes functions are given *implicitly*. For example, we might define $\mathrm{f}(x)$ to be the larger solution (for y) of $3y^2 + 4x^5y - 2x^2 - 4 = 0$.

The word *larger* is needed since the solution is not unique, and we want f to be a well-defined function.

In this case, we can obtain an explicit equation: $\mathrm{f}(x) = \dfrac{2x^5}{3} + \dfrac{1}{3}\sqrt{4x^{10} + 6x^2 + 12}$.

Sometimes a closed curve, which does not represent a true function but does express a relationship between two variables, is best described implicitly.

The **circle** with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

For example, consider the circle $x^2 + y^2 + 4x - 8y - 5 = 0$. Completing the square with respect to x and y, $(x + 2)^2 - 4 + (y - 4)^2 - 16 - 5 = 0$, i.e. $(x + 2)^2 + (y - 4)^2 = 25$, so the centre is (–2, 4) and the radius is 5.

Parametric Equations

It is often convenient to describe a curve (not necessarily representing a well-defined function) by two equations giving each of x and y in terms of a variable **parameter**.

For example, the circle $x^2 + y^2 = a^2$ has parametric equations $x = a\cos\theta, y = a\sin\theta$.

Many other types of curve can be described parametrically, such as the cycloid
$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$.

One way to sketch a graph defined by parametric equations is to work out x and y for a suitable range of values of the parameter.

Exercises

1. Find the amplitude and the period of

   (a) $3\sin 3x$ (b) $8\sin 3x\cos 3x$

   (c) $2\cos(2x + 4)$ (d) $6\cos x + 8\sin x$

2. Starting from the definitions, prove that $\cosh^2 x - \sinh^2 x \equiv 1$.

3. Solve, in terms of natural logarithms, (a) $4\sinh^2 x = \cosh^2 x$, (b) $7\sinh x = 24$.

4. Find the exact value of $\operatorname{arcosh}\dfrac{13}{12}$.

5. Find the coordinates of any points of intersection of the curves $y = \cosh 2x$ and $y = 3 - 2\cosh x$.

6. Find an identity relating $\coth^2 x$ and $\operatorname{cosech}^2 x$. Hence solve $\coth^2 x = 2\operatorname{cosech} x$.

7. Find the following limits:

   (a) $\displaystyle\lim_{x\to 0}\frac{2x^2 + 3x}{x}$, (b) $\displaystyle\lim_{x\to\infty}\frac{12x + 6}{3x - 4}$, (c) $\displaystyle\lim_{x\to 3}\frac{x - 3}{x^2 - 9}$.

8. The diameter of the pupil of an animal's eye is given by $\mathrm{f}(x)$ mm, where x is the intensity of the light on the pupil.

   If $\mathrm{f}(x) = \dfrac{80x^{-0.3}+60}{8x^{-0.3}+15}$, find (i) the diameter of the pupil when there is no light, (ii) the limit to which the diameter tends as the amount of light becomes large.

9. Sketch the following curves, and state the range of each function.

   (a) $y = \dfrac{x}{1+x}$, (b) $y = \dfrac{3x+2}{3-2x}$, (c) $y = \dfrac{2}{1-x^2}$, (d) $y = \dfrac{3x}{1-x^2}$,

   (e) $y = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}$, (f) $y = \dfrac{x^2+x-1}{x-1}$, (g) $y = \dfrac{x^2+1}{x^2+2}$.

10. $\mathrm{f}(x)$ is defined to be the larger solution (for y) of $2y^2 - 4xy - 3x^4 = 0$.
    Find an explicit formula for $\mathrm{f}(x)$.

11. The circle C has equation $x^2 + y^2 - 6x + 8y - 144 = 0$. Find the centre and radius of C. Show that the point A $(8, 8)$ lies on C and find the coordinates of B such that AB is a diameter of C.

12. The parametric equations of a curve C are $x = 1 + \sinh t$, $y = 5 - 4\cosh t$. Sketch C for $-1 \le t \le 1$. Show that C meets the x-axis at two points, and state their coordinates.

Differentiation

The gradient of the graph of a function describes the rate at which the function is changing at this point. For example, if the function describes the displacement of an object, then the gradient of a tangent to the graph gives its velocity.

Geometrically, the gradient of a graph at P $(x, \mathrm{f}(x))$ is found by drawing a chord from P to a nearby point $Q = (x+\delta x, \mathrm{f}(x+\delta x))$, where δx means a small increment in the x direction (NOT $\delta \times x$).

The gradient of this chord is $\dfrac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$.

As $\delta x \to 0$, the gradient of the chord tends to the gradient of the tangent.

Definition

Let f be a function defined in some neighbourhood of a point $(x, \mathrm{f}(x))$. If the limit

$$\lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$$

exists then f is said to be *differentiable* at $(x, \mathrm{f}(x))$. The value of the limit is the *derivative* or *derived function* or *differential coefficient* of $\mathrm{f}(x)$ at this point, which we can write as $\mathrm{f}'(x)$ or $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x))$.

f is a **differentiable function** if it is differentiable (i.e. the above limit exists) at every point where it is defined.

If $\mathrm{f}(x) = y$, the derivative is denoted by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, which can be understood as $\displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

When δy and δx are corresponding small increments in y and x, $\delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x}\,\delta x$.

An alternative version of the definition, giving the derivative where $x = a$, is

$$\mathrm{f}'(a) = \lim_{x \to a} \frac{\mathrm{f}(x) - \mathrm{f}(a)}{x - a}.$$

These formulae are used to differentiate functions from **first principles**.

Derivatives of Hyperbolic Functions

From the definitions in terms of exponentials, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x.$$

Using these results, the quotient and composite function rules give the following standard derivatives of hyperbolic functions:

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\sinh(ax+b)$	$a\cosh(ax+b)$
$\cosh(ax+b)$	$a\sinh(ax+b)$
$\tanh(ax+b)$	$a\ \mathrm{sech}^2(ax+b)$
$\coth(ax+b)$	$-a\ \mathrm{cosech}^2(ax+b)$
$\mathrm{sech}(ax+b)$	$-a\ \mathrm{sech}(ax+b)\tanh(ax+b)$
$\mathrm{cosech}(ax+b)$	$-a\ \mathrm{cosech}(ax+b)\coth(ax+b)$

Differentiating Inverse Functions

If y is a function of u which is a function of x, the rule for composite functions states: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$. Putting $u = x$ and $x = y$ gives $\frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y}$, so $\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = 1$.

If $y = \mathrm{f}(x)$ is an invertible function of x, and $\mathrm{f}^{-1}(x)$ is easier to differentiate than $\mathrm{f}(x)$, this technique can be used: express x in terms of y, differentiate with respect to y to get $\frac{\mathrm{d}x}{\mathrm{d}y}$ and then use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$. (**First** derivatives obey the laws of fractions.)

This method gives the following standard results:

$f(x)$	$f'(x)$	
$\arcsin x \equiv \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$,	$x \in (-1, 1)$
$\arccos x \equiv \cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$,	$x \in (-1, 1)$
$\arctan x \equiv \tan^{-1} x$	$\frac{1}{1+x^2}$	
$\text{arsinh}\ x \equiv \sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$	
$\text{arcosh}\ x \equiv \cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$x \in (1, \infty)$
$\text{artanh}\ x \equiv \tanh^{-1} x$	$\frac{1}{1-x^2}$,	$x \in (-1, 1)$

Parametric Differentiation

If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$.

For second derivatives, use $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx}$, where $\dfrac{dt}{dx} = \dfrac{1}{dx/dt}$.

Implicit Differentiation

An **implicit equation** $f(x, y) = g(x, y)$ can be differentiated term-by-term with respect to x. Whenever y occurs, it must be differentiated to get $\dfrac{dy}{dx}$, which can be abbreviated as y'. Thus, for example, y^2 differentiates (with respect to x) as $2yy'$.

Second derivatives: differentiate through the equation again; y' differentiates to y''.

Logarithmic Differentiation

Taking logarithms of both sides of an equation sometimes helps in differentiation.

If $y > 0$, $y = f(x)$ becomes $\ln y = \ln f(x)$, which differentiates to $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{f(x)} f'(x)$.

By this method we can show that if $f(x) = a^x$, where $a > 0$, then $f'(x) = a^x \ln a$.

Leibniz's Rule

This is a method for finding the nth derivative of a *product* of two functions.
Let f and g be n-times differentiable functions. Let h be defined by $h(x) = f(x)\, g(x)$. Then h is also n times differentiable and it can be proved that

$$h^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} f^{(r)}(x)\, g^{(n-r)}(x),$$

where $h^{(n)}(x)$ means the nth derivative of $h(x)$ with respect to x.

Radius of Curvature

You should be familiar with the application of differentiation to gradients, tangents, normals, turning points, maxima and minima, rates of change.

Another feature of a graph that can be found by differentiation is the **radius of curvature** of $y = \mathrm{f}(x)$, defined as

$$\rho = \frac{[1 + (\mathrm{d}y/\mathrm{d}x)^2]^{\frac{3}{2}}}{|\mathrm{d}^2y/\mathrm{d}x^2|}. \qquad \text{The } \textbf{curvature} \text{ is defined to be } \kappa = \frac{1}{\rho}.$$

l'Hôpital's Rule

Suppose f and g are differentiable real-valued functions with $\mathrm{f}(a) = \mathrm{g}(a) = 0$.

Then $\displaystyle\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \lim_{x\to a} \frac{\mathrm{f}(x) - \mathrm{f}(a)}{\mathrm{g}(x) - \mathrm{g}(a)} = \lim_{x\to a} \left(\frac{\mathrm{f}(x) - \mathrm{f}(a)}{x - a} . \frac{x - a}{\mathrm{g}(x) - \mathrm{g}(a)} \right)$ [for $x \neq a$]

$\displaystyle= \lim_{x\to a} \frac{\mathrm{f}(x) - \mathrm{f}(a)}{x - a} \div \lim_{x\to a} \frac{\mathrm{g}(x) - \mathrm{g}(a)}{x - a} = \frac{\mathrm{f}'(a)}{\mathrm{g}'(a)}$, provided $\mathrm{g}'(a) \neq 0$.

If $\mathrm{f}'(a) = \mathrm{g}'(a) = 0$ we can repeat the process to get $\displaystyle\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{\mathrm{f}''(a)}{\mathrm{g}''(a)}$, provided $\mathrm{g}''(a) \neq 0$.

Exercises

1. Differentiate from first principles : (a) x^5, (b) $\dfrac{1}{x}$, (c) $\sin 2x$.

2. Differentiate with respect to x:

   (a) $(x^2 + 3)^{5/2}$, (b) $\sinh(\cosh x)$, (c) $\arcsin\left(\dfrac{x+3}{2}\right)$,

   (d) $\mathrm{artanh}\,(1 - x^2)$, (e) $\coth^2(\mathrm{e}^x)$, (f) $\mathrm{arsech}\; x$ (i.e. $\mathrm{sech}^{-1}x$).

3. Use logarithmic differentiation to find the derivative with respect to x of

   (a) $(\sin x)^x$, (b) $3^{\cosh x}$, (c) $\dfrac{x \tan^2 x}{x^3 - 1}$, where $x > 1$.

4. Find the maximum and minimum values of $\arcsin(x^2 - 1)$ for $-1 \leq x \leq 1$.

5. A curve has parametric equations $x = a(1 - \cos 2t), y = a(2t + \sin 2t)$, where a is a non-zero real constant and $0 \leq t \leq \dfrac{\pi}{2}$. Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $t = \dfrac{\pi}{4}$.

6. A circle has equation $x^2 + y^2 - 2x + 6y - 15 = 0$. Find the gradient of the tangent to this circle at the point where $x = 5$ and $y < 0$,

   (a) by implicit differentiation,

   (b) by finding the centre and radius and using coordinate geometry.

7. The implicit equation of a curve is $2y^2 - 3xy + x = 6$. Find equations of the tangent and the normal to the curve at the point where $x = 1$ and $y > 0$.

   Also find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x, y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

8. Given that $\mathrm{f}(x) = \dfrac{1}{3}\sinh x(2+\cosh^2 x)$, show that $\mathrm{f}\,'(x) = \cosh^3 x$.

   Find, to the nearest integer, the radius of curvature of the curve $y = \mathrm{f}(x)$ at the point where $x = \ln 2$.

9. Use Leibniz's Rule to find (a) the third derivative of $x^4 \cos 2x$,

   (b) an expression for the nth derivative of $x^3 \sinh x$.

10. Use l'Hôpital's rule to find the following limits:

    (a) $\displaystyle\lim_{x\to a} \frac{x^2-a^2}{x^3-a^3}$, (b) $\displaystyle\lim_{x\to 2} \frac{\sin \pi x}{x^2-4}$, (c) $\displaystyle\lim_{x\to 0} \frac{\arctan x}{\ln(x+1)}$, (d) $\displaystyle\lim_{x\to 1} \frac{\ln x}{\mathrm{e}^x - \mathrm{e}}$.

Maclaurin Series

A series of the form $a_0 + a_1 x + a_2 x^2 + \ldots = \displaystyle\sum_{r=0}^{\infty} a_r x^r$ is called a **power series** in x.

Let f be a function such that $\mathrm{f}^{(n)}(0)$ exists for all $n \in \mathbb{N}$, and suppose $\mathrm{f}(x)$ can be expressed as a power series in x: $\mathrm{f}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

Then $\mathrm{f}(0) = a_0$. Now

$\mathrm{f}'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$ so $\mathrm{f}'(0) = a_1$. Also,

$\mathrm{f}''(x) = 2a_2 + 6a_3 x + \ldots$ so $\mathrm{f}''(0) = 2a_2$.

Similarly $\mathrm{f}'''(0) = 6a_3$, etc.

Thus $\mathrm{f}^{(r)}(0) = r!(a_r)$, so $a_r = \dfrac{1}{r!}\mathrm{f}^{(r)}(0)$.

The series $\displaystyle\sum_{r=0}^{\infty} \frac{1}{r!}\mathrm{f}^{(r)}(0)x^r$ is called the **Maclaurin Series** for $\mathrm{f}(x)$. It is also sometimes called a **Taylor Series**, but this name is usually reserved for expansions about values other than 0.

The expansion is valid for such values of x as make the series converge to $\mathrm{f}(x)$. This set of values can be written as $|x| < R$, where $R = \lim_{n\to\infty} |a_n/a_{n+1}|$ is called the **radius of convergence**.

It can be proved that a power series can be differentiated or integrated term-by-term to give another valid power series with the same radius of convergence.

We have already seen the series for e^x. Here are some other standard Maclaurin series. They are convergent for all real x unless otherwise stated. Note that in trigonometric series, x must be in radians.

- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots = \displaystyle\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots = \displaystyle\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}$
- $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots = \displaystyle\sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}$

- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{r=0}^{\infty} \frac{x^{2r}}{(2r)!}$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{r} \quad (-1 < x \leq 1)$
- $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots = -\sum_{r=1}^{\infty} \frac{x^r}{r} \quad (-1 \leq x < 1)$

Note that $\ln x$ itself has no Maclaurin series, as it is not defined at $x = 0$.

Other series can be obtained from these, e.g. $\ln(\cos x) = \ln\left(1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)\right)$

$= -\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) - \frac{1}{2}\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right)^2 - \ldots = -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$

Further Complex Numbers

Many power series are valid for complex numbers also. In particular, if $z \in \mathbb{C}$ then the series $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$ converges for all z. We call its limit $\exp(z)$ or e^z.

$\mathrm{e}^w\mathrm{e}^z = \left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots\right)\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)$

$= 1 + (w+z) + \frac{(w+z)^2}{2!} + \ldots = \mathrm{e}^{w+z}$.

The complex exponential function has the usual properties of exponentials.

Now if $\theta \in \mathbb{R}$, $\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{\mathrm{i}^2\theta^2}{2!} + \frac{\mathrm{i}^3\theta^3}{3!} + \frac{\mathrm{i}^4\theta^4}{4!} + \ldots = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$

$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \ldots\right) = \cos\theta + \mathrm{i}\sin\theta$.

From this we get the well-known relation $\mathrm{e}^{\mathrm{i}\pi} = -1$.

The complex number $z = r(\cos\theta + \mathrm{i}\sin\theta)$, with modulus r and argument θ, can be written in the **exponential form** $z = r\mathrm{e}^{\mathrm{i}\theta}$.

De Moivre's Theorem can be stated as $(r\mathrm{e}^{\mathrm{i}\theta})^n = r^n\mathrm{e}^{\mathrm{i}n\theta}$. This is valid for any $n \in \mathbb{R}$.

Complex roots of real numbers can be found using de Moivre's Theorem. The **cube roots of unity** (1) are the three roots of $z^3 = 1$; these are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$. They are usually denoted by $1, \omega, \omega^2$. Their sum is 0.

We can write 1 as $\cos 0 + \mathrm{i}\sin 0$, or $\mathrm{e}^{0\mathrm{i}}$.

Also $1 = \cos 2\pi + \mathrm{i}\sin 2\pi = \mathrm{e}^{2\pi\mathrm{i}}$ and $1 = \cos 4\pi + \mathrm{i}\sin 4\pi = \mathrm{e}^{4\pi\mathrm{i}}$.

Thus $1^{1/3} = \mathrm{e}^{0\mathrm{i}/3}, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}$. Now $\mathrm{e}^{2\pi\mathrm{i}/3} = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = \omega$ and $\mathrm{e}^{4\pi\mathrm{i}/3} = \omega^2$.

The cube roots of unity divide the unit circle into three equal parts. Similarly, the nth roots of unity are the vertices of a regular n-sided polygon with one vertex at 1; for example, the seventh roots of 1 are $1, \mathrm{e}^{2\pi\mathrm{i}/7}, \mathrm{e}^{4\pi\mathrm{i}/7}, \mathrm{e}^{6\pi\mathrm{i}/7}, \mathrm{e}^{8\pi\mathrm{i}/7}, \mathrm{e}^{10\pi\mathrm{i}/7}, \mathrm{e}^{12\pi\mathrm{i}/7}$.

The nth roots of a real number x, of which there are n, are $x^{1/n}\mathrm{e}^{2k\pi\mathrm{i}/n}$, for $k = 0, 1, \ldots, n-1$.

Exercises

1. Obtain the first two non-zero terms in the Maclaurin series for $\tan x$.

2. Use the binomial theorem to find the first three non-zero terms in the series for $(1+x^2)^{-1}$, stating the radius of convergence. Integrate your series term-by-term, and state which function you have found a Maclaurin series for.

3. By multiplying two Maclaurin series together, find the first three non-zero terms in the series for $\cos x \sinh x$.

4. Express $z = \sqrt{6} + \mathrm{i}\sqrt{2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Hence find the three cube roots of $\sqrt{6} + \mathrm{i}\sqrt{2}$.

5. Find the four fourth roots of 16.

6. Express the 5 fifth roots of unity in the form $\mathrm{e}^{\mathrm{i}\theta}$. Let u be one of the complex roots and $v = u + \dfrac{1}{u}$. Show that $v^2 + v - 1 = 0$. Deduce the value of $\cos \dfrac{2\pi}{5}$ in surd form.

Integration

There are two (equivalent) ways of defining integration. The first is as the inverse of differentiation. Given a function f, we look for a function F such that $\mathrm{F}'(x) = \mathrm{f}(x)$. If this can be found, we denote $\mathrm{F}(x)$ by $\int \mathrm{f}(x)\ \mathrm{d}x$. This is an *indefinite integral.*

$\mathrm{F}(x)$ is sometimes called an **antiderivative** or **primitive** of $\mathrm{f}(x)$.
Clearly if $\mathrm{F}(x)$ is an antiderivative of $\mathrm{f}(x)$ then so is $\mathrm{F}(x) + c$ for any real constant c.

The second approach uses area. If f is defined on the interval (α, β) and the area enclosed by its graph and the x-axis between $x = \alpha$ and $x = \beta$ is finite, this area is called the **definite integral** of $\mathrm{f}(x)$ from α to β, denoted by $\int_\alpha^\beta \mathrm{f}(x)\ \mathrm{d}x$.

We then say that $\mathrm{f}(x)$ is **integrable** on (α, β).

(Area below the x-axis is regarded as negative.)

This yields the definition $\displaystyle\int_\alpha^\beta \mathrm{f}(x)\ \mathrm{d}x = \lim_{N\to\infty} \sum_{i=0}^{N} \mathrm{f}(\alpha + i\delta x)\delta x$ with $\delta x = (\beta - \alpha)/N$.

The relation between definite and indefinite integration is given by:

The Fundamental Theorem of Calculus

Let $\mathrm{f} : (\alpha, \beta) \to \mathbb{R}$ be an integrable function. Then there exists a function
$\mathrm{F} : (\alpha, \beta) \to \mathbb{R}$ such that $\mathrm{F}' = \mathrm{f}$, and

$$\int_\alpha^\beta \mathrm{f}(x)\ \mathrm{d}x = \mathrm{F}(\beta) - \mathrm{F}(\alpha).$$

It follows that $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_\alpha^x \mathrm{f}(t)\ \mathrm{d}t = \dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{F}(x) - \mathrm{F}(\alpha)) = \mathrm{f}(x)$.

Properties of Definite Integrals

- $\int_{\alpha}^{\beta} a\mathrm{f}(x)\ \mathrm{d}x = a\int_{\alpha}^{\beta} \mathrm{f}(x)\ \mathrm{d}x$, for $a \in \mathbb{R}$
- $\int_{\alpha}^{\beta} (\mathrm{f}(x) + \mathrm{g}(x))\ \mathrm{d}x = \int_{\alpha}^{\beta} \mathrm{f}(x)\ \mathrm{d}x + \int_{\alpha}^{\beta} \mathrm{g}(x)\ \mathrm{d}x$
- $\int_{\beta}^{\alpha} \mathrm{f}(x)\ \mathrm{d}x = -\int_{\alpha}^{\beta} \mathrm{f}(x)\ \mathrm{d}x$
- If $\alpha \leq \gamma \leq \beta$ then $\int_{\alpha}^{\beta} \mathrm{f}(x)\ \mathrm{d}x = \int_{\alpha}^{\gamma} \mathrm{f}(x)\ \mathrm{d}x + \int_{\gamma}^{\beta} \mathrm{f}(x)\ \mathrm{d}x$
- If f is odd, $\int_{-\alpha}^{\alpha} \mathrm{f}(x)\ \mathrm{d}x = 0$. If f is even, $\int_{-\alpha}^{\alpha} \mathrm{f}(x)\ \mathrm{d}x = 2\int_{0}^{\alpha} \mathrm{f}(x)\ \mathrm{d}x$
- If f is periodic with period k and $N \in \mathbb{Z}$ then $\int_{0}^{\alpha} \mathrm{f}(x)\ \mathrm{d}x = \int_{Nk}^{Nk+\alpha} \mathrm{f}(x)\ \mathrm{d}x$

The natural logarithm function can be defined as $\ln x = \int_{1}^{x} \frac{1}{t}\ \mathrm{d}t$.

Sometimes relationships between functions can be deduced from different forms of the same indefinite integral, e.g. for $x \in (-1, 1)$, $\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.
Thus both $-\arccos x$ and $\arcsin x$ are indefinite integrals of $\frac{1}{\sqrt{1-x^2}}$, so it must be that $-\arccos x = \arcsin x + c$.
As $\arccos 0 = \frac{\pi}{2}$ and $\arcsin 0 = 0$, we get $c = -\frac{\pi}{2}$, so $\quad \arcsin x = \frac{\pi}{2} - \arccos x$.

Standard Integrals

By inverting standard derivatives we get the following integrals, in addition to those in Part A; a and b are real constants. In each case a constant of integration c should be added for indefinite integrals.

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\ \mathrm{d}x$	
$\sinh(ax+b)$	$\frac{1}{a}\cosh(ax+b)$	
$\cosh(ax+b)$	$\frac{1}{a}\sinh(ax+b)$	
$a^x \ (a > 0)$	$\frac{a^x}{\ln a}$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin \frac{x}{a}$	$x \in (-a, a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan \frac{x}{a}$	
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh} \frac{x}{a}$	
$\frac{1}{\sqrt{x^2-a^2}}$,	$\operatorname{arcosh} \frac{x}{a}$	$x \in (a, \infty), a > 0$

Integration by Substitution

Often we cannot integrate directly, but have to transform an integral into a recognisable standard form. In this respect, integration is much less routine than differentiation.

In the substitution method we choose a suitable variable $u = \mathrm{g}(x)$, so $\mathrm{d}u = \mathrm{g}'(x)\ \mathrm{d}x$, and obtain the integral of a function of u with respect to u. Having integrated, the answer must then be expressed in terms of x.

Sometimes it is easier to express x in terms of the new variable u.

The limits on a definite integral can be converted into values of the new variable, e.g.

$\displaystyle\int_{\pi/2}^{\pi} \cos^3 x\ \mathrm{d}x = \int_{\pi/2}^{\pi} (1-\sin^2 x)\cos x\ \mathrm{d}x = \int_1^0 (1-u^2)\ \mathrm{d}u$ where $u = \sin x$.

Rational functions containing square roots of quadratic functions can often be converted (e.g. by completing the square) to one of the following types:

- If $\sqrt{a^2-x^2}$ occurs, try substituting $x = a\sin u$, so $\mathrm{d}x = a\cos u\ \mathrm{d}u$.
  Simplify using $a^2(1-\sin^2 u) \equiv a^2\cos^2 u$.
- If $\sqrt{x^2-a^2}$ occurs, try substituting
  $x = a\cosh u$, so $\mathrm{d}x = a\sinh u\ \mathrm{d}u$ or $x = a\sec u$, so $\mathrm{d}x = a\sec u\tan u\ \mathrm{d}u$.
- If $\sqrt{x^2+a^2}$ occurs, try substituting
  $x = a\sinh u$, so $\mathrm{d}x = a\cosh u\ \mathrm{d}u$ or $x = a\tan u$, so $\mathrm{d}x = a\sec^2 u\ \mathrm{d}u$.

(Sometimes one alternative will work better than the other.)

For integrals of the form $\displaystyle\int \frac{1}{x\sqrt{ax^2+bx+c}}\ \mathrm{d}x$, first put $x = \dfrac{1}{u}$, so $\mathrm{d}x = -\dfrac{1}{u^2}\ \mathrm{d}u$, etc.

Also be aware of: $\displaystyle\int \frac{1}{\sqrt{x+a}}\ \mathrm{d}x = 2\sqrt{x+a}+c$, $\displaystyle\int \frac{x}{\sqrt{x^2+a}}\ \mathrm{d}x = \sqrt{x^2+a}+c$.

The substitution $t = \tan\frac{1}{2}x$

If $t = \tan\dfrac{x}{2}$, then $\tan x = \dfrac{2t}{1-t^2}$, from which we get

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad \mathrm{d}x = \frac{2\ \mathrm{d}t}{1+t^2}.$$

This is useful for some integrals of the form $\displaystyle\int \frac{1}{a\sin x + b\cos x + c}\ \mathrm{d}x$.

Integration by Parts

This method is obtained from the product rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \text{hence} \qquad u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Integration of this formula gives

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x, \text{ or, equivalently } \int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u.$$

Integration by parts is normally used for a product in which one factor becomes simpler when differentiated. More than one application of the method may be needed.

Sometimes 'multiplication by 1' is used to express a single function as a product, i.e. $\mathrm{f}(x) = \mathrm{f}(x) \times 1$. For example, to find $\int \ln x \; \mathrm{d}x$ we can let $u = \ln x$, $\mathrm{d}v = 1 \; \mathrm{d}x$.

Then $\mathrm{d}u = \dfrac{1}{x} \; \mathrm{d}x, v = x$, so $\displaystyle\int \ln x \; \mathrm{d}x = x \ln x - \int 1 \; \mathrm{d}x = x \ln x - x + c$

Mixed Integration Worked Examples

1. $\displaystyle\int \frac{x^3+2}{x^2+2} \; \mathrm{d}x = \int x + \frac{2-2x}{x^2+2} \; \mathrm{d}x = \int x + \frac{2}{x^2+2} - \frac{2x}{x^2+2} \; \mathrm{d}x$

   $= \dfrac{x^2}{2} + \sqrt{2} \arctan \dfrac{x}{\sqrt{2}} - \ln(x^2+2) + c.$

2. Find $\displaystyle\int \frac{1}{x(x^2+1)^2} \; \mathrm{d}x.$

   In partial fractions, $\dfrac{1}{x(x^2+1)^2} \equiv \dfrac{1}{x} - \dfrac{x}{x^2+1} - \dfrac{x}{(x^2+1)^2}.$

   The second term is integrated using $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} = \ln|\mathrm{f}(x)| + c$ and the last term is done by the substitution $u = x^2+1, \mathrm{d}u = 2x \; \mathrm{d}x$.

   We get $\ln|x| - \dfrac{1}{2} \ln(x^2+1) + \dfrac{1}{2(x^2+1)} + c.$

3. Find $\displaystyle\int \frac{1}{\sqrt{-3x^2+12x-8}} \mathrm{d}x.$

   Completing the square,

   $-3x^2+12x-8 = -3(x^2-4x+4)+4 = -3(x-2)^2+4 = 4\,[\,1-(\sqrt{3}(x-2)/2)^2\,].$

   Substitute $u = \dfrac{\sqrt{3}}{2}(x-2)$, so $x = \dfrac{2}{\sqrt{3}} u + 2$, $\mathrm{d}x = \dfrac{2}{\sqrt{3}} \mathrm{d}u$,

   and the integral becomes

   $\displaystyle\int \frac{1}{\sqrt{1-u^2}} \frac{2}{\sqrt{3}} \mathrm{d}u = \frac{2}{\sqrt{3}} \arcsin u + c = \frac{2}{\sqrt{3}} \arcsin\left(\frac{\sqrt{3}(x-2)}{2}\right) + c.$

4. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}} \mathrm{d}x.$ Substitute $x = \tan\theta$, so $\mathrm{d}x = \sec^2\theta \; \mathrm{d}\theta$.

   $\displaystyle\int \frac{1}{\tan^2\theta \sec\theta} \frac{1}{\cos^2\theta} \mathrm{d}\theta = \int \frac{\cos\theta}{\sin^2\theta} \mathrm{d}\theta.$ Now substitute $u = \sin\theta$ to get

   $\displaystyle\int \frac{1}{u^2} \; \mathrm{d}u = -\frac{1}{u} + c = -\frac{1}{\sin\theta} + c = -\frac{\sqrt{1+x^2}}{x} + c.$

5. $\displaystyle\int \frac{\sin^2 x}{1+\cos x} \; \mathrm{d}x = \int \frac{(1+\cos x)(1-\cos x)}{1+\cos x} \; \mathrm{d}x = \int (1-\cos x) \; \mathrm{d}x = x - \sin x + c.$

6. $\int \sqrt{4-x^2} \; \mathrm{d}x$. Use the substitution $x = 2\sin\theta$, so $\mathrm{d}x = 2\cos\theta \; \mathrm{d}\theta$, to get

   $\int \sqrt{(4-4\sin^2\theta)} \; 2\cos\theta \; \mathrm{d}\theta = 4 \int \cos^2\theta \; \mathrm{d}\theta = \sin 2\theta + 2\theta + c = 2\sin\theta\cos\theta + 2\theta + c$

   $= \dfrac{2x}{2}\sqrt{1-(x/2)^2} + 2\arcsin\dfrac{x}{2} + c = \dfrac{1}{2}x\sqrt{4-x^2} + 2\arcsin\dfrac{x}{2} + c$

7. $\displaystyle\int \frac{1}{\cos^4 x}\,\mathrm{d}x = \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x}\,\mathrm{d}x = \int \tan^2 x \,\frac{1}{\cos^2 x}\,\mathrm{d}x + \int \frac{1}{\cos^2 x}\,\mathrm{d}x$
$\displaystyle = \int \tan^2 x \,\frac{\mathrm{d}(\tan x)}{\mathrm{d}x}\,\mathrm{d}x + \int \sec^2 x\,\mathrm{d}x = \frac{1}{3}\tan^3 x + \tan x + c.$

Reduction Formulae

There are many cases in which it is useful to reduce an integral involving a power of some function to one involving a lower power, e.g. if $I_n = \int \sin^n x \ \mathrm{d}x$ we can express I_n in terms of $I_{n-2} = \int \sin^{n-2} x \ \mathrm{d}x$. Such a **reduction formula** is commonly, but not always, found using integration by parts.

Example

Let $I_n = \int \sin^n x \,\mathrm{d}x$, where n is a positive integer. Take $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x \ \mathrm{d}x$, so $v = -\cos x$. Using the integration by parts formula,

$$\begin{aligned} I_n &= \sin^{n-1} x \cdot (-\cos x) - \textstyle\int (-\cos x)\cdot(n-1)\sin^{n-2} x \cos x \,\mathrm{d}x \\ &= -\cos x \ \sin^{n-1} x + (n-1)\textstyle\int (1-\sin^2 x)\ \sin^{n-2} x\,\mathrm{d}x \quad (\text{use } \cos^2 x = 1-\sin^2 x) \\ &= -\cos x \ \sin^{n-1} x + (n-1)(I_{n-2} - I_n). \end{aligned}$$

So $\displaystyle I_n = -\frac{1}{n}\cos x \ \sin^{n-1} x + \frac{n-1}{n} I_{n-2}.$

$$\begin{aligned} \text{Then } I_5 &= -\tfrac{1}{5}\sin^4 x \cos x + \tfrac{4}{5} I_3 \\ &= -\tfrac{1}{5}\sin^4 x \cos x + \tfrac{4}{5}\left(-\tfrac{1}{3}\sin^2 \cos x + \tfrac{2}{3}\textstyle\int \sin x \,\mathrm{d}x\right) \\ &= -\tfrac{1}{5}\sin^4 x \cos x - \tfrac{4}{15}\sin^2 x \cos x - \tfrac{8}{15}\cos x + c. \end{aligned}$$

Applications

The **mean value** of $\mathrm{f}(x)$ over the interval $[\alpha, \beta]$ is equal to $\displaystyle\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathrm{f}(x)\ \mathrm{d}x.$

The **length of the arc** joining the points on $y = \mathrm{f}(x)$ at which $x = \alpha$ and $x = \beta$ is

$$s = \int_\alpha^\beta \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x.$$

If this arc is rotated once about the x-axis, the **curved surface area** is

$$A = 2\pi \int_\alpha^\beta y \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x.$$

Improper Integrals

It can happen that the definite integral of a function is finite even though x or y tends to ∞ or $-\infty$ at one of the limits of integration.

We define $\displaystyle\int_\alpha^\infty \mathrm{f}(x)\ \mathrm{d}x$ to be $\displaystyle\lim_{\beta\to\infty}\int_\alpha^\beta \mathrm{f}(x)\ \mathrm{d}x$, if this limit exists and is finite.

Also if $\mathrm{f}(x)$ does not have a finite value at $x = \alpha$ we define $\int_\alpha^\beta \mathrm{f}(x)\,\mathrm{d}x$ to be $\lim_{\epsilon \to 0} \int_{\alpha+\epsilon}^\beta \mathrm{f}(x)\,\mathrm{d}x$ (where $\epsilon > 0$), if this limit exists. Otherwise the integral *diverges.*

Exercises

1. Find the integrals

   (a) $\int \sinh(3x-4)\,\mathrm{d}x$ (b) $\int_0^1 \frac{5}{\sqrt{4-x^2}}\,\mathrm{d}x$ (c) $\int 3^x + \frac{\sin x}{\cos x + 1}\,\mathrm{d}x$

   (d) $\int \frac{1}{(x^2+2x+2)}\,\mathrm{d}x$ (e) $\int_3^4 \frac{1}{4x^2-20x+25}\,\mathrm{d}x$

2. Using suitable substitutions, or otherwise, find the integrals

   (a) $\int \frac{1}{\sqrt{(x-1)(x-3)}}\,\mathrm{d}x$ (b) $\int \frac{x-2}{\sqrt{x^2-1}}\,\mathrm{d}x$

   (c) $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ (d) $\int \frac{x+1}{\sqrt{x^2-x+1}}\,\mathrm{d}x$

   (e) $\int \frac{\sqrt{x^2-9}}{x}\,\mathrm{d}x$ (f) $\int \sqrt{2x-x^2}\,\mathrm{d}x$

3. Find the integrals:

   (a) $\int \frac{\cos x}{\sin x + \cos x}\,\mathrm{d}x$ (b) $\int \frac{1}{2-\sin x}\,\mathrm{d}x$ (c) $\int_{-1}^1 \sqrt{(1-x^2)}\,\mathrm{d}x$

4. Using partial fractions or otherwise, find the integrals

   (a) $\int \frac{x^2}{(x-1)(x-3)}\,\mathrm{d}x$ (b) $\int \frac{x+1}{x(x+5)^2}\,\mathrm{d}x$

   (c) $\int \frac{x^3}{(x+1)(x^2-4)}\,\mathrm{d}x$ (d) $\int \frac{x^3+4x^2-x+3}{(x-2)(x^2+1)}\,\mathrm{d}x$

5. Prove that, for a, $b > 0$, $\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}\,\mathrm{d}x = \frac{\pi}{2ab}$.

6. Find the mean value of $y = \frac{1}{x^2+5x+6}$ over the interval $0 \le x \le 6$. By reference to a sketch, explain why the mean value you have found is reasonable.

7. A curve is given by the parametric equations $x = 2\sinh^3 t, y = 3\cosh^2 t$, for $0 \le t \le \ln 3$. Find the total length of the curve.

8. The arc of the curve $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated once about the x-axis. Find the area of the curved surface formed.

9. Given that $I_n = \int_0^1 x^n \cosh x\,\mathrm{d}x$, where n is a positive integer, prove that for $n \ge 2$,

   $$I_n = \sinh 1 - n\cosh 1 + n(n-1)I_{n-2}.$$

   Hence find, in terms of hyperbolic functions, the value of I_4.

10. Evaluate the following, if they exist.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\, \mathrm{d}x$ (b) $\displaystyle\int_0^2 x^{-1/3}\, \mathrm{d}x$ (c) $\displaystyle\int_{-\infty}^{-2} \frac{-1}{x^2+4}\, \mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/2} \tan x\, \mathrm{d}x$ (e) $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-4x+3}}\, \mathrm{d}x$

Ordinary Differential Equations

In this module we cover first order ordinary differential equations (o.d.e.s). These contain only first derivatives, $\left(\frac{dy}{dx}\right)$. We will focus on the following generic types;

- Variables separable equations
$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$
- Linear equations
$$\frac{dy}{dx} + yP(x) = Q(x)$$
- Homogeneous Equations
$$\frac{dy}{dx} = H\left(\frac{y}{x}\right)$$
- Bernoulli Equation
$$\frac{dy}{dx} + yR(x) = y^n S(x)$$
- Equations that can be transformed to one of the types above through substitution.

Solving a first order o.d.e such as $y' = 2y$ gives $y(x) = Ce^{2x}$ and thus gives rise to a **family of solutions** depending on the parameter C. In this case y is defined for all $x \in \mathbb{R}$ and $y \in (0, \infty)$. Note that for brevity we frequently use y' to denote differentiation of y with respect to its independent variable. Thus every point (x, y) in the positive half plane $y > 0$ lies on a solution curve, moreover it lies on only one solution curve. To find a specific, unique solution we need only specify a particular value of (x, y) such as $y(x_0) = y_0$. Thus $y' = 2y, \quad y(0) = 4$ has the solution $y(x) = 4e^{2x}$.

Always check that your solution is correct by verifying by differentiation and rearrangement that it satisfies the original differential equation

Many o.d.e.s will also have y = constant as a solution (e.g. $y' = 2y$ has $y \equiv 0$). This is known as the trivial solution, you should therefore always check to see if there is a trivial solution to the o.d.e. you are solving.

Variables separable equations

Write $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ in the form $g(y)\frac{dy}{dx} = f(x)$ and integrate both sides with respect to x to give

$$\int g(y)dy = f(x)dx$$

and then make y the subject of the equation. In principle this type of o.d.e. can always be solved, provided that f and g can be integrated to give closed form functions. If they can't there are of course powerful numerical and qualitative techniques available.

Here are some examples (we use c as the constant of integration)

1.

$$\frac{dy}{dt} = 5t^4y^2 \quad y(0) = 1 \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dt} = 5t^4 \quad \Rightarrow \quad \int \frac{dy}{y^2} = \int 5t^4 dt$$

$$\Rightarrow -\frac{1}{y} = t^5 + c, \quad \text{and since} \quad y(0) = 1, \quad c = -1 \quad \Rightarrow \quad -\frac{1}{y} = t^5 - 1 \quad \Rightarrow \quad y(t) = \frac{1}{t^5 - 1}$$

2.

$$\frac{dy}{dx} = \cos x \tan y \quad \Rightarrow \quad \frac{1}{\tan y}\frac{dy}{dx} = \cos x \quad \Rightarrow \quad \int \frac{dy}{\tan y} = \int \cos x dx$$

$$\Rightarrow \ln|\sin y| = \sin x + c \quad \Rightarrow \quad \sin y = Ae^{\sin x} \quad (\text{where} \quad A = e^c) \quad \Rightarrow \quad y(x) = \sin^{-1} Ae^{\sin x}$$

Linear equations - the integrating factor method

We transform the equation into one where the variables are separable and then apply the method of the previous section. The equation must be in the following specific form (it is linear in y and y');

$$\frac{dy}{dx} + yP(x) = Q(x)$$

The steps are as follows

1. Make sure the the o.d.e. is in the specific form above (by rearranging or substitution if necessary).
2. Work out the integrating factor $e^{\int P(x)dx}$ (Note that there is no constant of integration at this stage).
3. Multiply both sides of the equation by the integrating factor,

$$e^{\int P(x)dx}\frac{dy}{dx} + ye^{\int P(x)dx}P(x) = Q(x)e^{\int P(x)dx},$$

   the left hand side can be written as $\frac{d}{dx}ye^{\int P(x)dx}$ (check this by using the rule for differentiating products).
4. Now the equation is

$$\frac{d}{dx}ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

5. Integrate the right hand side

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx} + c.$$

6. Make y the subject of the equation

$$y(x) = \frac{\int Q(x)e^{\int P(x)dx} + c}{e^{\int P(x)dx}}.$$

Here are some examples of this method;

1.

$$ty' = y + t^3 \quad \Rightarrow \quad y' - \frac{y}{t} = t^2 \quad \text{(now the equation is in the correct form.)}$$

The integrating factor is $e^{-\int \frac{1}{t}dt} = e^{-\ln(t)} = e^{\ln \frac{1}{t}} = \frac{1}{t}$, multiply both sides by $\frac{1}{t}$

$$\frac{1}{t}y' - \frac{y}{t^2} = t \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{y}{t}\right) = \frac{t^2}{2} + c \quad \Rightarrow \quad y(t) = \frac{t^3}{2} + ct$$

2.

$$y' + 2xy = 4x \quad y(0) = 3 \quad \Rightarrow \quad \text{The integrating factor is} \quad e^{\int 2xdx} = e^{x^2}$$

multiply both sides by $e^{x^2} \Rightarrow e^{x^2}y' + 2xye^{x^2} = 4xe^{x^2} \quad \Rightarrow \quad \frac{d}{dx}ye^{x^2} = 4xe^{x^2}$

$$\Rightarrow \quad ye^{x^2} = \int 4xe^{x^2}dx \quad = 2e^{x^2} + c \Rightarrow y = 2 + ce^{-x^2}$$

we have $y(0) = 3 \quad \Rightarrow \quad c + 2 = 3, \quad c = 1$ thus $y(x) = 2 + e^{-x^2}$

Homogeneous equations

These are equations where each of the polynomial expressions have the same order (e.g. x^2y^3 and xy^4 are both of order 5). They can be rearranged into the form $\frac{dy}{dx} = H\left(\frac{y}{x}\right)$. With the substitution $u = \frac{y}{x}$ or $y = ux$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$ by the chain rule. Thus we can rewrite the equation as

$$u + x\frac{du}{dx} = H(u)$$

which we can rearrange to give us a variables separable equation

$$x\frac{du}{dx} = H(u) - u \quad \Rightarrow \quad \int \frac{du}{H(u) - u} = \int \frac{dx}{x}$$

which we now integrate and complete the solution by substituting for u.

The following example shows how this method works;

$2xyy' - y^2 + x^2 = 0$

$\Rightarrow y' = \dfrac{y^2 - x^2}{2xy} = \dfrac{\frac{y^2}{x^2} - 1}{2\frac{y}{x}}$ (dividing numerator and denominator of the right hand side by x^2)

let $u = \dfrac{y}{x}, y = ux \quad y' = u + x\dfrac{du}{dx} \quad \Rightarrow \quad u + x\dfrac{du}{dx} = \dfrac{u^2 - 1}{2u}$

$x\dfrac{du}{dx} = -\dfrac{1 + u^2}{2u} \quad \Rightarrow \quad \displaystyle\int \frac{2u\, du}{1 + u^2} = -\int \frac{dx}{x}$ now integrate both sides

$\Rightarrow \ln(1 + u^2) = -\ln|x| + c \quad \Rightarrow \quad 1 + u^2 = e^{-\ln|x| + c} = \dfrac{A}{x}$ with $A = e^c$

(substituting for u) $1 + \dfrac{y^2}{x^2} = \dfrac{A}{x} \quad \Rightarrow \quad y^2 = Ax - x^2$

Bernoulli equations

$$\frac{dy}{dx} + yR(x) = y^n S(x)$$

The right hand side contains y^n so that the integrating factor method will not work without further transformation of the equation. We now make the substitution $u = y^{1-n}$ with

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}.$$

We can now transform the original equation in x and y into an equation in x and u as follows;

$$\frac{y^n}{1-n}\frac{du}{dx} + R(x)y = S(x)y^n$$

$$\frac{1}{1-n}\frac{du}{dx} + R(x)y^{1-n} = S(x) \quad \Rightarrow \quad \frac{1}{1-n}\frac{du}{dx} + R(x)u = S(x)$$

$$\frac{du}{dx} + u(1-n)R(x) = (1-n)S(x).$$

The o.d.e. is now in the correct form to be solved by the integrating factor method, as the following example shows.

$$x\frac{dx}{dy} = x^2y^2\ln x \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = xy^2\ln x$$

$$\text{we have} \quad R(x) = \frac{1}{x}, \quad S(x) = x\ln x, \quad n = 2. \quad \text{if} \quad u = y^{-1}, \quad \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

$$\text{substitute for } y \quad -\frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x}{u^2}\ln x \quad \Rightarrow \quad \frac{du}{dx} - \frac{u}{x} = -lnx$$

$$\text{the integrating factor is} \quad e^{-\int \frac{1}{x}dx} = \frac{1}{x} \quad \Rightarrow \quad \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} = \frac{d}{dx}\left(\frac{u}{x}\right) = -\ln x$$

$$\Rightarrow \frac{u}{x} = -\int \ln x dx = x - x\ln x + c \quad \Rightarrow \quad u = x^2(1 - \ln x) + cx$$

$$\text{substituting for } u; \quad y(x) = \left(x^2(1 - \ln x) + cx\right)^{-1}$$

Equations that can be transformed to one of the types above through substitution.

Just as the substitution $u = \dfrac{y}{x}$ can be used to transform a homogeneous equation into one that can, in principle, be solved, other substitutions may also be used to transform what appear to be intractable equations, as the following miscellaneous examples show;

1. $$y' = \frac{y}{t} + \frac{t-1}{2y}$$

   If we multiply both sides by y we obtain $yy' = \dfrac{y^2}{t} + \dfrac{t-1}{2}$, we observe that if the y^2 term were only y the right hand side of the equation would be in the form of a linear o.d.e. We therefore try the substitution $u = y^2$, so that $u' = 2yy'$. If we now substitute for y and y' in the original equation, multiply through by 2 and rearrange we obtain
   $$u' + \frac{2}{t}u = t - 1,$$
   which is a linear o.d.e. and we can solve it straightforwardly and substitute for u to obtain $y = \dfrac{t^2}{4} - \dfrac{t}{3} + \dfrac{c}{t^2}$

2. $$y' = y - 4t + y^2 - 8yt + 16t^2 + 4$$

   We can rearrange the right hand side of the equation to give $y' = (y-4t)+(y-4t)^2+4$ which suggests trying the substition $u = y - 4t$, with $u' = y' - 4$. Substituting into the original equation we obtain $u' = u + u^2$ which is a variables separable equation and can be solved to give, after substituting for u,
   $$y(t) = \frac{4t + 4cte^{-t} + 1}{c(1 + e^{-t})}$$

As an exercise, check the two solutions above.

Exercises

Solve the following differential equations. Verify that your solutions do indeed solve the given o.d.e.s

1. (a) $y' = \dfrac{e^{-2x}}{y^2}$
   (b) $xy' = (x-1)y \quad y(1) = 1$
   (c) $\sqrt{1+t^2}y' = ty^3 \quad y(0) = 2$
   (d) $y' = y^2 - 1 \quad y(0) = 2$
   (e) $\sin xy' = 2y\cos x$
   (f) $yy' = 2(xy + x)$
   (g) $ye^{x+y}y' = 1$
   (h) $y' = \dfrac{2(y^2+y-2)}{x^2+4x+3}$
   (i) $y'' + (y')^2 + 1 = 0$ (Hint; substitute $y' = u$)
   (j) $xy'' = y'$
   (k) $2xy' + y = 0, \quad y(4) = 1$
   (l) $(1-x^2)y' + 4xy = 0, \quad y(0) = 2$

2. (a) $y' + y = 5e^x$
   (b) $xy' + y = x^4 - x \quad y(1) = 2$
   (c) $y' + \dfrac{2y}{x+1} = 1$
   (d) $ty' + 2y = e^t$
   (e) $xy' = 2y + x^2$
   (f) $y' + 2xy + x = e^{-x^2}$
   (g) $y' + y\tan x = \sec x$
   (h) $x^2y' + 2xy - x + 1 = 0$
   (i) $(1-x^2)y' + xy = 2x$
   (j) $y' + \dfrac{y}{1-x} = x^2 - x$
   (k) $xy' + (1+x)y = e^{-x}$
   (l) $y + y' = e^x, \quad y(0) = 2$
   (m) $(1+x^2)y' = 1 + xy, \quad y(1) = 0$
   (n) $xy' + x^2 - 3y = 0$ with (a) $y(1) = -1$ and (b) $y(-1) = 1$
   (o) $y^2 + (3xy - 4y^3)y' = 0$ (Hint; consider x as the dependant variable.)

3. (a) $x^2y' = x^3 - y^3 \quad y(1) = 1$
   (b) $xy' = y + \sqrt{x^2+y^2}, \quad y(4) = 3$
   (c) $y' = \dfrac{2x-y}{x-2y}$
   (d) $y' = \dfrac{x+y}{x-y}$

(e) $y' = \dfrac{x-y+5}{x+y-1}$ (Hint; substitute $x = p+A, y = q+B$ where A and B are suitable constants to transform this into a homogeneous equation.)

(f) $y' = \dfrac{2x+2y-1}{3x+y-2}$

4. Use the substitutions given to solve the following equations for y

(a) $y' = (y-t)^2 - (y-t) - 1 \quad \text{let} \quad u = y - t$

(b) $y' = \dfrac{yt}{2} + \dfrac{e^{\frac{t^2}{2}}}{2y} \quad \text{let} \quad u = y^2$

(c) $y' = \dfrac{y}{1+t} - \dfrac{y}{t} + t^2(1+t) \quad \text{let} \quad u = \dfrac{y}{1+t}$

Introduction to Fourier Series

Maclaurin series are very useful polynomial approximations of functions, however they do not approximate periodic functions effectively. For this we use Fourier series.
Consider $f(x)$ where f is 2π periodic, i.e. for all x we have $f(x) = f(x+2\pi)$. We can approximate $f(x)$ by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx f(x)dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx f(x)dx$$

(The derivation of the Fourier series will be covered in the lectures). If the function is $2L$ periodic then with a scale change we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} f(x)dx \quad b_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)dx$$

Products of odd and even functions

Suppose that $f(x)$ is even and $g(x)$ is odd. Then $f(x) = f(-x)$ and $g(x) = -g(-x)$ so that $f(x)g(x) = f(-x)\left(-g(-x)\right) = -f(-x)g(-x)$. Thus $f(x)g(x)$ is odd.
If both $f(x)$ and $g(x)$ are even, then $f(x) = f(-x)$ and $g(x) = g(-x)$ so that $f(x)g(x) = f(-x)g(-x)$ and so $f(x)g(x)$ is even.
Lastly, if both $f(x)$ and $g(x)$ are odd then $f(x) = -f(-x)$ and $g(x) = -g(-x)$ so that $f(x)g(x) = \left(-f(-x)\right)\left(-g(-x)\right) = f(-x)g(-x)$ and so $f(x)g(x)$ is even.
The rule is odd$\times$odd=even, even$\times$even=even and odd$\times$even=odd. This rule helps considerably in reducing the work involved in computing Fourier series.

Integrating odd and even functions

If $f(x)$ is even then $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$

If $g(x)$ is odd then $\int_{-L}^{L} g(x)dx = 0$ so $a_0 = 0$. in this case

Moreover, if $f(x)$ is even, then $\cos \frac{n\pi x}{L} f(x)$ is even while $\sin \frac{n\pi x}{L} f(x)$ is odd, hence

$$\frac{1}{L}\int_{-L}^{L} \cos \frac{n\pi x}{L} f(x)dx = \frac{2}{L}\int_{0}^{L} \cos \frac{n\pi x}{L} f(x)dx, \quad \text{and} \quad \frac{1}{L}\int_{-L}^{L} \sin \frac{n\pi x}{L} f(x)dx = 0.$$

The Fourier series of an even function contains only the cosine terms.

If $g(x)$ is odd then $\cos \frac{n\pi x}{L} g(x)$ is odd while $\sin \frac{n\pi x}{L} g(x)$ is even, hence

$$\frac{1}{L}\int_{-L}^{L} \sin \frac{n\pi x}{L} f(x)dx = \frac{2}{L}\int_{0}^{L} \sin \frac{n\pi x}{L} f(x)dx, \quad \text{and} \quad \frac{1}{L}\int_{-L}^{L} \cos \frac{n\pi x}{L} g(x)dx = 0.$$

The Fourier series of an odd function contains only the sine terms. In summary;

- if f is even then
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$
- if f is odd then
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Examples

We obtain the Fourier series for two simple examples

1.
$$f(x) = \begin{cases} 0 & : \quad x \in ((4j+1),(4j+3)] \\ 1 & : \quad x \in (4j-1, 4j+1] \end{cases}$$

for all $j \in \mathbb{Z}$. Clearly $f(x)$ is a rectangular wave of period $2L = 4$, moreover $f(x)$ is even, so we require only the cosine terms of the Fourier series. We consider the interval $(-2, 2)$

$$a_0 = \frac{1}{2}\int_{0}^{1} 1 \; dx = \frac{1}{2} \quad a_n = \int_{0}^{1} \cos \frac{n\pi x}{2} dx = \frac{2}{n\pi} \sin \frac{n\pi x}{2}\bigg|_{0}^{1}$$

$$\sin \frac{n\pi x}{2} = \begin{cases} 0 & : \quad \text{for} \quad \text{n} \quad \text{even} \\ 1 & : \quad \text{for} \quad \text{n} = \quad 1,5,9\ldots \\ -1 & : \quad \text{for} \quad \text{n} = \quad 3,7,11\ldots \end{cases}$$

We thus have

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos \frac{\pi x}{2} - \frac{1}{3}\cos \frac{3\pi x}{2} + \frac{1}{5}\cos \frac{5\pi x}{2} \ldots\right)$$

which we can express more succinctly as

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2} \qquad n \in \mathbb{N}$$

2. $g(x) = x - 2jL$ for each interval $((2j-1)L, (2j+1)L]$, for all $j \in \mathbb{Z}$.

   g is $2L$ periodic and is an odd function (draw the graph to check). We consider the interval $(-L, L]$. Since g is an odd function we only require the sine terms of the Fourier series.

$$b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx \qquad = -\frac{2}{L} \left[\frac{L}{n\pi} x \cos \frac{n\pi x}{L} \right]_0^L + \frac{2}{L} \int_0^L \frac{L}{n\pi} \cos \frac{n\pi x}{L} dx$$

$$= -\frac{2L}{n\pi} + \frac{2}{L} \frac{L^2}{n^2\pi^2} \left[sin \frac{n\pi x}{L} \right]_0^L \qquad = -\frac{2L}{n\pi} + \frac{2L}{n^2\pi^2} \sin n\pi$$

   $\cos n\pi$ is 1 for n odd and -1 for n even, while the $\sin n\pi$ is zero. Thus we have

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin \frac{n\pi x}{L} \qquad n \in \mathbb{N}$$

Complex Fourier series

We have the identities $e^{\pm inx} \equiv \cos nx \pm i \sin nx$ and

$$\cos nx \equiv \frac{1}{2} \left(e^{inx} + e^{-inx} \right) \qquad \sin nx \equiv \frac{1}{2i} \left(e^{inx} - e^{-inx} \right)$$

We can therefore express the general term of the Fourier series for a 2π periodic function $f(x)$ as

$$a_n \cos nx + b_n \sin nx = \frac{a_n}{2} \left(e^{inx} + e^{-inx} \right) + \frac{b_n}{2i} \left(e^{inx} - e^{-inx} \right)$$

and we can write the right hand side as

$$c_n e^{inx} + k_n e^{-inx} \quad \text{where} \quad c_n = \frac{1}{2}(a_n - ib_n) \quad \text{and} \quad k_n = \frac{1}{2}(a_n + ib_n)$$

Hence we can calculate

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) \cos nx - i f(x) \sin nx)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$k_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) \cos nx + i f(x) \sin nx)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx.$$

Since $k_{-n} = c_n$ we can therefore express the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

In some circumstances this form of the Fourier series is more convenient to work with than the sine and cosine series, as the following example shows.

Example

We require the Fourier series of $f(x) = e^x, \quad x \in (-\pi, \pi)$; in fact if we define $f(x) = e^{x-2j\pi x}, \quad j \in \mathbb{Z}$ for each interval $x \in [(2j-1)\pi, (2j+1)\pi)$ we see that f is 2π periodic and neither odd nor even.

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\frac{1}{1-in}\left[e^x e^{-inx}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}\left(e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right)$$

Now, $e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = (-1)^n$ for $n \in \mathbb{Z}$ so $c_n = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{2\pi(1-in)} = \frac{(-1)^n \sinh\pi}{\pi(1-in)}$ and therefore

$$e^x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+in)\sinh\pi}{\pi(1+n^2)}, \quad x \in (-\pi, \pi)$$

We now need to express this in terns of real expressions, since $e^x \in \mathbb{R}$ for $x \in \mathbb{R}$. We have $(1+in)e^{inx} = (1+in)(\cos nx + \sin nx) = (\cos nx - n\sin nx) + i(\sin nx + n\cos nx)$, while $(1-in)e^{-inx} = (1-in)(\cos nx - \sin nx) = (\cos nx - n\sin nx) - i(\sin nx + n\cos nx)$.

The imaginary term has matching terms in n and $-n$ and so vanishes while the real term has equal values for n and $-n$. When $n = 0$ we have that $c_0 = \frac{\sinh\pi}{\pi}$, thus

$$e^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1+n^2)}(\cos nx - n\sin nx), \quad x \in (-\pi, \pi)$$

Exercises

1. Find the Fourier series of the following 2π periodic functions;

   (a) $$f(x) = x, \quad x \in (-\pi, \pi]$$

   (b) $$f(x) = x^2, \quad x \in (-\pi, \pi]$$

   (c) $$f(x) = \begin{cases} -1 & : \quad x \in (-\pi, 0] \\ 1 & : \quad x \in (0, \pi] \end{cases}$$

   (d) $$f(x) = \begin{cases} x & : \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ & : \\ 1 & : \quad x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases}$$

   (e) $f(x) = e^x, \quad x \in (-\pi, \pi]$ (Use the real form of the Fourier series and compare the difficulty with that using the complex form)

2. Find the Fourier series of the following $2L$ periodic functions;

   (a) $$f(x) = |x|, \quad x \in (-2, 2], \quad L = 2$$

(b)
$$f(x) = \frac{\pi x^3}{2}, \quad x \in (-1, 1], \quad L = 1$$

(c)
$$f(x) = \begin{cases} \frac{1}{2} + x & : \quad x \in \left(-\frac{1}{2}, 0\right] \\ & : \\ \frac{1}{2} - x & : \quad x \in \left(0, \frac{1}{2}\right] \end{cases} \qquad L = \frac{1}{2}$$

(d)
$$f(x) = \pi \sin \pi x, \quad x \in (0, 1), \quad L = \frac{1}{2}$$

3. Find the complex Fourier series for the following 2π periodic functions;

(a)
$$f(x) = |x|, \quad x = (-\pi, \pi]$$

(b)
$$f(x) = \begin{cases} -1 & : \quad x \in (-\pi, 0] \\ 1 & : \quad x \in (0, \pi] \end{cases}$$

MAT1015: Techniques in Calculus

PART C

Specimen tests and Answers

specimen test 1

1. Solve $\frac{12}{x} = \frac{1}{2} + \frac{10}{x}$
2. The solution of the inequality $2(1-3x) - 3(2-x) < 5$ is $x > a$. What is the value of a?
3. Simplify $y = \frac{2x}{x^2-4} - \frac{1}{x+2}$, where $|x| \neq 2$ as much as possible.
4. Factorise $x^3 - 8y^3$
5. Given that $\frac{x-1}{x^2-7x+12} \equiv \frac{A}{x-3} + \frac{B}{x-4}$, what is the value of A?
6. If $x^2 + 8x + 21 = (x+p)^2 + q$ for all values of x, what is the value of q?
7. What is the numerical remainder when $(x^3 - x^2 + 2x - 1)$ is divided by $(x-3)$?
8. Make y the subject of the formula $x - \frac{y}{q} = y + 2z$, where $|q| \neq 0, |q| \neq 1$.
9. Given that $x^4 - 7x^2 - 18 \equiv (x-a)(x+a)(x^2+b)$, where $a > 0$, what is the value of a?
10. Given that the quadratic equation $3x^2 + 12x - 7 = 0$ has roots u and v, what is the value of $u + v$?
11. Find the range of values for p such that equation $3x^2 + px + 3 = 0$ has **no** real roots for x.
12. The graphs with equations $y = 3x - 2$ and $xy = 1$ meet at two points. At one of these points, $x = -\frac{1}{3}$. What is the value of y at the other point?

specimen test 1 - answers

1. $x = 4$
2. $a = 3$
3. $\dfrac{1}{x-2}$
4. $(x-2y)(x^2+2xy+4y^2)$
5. $A = -2$
6. $q = 5$
7. 23
8. $y = \dfrac{q(x-2z)}{1+q}$
9. $a = 3$
10. -4
11. $-6 < p < 6$
12. $x = -1$

specimen test 2

1. If $x > 0$ and $(x^{-2})^{-3} \equiv x^a$, what is the value of a?
2. Given that $\log_2(a+2) = 4$, what is the value of a?
3. Find the value of $\sum_{r=0}^{21}(7-3r)$.
4. Find the coefficient of x^4 in the binomial expansion of $(2x-3)^6$.
5. make 4x4 the subject of the formula $y = \mathrm{e}^{x+2}$, where $x > -2$,
6. What is $\cos 2x \sin 5x - \sin 2x \cos 5x$ in its simplest form?
7. Given that $\tan \frac{\pi}{n} = 1$, where $n > 1$, what is the value of n?
8. Given that $\frac{1}{4-2\sqrt{3}}$ equals $a + b\sqrt{3}$ where a and b are rational numbers, what is the value of b?
9. For any positive value of x, $5\ln x + 6\ln x - \ln(x^3) = \ln(x^n)$.
   What is the value of n?
10. In the binomial series for $(1-2x)^{-3}$, what is the coefficient of x^3?
11. The smallest positive solution of the equation $\cos 2x = 6\sin^2 x - 5$ is $x = n$ **degrees**.
    What is the value of n?
12. Find the non-zero value of x for which $3^{2x+1} - 4(3^x) + 1 = 0$.

specimen test 2 - answers

1. $a = 6$
2. $a = 14$
3. 2160
4. $x = \ln y - 2$
5. $\sin 3x$
6. $n = 4$
7. $b = \dfrac{1}{2}$
8. $n = 8$
9. 80
10. $n = 60$
11. $x = \frac{1}{3}$

specimen test 3

1. At the point where $x = 1$, what is the gradient of the curve $y = \mathrm{e}^{2x+3}$
2. If $\mathrm{f}(x) = x^2\mathrm{e}^{3x}$, find the derived function $\mathrm{f}\,'(x)$
3. The minimum point on the graph of $y = 3x^4 - 4x^3 + 7$ is at (a, b).
   What is the value of b?
4. Find the gradient at the point where $x = -2$ on the curve $y = \dfrac{x}{x+1}$.
5. If $y = \tan x$ then find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$
6. Find the exact value of $\displaystyle\int_0^1 \mathrm{e}^{3x+1}\ \mathrm{d}x =$
7. Using the substitution $u = \sin x$, or otherwise, find
   $\displaystyle\int_0^{\pi/2} 12\sin^3 x\cos x\ \mathrm{d}x.$
8. Find the exact value of $\displaystyle\int_{-1}^{\mathrm{e}-2} \frac{1}{x+2}\ \mathrm{d}x.$
9. Find the area, in square units, of the finite region bounded by the curve $y = 3(x-2)^2$, the x-axis and the y-axis.
10. The part of the graph $y = \dfrac{1}{x}$ between $x = 1$ and $x = 3$ is rotated about the x-axis. Find the volume, in cubic units, of the solid formed.
11. Find the value of k for which $\displaystyle\int_3^5 \frac{2}{(x-2)(x+1)}\ \mathrm{d}x = \frac{2}{3}\ln k.$
12. If $y = \sin^2 2x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin nx$. What is the value of n?

specimen test 3 - Answers

1. $2e^5$
2. $xe^{3x}(2+3x)$
3. $b=6$
4. $2\sec^2 x \tan x$
5. $x\frac{1}{3}(e^4-e)$
6. 3
7. 1
8. 8
9. $\frac{2}{3\pi}$
10. 2
11. 4

CLASS TEST 1 - SPECIMEN PAPER

Working and answers should be written in the spaces provided. You may use rough paper for extra working, but only what you write on this sheet will be marked.

1. The complex numbers w and z are defined by $w = 3 + 4\mathrm{i}$, $z = -5 - 12\mathrm{i}$.

   (a) Find the modulus of w. .... [2]

   (b) Find the argument of z, in radians between $-\pi$ and π, to 2 decimal places.

   .... [2]

   (c) Find the value of $z\overline{z}$.

   .... [2]

   (d) Express $\dfrac{w}{z}$ in the form $a + b\mathrm{i}$ where a and b are real.

   .... [3]

   (e) Find $|z - w|$ in the form $p\sqrt{q}$ where q is prime.

   .... [2]

2. Expand $(\cos\theta + \mathrm{i}\sin\theta)^3$. Hence express $\sin 3\theta$ in terms of powers of $\sin\theta$.

   .... [5]

3. Find the exact values of the following:

   (a) $\left|\tan\dfrac{3\pi}{4}\right|$ .... [2]

   (b) The amplitude of $3\sin 3x\cos 3x$ .... [2]

   (c) $\mathrm{sgn}(\arcsin(-1))$ .... [2]

   (d) $\lfloor\sinh(\ln 7)\rfloor$ .... [2]

TURN OVER

4. The functions f and g are defined, for $x \in [0, \infty)$, by $f(x) \equiv 1-2x$, $g(x) \equiv x^2+2x-1$. For each of f and g, the codomain is equal to the range.

Find the inverse functions f^{-1} and g^{-1}, stating their domains.

$f^{-1}(x) =$ .... Domain of f^{-1} = ... [3]

$g^{-1}(x) =$ .... Domain of g^{-1} = ... [4]

5. (a) Express $\dfrac{2x+1}{(x^2+1)(x-2)}$ in partial fractions.

.... [4]

Hence integrate $\dfrac{2x+1}{(x^2+1)(x-2)}$ with respect to x.

.... [3]

6. Complete the following definitions.

(a) $f: X \to Y$ is an odd function if .... [2]

(b) $f: X \to Y$ is injective (1-to-1) if .... [2]

(c) $f: X \to Y$ is surjective (onto) if .... [2]

(d) $f \circ f^{-1}(x)$ is .... [2]

(e) The first 4 terms in the power series for $\exp(x)$ are .... [2]

(f) If $a > 0$ is real, a^x can be expressed as .... [2]

CLASS TEST 1 - SPECIMEN PAPER : Answers

1. $w = 3 + 4\mathrm{i}$, $z = -5 - 12\mathrm{i}$.

   (a) $|w| = \sqrt{3^2 + 4^2} = 5.$ [2]

   (b) $\tan(\arg z) = \dfrac{12}{5}$. z is in third quadrant so $\arg z = -1.97$. [2]

   (c) $z\overline{z} = (-5 - 12\mathrm{i})(-5 + 12\mathrm{i}) = 25 + 144 = 169.$ [2]

   (d) $\dfrac{w}{z} = \dfrac{(3 + 4\mathrm{i})(-5 + 12\mathrm{i})}{169} = -\dfrac{63}{169} + \dfrac{16}{169}\mathrm{i}.$ [3]

   (e) $|z - w| = |8 + 16\mathrm{i}| = 8\sqrt{5}$ [2]

2. $(\cos\theta + \mathrm{i}\sin\theta)^3 \equiv \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta.$ [2]

   By de Moivre's Theorem, also $(\cos\theta + \mathrm{i}\sin\theta)^3 \equiv \cos 3\theta + \mathrm{i}\sin 3\theta$, so equating imaginary parts, $\sin 3\theta \equiv 3\cos^2\theta\sin\theta - \sin^3\theta$. [2]

   Using $\sin^2\theta + \cos^2\theta \equiv 1$, we have $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$. [1]

3. (a) $\left|\tan\dfrac{3\pi}{4}\right| = 1$ [2]

   (b) Amplitude of $3\sin 3x\cos 3x$ is amplitude of $(3/2)\sin 6x = 3/2$. [2]

   (c) $\mathrm{sgn}(\arcsin(-1)) = \mathrm{sgn}(-\pi/2) = -1$ [2]

   (d) $\lfloor\sinh(\ln 7)\rfloor = \left\lfloor\dfrac{7 + 1/7}{2}\right\rfloor = \left\lfloor\dfrac{24}{7}\right\rfloor = 3$ [2]

4. $\mathrm{f}^{-1}(x) = \dfrac{1 - x}{2}$ [2]

   Domain of f^{-1} is $(-\infty, 1]$ or $\mathrm{f}^{-1}(x) \leq 1$ [1]

   $\mathrm{g}(x) = (x + 1)^2 - 2$ so $\mathrm{g}^{-1}(x) = \sqrt{x + 2} - 1$ [3]

   Domain of g^{-1} is $[-1, \infty)$ or $\mathrm{g}^{-1}(x) \geq -1$ [1]

5. (a) Let $\dfrac{2x + 1}{(x^2 + 1)(x - 2)} \equiv \dfrac{Ax + B}{x^2 + 1} + \dfrac{C}{x - 2}.$ [1]

   Then $(Ax + B)(x - 2) + C(x^2 + 1) \equiv 2x + 1$. [1]

   $x = 2 \Rightarrow C = 1$. x^2 terms $\Rightarrow A = -1$. Constant terms $\Rightarrow B = 0$

   Thus $\dfrac{2x + 1}{(x^2 + 1)(x - 2)} \equiv \dfrac{1}{x - 2} - \dfrac{x}{x^2 + 1}.$ [2]

   $\displaystyle\int \frac{2x + 1}{(x^2 + 1)(x - 2)}\,\mathrm{d}x = \ln|x - 2| - \frac{1}{2}\ln(x^2 + 1) + c.$ [3]

6. (a) $\mathrm{f} : X \to Y$ is an odd function if for all $x \in X$, $\mathrm{f}(-x) = -\mathrm{f}(x)$. [2]

   (b) $\mathrm{f} : X \to Y$ is injective if $\mathrm{f}(a) = \mathrm{f}(b) \Rightarrow a = b$. [2]

   (c) $\mathrm{f} : X \to Y$ is surjective if the range of f is the codomain Y. [2]

   (d) The identity function on $\mathbb{R}$ OR $\mathrm{id} : x \mapsto x$ for all $x \in \mathbb{R}$ OR x. [2]

   (e) The first four terms in the series for $\exp(x)$ or e^x are $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ [2]

   (f) If a is any positive real number, a^x is defined to be $\exp(x\ln a)$ [2]

CLASS TEST 2-SPECIMEN PAPER

Working and answers should be written in the spaces provided. You may use rough paper for extra working, but only what you write on this sheet will be marked.

1. Find the following limits:

   (a) $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x}$ ....[2]

   (b) $\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 8x + 15}$ ....[2]

   (c) $\lim_{x \to \infty} \frac{e^x}{\cosh x}$ ....[2]

2. The parametric equations for a curve C are $x = t^2 - 1, y = t + 1$.

   (a) Find the coordinates of all the points where C intersects the y-axis.

   ....[2]

   (b) Find the gradient of C at the point (3, 3) ....[3]

   (c) Hence find the equation of the tangent to C at (3, 3) ....[2]

3. If the curve K is defined by the equation

$$y = \frac{x^2 + 5}{x + 2},$$

   then K has

   (a) — a vertical asymptote whose equation is ....[2]

   (b) — turning points at $x =$ .... and ....[4]

   (c) — a local maximum at $x =$ ....[3]

   (d) — a local minimum at $x =$ ....[3]

   (e) — an oblique asyptote whose equation is ....[4]

TURN OVER

4. Let B be a constant and let the Maclaurin series for $f(x) = e^x/(1 + Bx)$, truncated after the third term, be $S = a_0 + a_1x + a_2x^2$, where a_0, a_1 and a_2 depend on B. Then

(a) $a_0 =$ ....[1]

(b) $a_1 =$ ....[2]

(c) $a_2 =$ ....[3]

(d) For what value of B is S identical to the first three terms in the series for $(1 - x/2)^{-1} = \sum_{k=0}^{\infty}(x/2)^k$?

..............................[2]

5. In each case simplifying your answer as far as possible, find

(a) $\displaystyle\int \frac{dx}{x^2 + 6x + 25}$

..............................[4]

(b) $\displaystyle\int_1^2 x \ln x \, dx$

..............................[4]

(c) $\displaystyle\int \cos^5 x \, dx$ [Hint: substitute $u = \sin x$]

..............................[5]

CLASS TEST 2 SPECIMEN PAPER: Answers

1. Limits are:
   (a) l'Hopital: $\boxed{-\pi}$ [2]
   (b) $\lim_{x\to 0} \frac{x^2-7x+10}{x^2-8x+15} = \lim_{x\to 5} \frac{x-2}{x-3} = \boxed{\frac{3}{2}}$ (or l'Hopital) [2]
   (c) Def. of cosh: $\boxed{2}$ [2]
2. Parametric curves.
   (a) $x = 0$ so $t = \pm 1$ so $y = 0, 2$; hence, points are $\boxed{(0,0) \text{ and } (0,2)}$ [2]
   (b) $dy/dx = dy/dt \times dt/dx = 1/(2t) = 1/(2y-2)$ $\boxed{\frac{1}{4}}$ [3]
   (c) Eq. is $y = mx + c$ where $m = 1/4$ and $3 = 3/4 + c$, so $\boxed{y = (x+9)/4}$ [2]
3. Curve sketching.
   (a) Denominator $= 0$ when $\boxed{x = -2}$ [2]
   (b) $y' = (x-1)(x+5)/(x+2)^2$ so turning points are $\boxed{x = -5, 1}$ [4]
   $y'' = [2(x+2)^3 - 2(x-1)(x+2)(x+5)]/(x+2)^4$ so
   $y'' = 2/3 \ \ (x = 1), \ \ y'' - 2/3 \ \ (x = -5)$. Hence
   (c) Maximum at $\boxed{x = -5}$ [3]
   (d) Minimum at $\boxed{x = 1}$ [3]
   (e) Algebraic long division gives $y = x - 2 + 9/(x+2)$.
   Hence oblique asymptote is $\boxed{y = x - 2}$ [4]
4. Maclaurin series.
   (a) $f(0) = 1$ so $\boxed{a_0 = 1}$ [1]
   (b) $f'(x) = e^x(1 - B + Bx)/(1 + Bx)^2$ so $f'(0) = 1 - B$ and so $\boxed{a_1 = 1 - B}$ [2]
   (c) $f''(0) = 1 - 2B + 2B^2$ and so $\boxed{a_2 = (1 - 2B + 2B^2)/2}$ [3]
   (d) $(1 - x/2)^{-1} = 1 + x/2 + x^2/4 + \ldots$.
   If $B = 1/2$ then $a_1 = 1/2$ and $a_2 = 1/4$; hence $\boxed{B = 1/2}$ [2]
5. Integrals
   (a) Complete square and subs. $u = x + 3$ to get $\boxed{(1/4)\arctan[(x+3)/4] + c}$ [4]
   (b) By parts to get $x^2(2\ln x - 1)/4$ giving $\boxed{2\ln 2 - 3/4}$ [4]
   (c) Substituting $u = \sin x$ gives $I = \int (1-u^2)^2 \, du$,
   giving $\boxed{\sin x - (2/3)\sin^3 x + (1/5)\sin^5 x + c}$ [5]