# MAT 1015 Techniques in Calculus I Autumn 2009

## Coursework 3 SOLUTIONS

1. Find the first four non-zero terms and the radius of convergence of the Maclaurin Series for the following functions;

$$(a) \quad \tan^{-1} x \qquad\qquad (b) \quad \ln\left(\frac{1-x}{1+x}\right) \; (c) \quad \frac{e^x}{1+x}$$

(a) By integrating both sides of the series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + O\left(x^8\right) \qquad |x| < 1$$

$$\tan^{-1} x = x - \frac{1}{3}\,x^3 + \frac{1}{5}\,x^5 - \frac{1}{7}\,x^7 + O\left(x^8\right) \qquad |x| < 1$$

(b) Write the expression as $\ln(1-x) - \ln(1+x)$ to get

$$\ln\left(\frac{1-x}{1+x}\right) = -2\,x - \frac{2}{3}\,x^3 - \frac{2}{5}\,x^5 - \frac{2}{7}\,x^7 + O\left(x^8\right) \qquad |x| < 1$$

(c)

$$\frac{e^x}{1+x} = 1 + 1/2\,x^2 - 1/3\,x^3 + 3/8\,x^4 + O\left(x^5\right) \qquad |x| < 1$$

2. Prove that

$$\lim_{x\to 0} x^2 \sin\left(\frac{3+x)}{x^4}\right) = 0.$$

As $x \to 0$, $x^2 \to 0$ while $\sin\left(\frac{3+x}{x^4}\right) \to$ some undetermined value in the interval $[-1..1]$. Since the limit of a product is the product of a limit it must follow that the function goes to zero as $x \to 0$.

Alternatively use the sandwich theorem

$$-1 \le \sin\left(\frac{3+x)}{x^4}\right) \le 1 \qquad \Rightarrow \qquad -x^2 \le x^2 \sin\left(\frac{3+x)}{x^4}\right) \le x^2$$

and then the result follows as we let $x \to 0$

3. (a) **Without** using l'Hôpital's rule find:

$$(i) \quad \lim_{x\to 5} \frac{3x^2 - 16x + 5}{x^2 - 3x - 10} \qquad (ii) \quad \lim_{x\to 0} \frac{17x^{-0.25} + 8}{3x^{-0.33} + 2} \qquad (iii) \quad \lim_{x\to\infty} \frac{3x^4 - 7x^2 - 1}{8x^4 + 4x^2 + 5}.$$

(i) Factor the numerator and denominator to get $\frac{(3x-1)(x-5)}{(x-5)(x+2)}$ and cancel the $(x-5)$ factor. Then

$$\lim_{x\to 5} \frac{3x^2 - 16x + 5}{x^2 - 3x - 10} = \frac{14}{7} = 2$$

(ii) Multiply numerator and denominator by $x^{0.33}$ to get $\dfrac{17x^{0.08} + 8x^{0.33}}{3 + 2^{0.33}}$. It is clear that this expression tends to zero as $x \to 0$.

(iii) Dividing through by $x^{-4}$

$$\lim_{x\to\infty} \frac{3x^4 - 7x^2 - 1}{8x^4 + 4x^2 + 5} = \lim_{x\to\infty} \frac{3 - 7x^{-2} - x^{-4}}{8 + 4x^{-2} + 5x^{-4}}$$

The limit of the function as $x \to \infty = \dfrac{3}{8}$.

(b) **Using** l'Hôpital's rule find:

$$(i) \quad \lim_{x\to 1} \frac{\tan(x-1)}{x^2 - 1} \qquad (ii) \quad \lim_{x\to 0} \frac{10^x - e^x}{x} \qquad (iii) \quad \lim_{x\to 0} \frac{2\sin x - \sin 2x}{2e^x - 2 - 2x - x^2}.$$

(1) Applying l'Hôpital's rule once we have:

$$\lim_{x\to 1} \frac{\tan(x-1)}{x^2 - 1} = \lim_{x\to 1} \frac{\sec^2(x-1)}{2x} = \frac{1}{2}.$$

(ii)

$$\lim_{x\to 0} \frac{10^x - e^x}{x} = \lim_{x\to 0} \frac{\ln 10 \times 10^x - e^x}{1} = \ln 10 - 1.$$

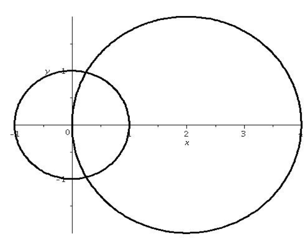
(iii) Applying l'Hôpital's rule three times:

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{2e^x - 2 - 2x - x^2} = \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{2e^2x - 2 - 2x}$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{2e^x - 2} = \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{2e^x} = 3.$$

4. Find the area lying inside both of the circles $x^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

First sketch the graph to see what the area looks like



The two circles intersect at $x = 0.25$ and the small circle cuts the $x-$ axis at $x = 1$. The area is thus given by

$$2\int_0^{0.25} \sqrt{4 - (x-2)^2}dx + 2\int_{0.25}^1 \sqrt{1 - x^2}dx.$$

Taking the first integral $\int \sqrt{4 - (x-2)^2}dx$, put $x - 2 = 2\sin\theta$, then $dx = 2\cos\theta d\theta$ and the integral is transformed into

$$\int 4\cos^2\theta d\theta \quad = \quad 2\int (1+2\cos 2\theta)d\theta \quad = \quad 2\theta + 2\sin\theta\cos\theta.$$

Now substitute back to obtain

$$2\sin^{-1}\left(\frac{x}{2}-1\right) + (x-2)\sqrt{1-\left(\frac{x}{2}-1\right)^2}.$$

Thus

$$2\int_0^{0.25}\sqrt{4-(x-2)^2}dx = 2\left(2\sin^{-1}\left(\frac{x}{2}-1\right) + (x-2)\sqrt{1-\left(\frac{x}{2}-1\right)^2}\right)\Bigg|_0^{0.25}$$

We can evaluate this to be $2\,\pi - \frac{7}{16}\,\sqrt{15} - 4\,\sin^{-1}\left(\frac{7}{8}\right) = 0.327$.

The second integral is evaluated in a similar way using the substitution $x = \sin\theta$ to give

$$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} + \sin^{-1}(x)$$

Thus

$$2\int_{0.25}^{1}\sqrt{1-x^2}dx = -\frac{1}{32}\,\sqrt{15} - \frac{1}{2}\,\arcsin\left(\frac{1}{4}\right) + \frac{1}{4}\,\pi = 1.077.$$

The area in both circles is thus 1.404 sq.units to 3 dps.

5. Find the mean value of $y = \sqrt{4-x^2}$ over $[0,2]$.

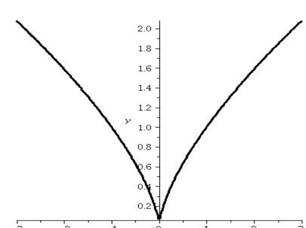
$$\int_0^2 \sqrt{4-x^2}dx = \frac{1}{2}\,x\sqrt{4-x^2} + 2\,\arcsin\left(\frac{x}{2}\right)\Bigg|_0^2 = \pi$$

Either by the substitution $x = 2\sin\theta$ or using integration by parts of $\int_0^2 1\times\sqrt{4-x^2}dx$

The mean value is thus $\frac{\pi}{2}$.

6. Find the length of the curve $y^3 = x^2$ from $x = -1$ to $x = 1$.

The graph is:



If we calculate $y' = \dfrac{2}{3x^{\frac{1}{3}}}$ the formula for the length is then

$$\int_{-1}^{1} \sqrt{1 + \frac{4}{9x^{\frac{2}{3}}}}dx,$$

which looks like a very difficult integral. But we also have $x = y^{\frac{3}{2}}$ so that $x' = \dfrac{3\sqrt{y}}{2}$ The curve is symmetrical about the y axis so our required length is

$$2\int_{0}^{1} \sqrt{1 + \frac{9y}{4}}dy = 2\frac{1}{27}\,(4 + 9\,y)^{\frac{3}{2}}\Bigg|_{0}^{1} = \frac{13}{27}\sqrt{13} - \frac{8}{27} = 2.879$$

7. Find the area of the surface produced by rotating the ellipse $x^2 + 4y^2 = 4$, around the $x$ axis. The formula we need is

$$A = \int 2\pi y\sqrt{1 + (y')^2}dx$$

Now $y = \dfrac{1}{2}\sqrt{4 - x^2}$ so $y' = -\dfrac{1}{2}\dfrac{x}{\sqrt{4 - x^2}}$ and the integral we need to evaluate is

$$2\int \frac{\pi}{4}\sqrt{16 - 3x^2}dx.$$

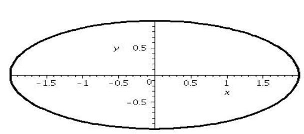
We use the substitution $\sqrt{3}x = 4\sin\theta$ to transform the integral into

$$\int \frac{4\pi}{\sqrt{3}}\cos^2\theta d\theta = \frac{2\pi}{\sqrt{3}}\left(\theta + \frac{1}{2}\sin 2\theta\right).$$

Substituting back in terms of $x$ we obtain

$$\int \frac{\pi}{4}\sqrt{16 - 3x^2}dx = \frac{1}{8}\,\pi\,x\sqrt{16 - 3\,x^2} + \frac{2}{3}\,\pi\,\sqrt{3}\,\mathrm{arcsin}\left(\frac{\sqrt{3}x}{4}\right).$$

The graph is




So we need to evaluate the integral from $x = -2$ to $x = 2$. Thus the area is

$$2\int_{-2}^{2} \frac{\pi}{4}\sqrt{16 - 3x^2}dx = \pi + \frac{4}{9}\sqrt{3}\pi^2 = 21.478$$

8. Find the following integrals:

$$(a)\quad \int \frac{d\theta}{1+\sin\theta+\cos\theta} \qquad (b)\quad \int \left(x^2+2\,x+5\right)^{-1} dx$$

$$(c)\quad \int \frac{1}{\sqrt{x^2+2\,x+5}} dx \qquad (d)\quad \int \left(x^2+2\,x+5\right)^{-2} dx$$

$$(e)\quad \int \frac{2x^3-5x^2+4x-4}{x^2-x} dx$$

(a) We use the substitution $\tan\left(\frac{\theta}{2}\right) = x$ and transform the integral

$$\tan\left(\frac{\theta}{2}\right) = x, \quad \sin\left(\frac{\theta}{2}\right) = \frac{x}{\sqrt{1+x^2}} \quad \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\text{thus} \quad \sin\theta = \frac{2x}{1+x^2} \quad \cos\theta = \frac{1-x^2}{1+x^2} \quad \text{and} \quad d\theta = \frac{2dx}{1+x^2}$$

Our integral becomes

$$\int \frac{1}{1+x} dx = \ln|1+x| = \ln\left|1+\tan\left(\frac{\theta}{2}\right)\right| + C.$$

(b) The denominator is written as $(x+1)^2+4$, completing the square. Then either directly or using the substitution $x+1 = 2\tan\theta$ we have

$$\int \left(x^2+2\,x+5\right)^{-1} dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

(c) Completing the square under the square root sign and substituting $x+1 = 2\sinh u$

$$\int \left(x^2+2\,x+5\right)^{-\frac{1}{2}} dx = \sinh^{-1}\left(\frac{x+1}{2}\right) + C$$

(d) Once more we complete the square and substitute $x+1 = 2\tan\theta$ to transform the integral into

$$\frac{1}{8}\int \cos^2\theta d\theta = \frac{1}{32}\sin 2\theta + \frac{\theta}{16} + C$$

Then we calculate $\sin\theta = \frac{x+1}{(x+1)^2+4}$, $\qquad \cos\theta = \frac{2}{(x+1)^2+4}$ so the result becomes

$$\int \left(x^2+2\,x+5\right)^{-2} dx = -\frac{1}{16}\,\frac{2\,x+2}{x^2+2\,x+5} + \frac{1}{16}\tan^{-1}\left(\frac{x+1}{2}\right)$$

(e) First we use long division to simplify the integrand to $2x - 3 + \frac{x-4}{x^2-x}$. Then we use partial fractions to obtain

$$\frac{x-4}{x^2-x} = \frac{4}{x} - \frac{3}{x-1}.$$

We can integrate our expression straightforwardly to give

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} dx = x^2 - 3\,x + 4\,\ln|x| - 3\,\ln|x-1| + C$$

9. Find the series representation of the following integrals, stating the range of values of $x$ for which the series converge:

$$(a) \quad \int \frac{\sin x}{x} dx, \quad (b) \quad \int \frac{e^x - 1}{1 - x^2} dx.$$

(a)

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots$$

The series converges for all $x$ so we can integrate term by term to get

$$\int \frac{\sin x}{x} dx = C + x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} \dots$$

We know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, so we have $C = 1$ and

$$\int \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} (-1)^{n+2} \frac{x^{2n+1}}{(2n+1) \cdot (2n+1)!}$$

(b)

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \qquad \text{for all} \quad x$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \dots \qquad \text{for} \quad |x| < 1$$

Now we multiply the two series together to arrive at a series expansion for the integrand, valid for $|x| < 1$

$$\left( x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \right) \times \left( 1 + x^2 + x^4 + x^6 \dots \right) = x + \frac{1}{2}\,x^2 + \frac{7}{6}\,x^3 + \frac{13}{24}\,x^4 \dots$$

Finally we integrate to obtain

$$\int \frac{e^x - 1}{1 - x^2} dx = C + \frac{x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{24} + \frac{13x^5}{120} \dots$$

When $x = 0$ the left hand side is zero so $C = 0$. There is no obvious general form for this series.

10. By direct multiplication of the Maclaurin Series for $e^x$ and $e^y$, show that $e^x e^y = e^{x+y}$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} \dots$$

multiplying the two convergent series

$$e^x e^y = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots\right) \times \left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} \dots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots$$

$$+ y + xy + \frac{x^2 y}{2} \dots$$

$$+ \frac{y^2}{2} + \frac{xy^2}{2} \dots$$

$$+ \frac{y^3}{3}$$

$$= 1 + (x + y) + \frac{1}{2}(x + y)^2 + \frac{1}{6}(x + y)^3$$

$$= e^{x+y}$$

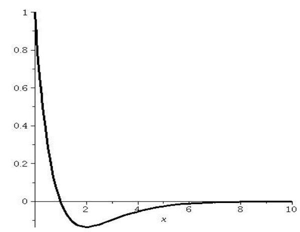
11. * Evaluate the following integrals:

$$(a) \quad \int_0^{\infty} (1-x)e^{-x}dx \quad (b) \quad \int x\cos^{-1} x dx.$$

(a)

$$\int_0^{\infty} (1-x)e^{-x}dx = \lim_{a\to\infty} \int_0^{a} (1-x)e^{-x}dx = \lim_{a\to\infty} xe^{-x}\Big|_0^a = \lim_{a\to\infty} ae^{-a} = 0.$$

The graph of the function is as below:




(b)

$$\int x\cos^{-1} x dx = \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}}dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \sqrt{1-x^2}dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \left(\frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x\right) + \frac{1}{2}\sin^{-1} x$$

$$= \frac{1}{2}x^2\cos^{-1}(x) - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}(x) + C$$

Recall that $\int \sqrt{1-x^2}dx$ can be evaluated with the substitution $x = \sin\theta$

12. * Find the value of $c$ for which the following integral converges and evaluate the integral at that value.

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx.$$

We express the improper integral as

$$\lim_{a\to\infty} \int_0^{a} \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx = \lim_{a\to\infty} \left(\sinh^{-1} x - c\ln|x+1|\Big|_0^a\right)$$

$$= \lim_{a\to\infty} \left(\ln(a+\sqrt{1+a^2})\right) - c\ln a + 1 = \lim_{a\to\infty} \ln\left(\frac{a+\sqrt{1+a^2}}{(a+1)^c}\right)$$

We can now substitute $a = \frac{1}{b}$

$$\lim_{b \to 0} \ln \left( \frac{\frac{1}{b} + \sqrt{1 + \frac{1}{b^2}}}{(\frac{1}{b} + 1)^c} \right)$$

if $c = 1$ the expression simplifies to

$$\lim_{b \to 0} \ln \left( \frac{\frac{1}{b} + \sqrt{1 + \frac{1}{b^2}}}{(\frac{1}{b} + 1)} \right) = \ln 2$$

For any other value of $c$ the integral does not converge.

13. Let $y = f^{-1}(x)$ be the inverse of function $f$. Prove that

$$\int f^{-1}(x)dx = xf^{-1}(x) - \int f(y)dy.$$

We have $x = f(y)$ and

$$\int f^{-1}(x)dx = xf^{-1}(x) - \int x \frac{d}{dx} f^{-1}(x)dx.$$

But

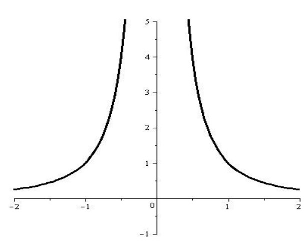
$$\int x \frac{d}{dx} f^{-1}(x)dx = \int f(y)\frac{dy}{dx}dx = \int f(y)dy.$$

14. * What is wrong with the following calculation;

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1 + \frac{1}{-1} = -2,$$

where did the error occur and why is $-2$ an unreasonable value for the integral?

The graph of the function is as follows




The function $\frac{1}{x^2}$ is not defined for $x = 0$ which is the middle of the interval over which the integral is evaluated. Moreover $\frac{1}{x^2}$ is positive for all $x$ and the integral of a positive function must be positive.