**CM 1021 Mathematical Methods for Computing I**

**Exercise Sheet 6 - Solutions**

When answering these questions, work through the calculations involved rather than simply substituting numbers into the formulas developed in the lectures.

1. The supply ($Q_S$) and demand ($Q_D$) for a product are related to its price ($P$) by

$$\begin{aligned} Q_D &= 8 - 2P \\ Q_S &= 1 + P. \end{aligned}$$

   (a) Find the equilibrium price $P_E$ and quantity $Q_E$.

   $8 - 2P = 1 + P$, so $P_E = 4$ and $Q_E = 5$

   (b) An excise tax $t$ is applied to this product.
   State the new supply equation which includes the tax $t$.

   $Q'_S = 1 + P - t$
   Find the equilibrium price $P_E(t)$ and quantity $E(t)$ as functions of $t$.

$$8 - 2P = 1 + P - t \Longrightarrow 3P = 7 + t \Longrightarrow P_E(t) = \frac{7}{3} + \frac{t}{3}, \quad Q_E(t) = \frac{10}{3} - \frac{2t}{3}$$

   (c) Find the value of the tax $t$ which will result in the maximum tax yield being generated.

$$Y(t) = t\left(\frac{10}{3} - \frac{2t}{3}\right) \Longrightarrow Y' = \frac{10}{3} - \frac{4t}{3}$$

   If $Y' = 0, \quad t_{max} = \dfrac{5}{2}$

2. The demand and supply functions of a product are given by

$$\begin{aligned} P &= -4Q_D + 120 \\ P &= \frac{1}{3}Q_S + 29. \end{aligned}$$

   (a) Calculate the equilibrium price and quantity

$$-4Q + 120 = \frac{Q}{3} + 29 \Longrightarrow Q_E = 21, \quad P_E = 36$$

   (b) Calculate the new equilibrium price and quantity after the imposition of a fixed excise tax of £13 per item. Who pays this tax?

With an excise tax of £13 the new supply equation is $P = \frac{Q_S}{3} + 29 + 13$.
The new equilibrium is thus found as
$-4Q + 120 = \frac{Q}{3} + 29 + 13 \Longrightarrow Q'_E = 18, \quad P'_E = 48$
The price has increased from £36 to £48. Thus $\frac{12}{13}$ of the tax is paid by the consumer and $\frac{1}{13}$ by the supplier.

3. The supply and demand functions for a product are

$$P = 2Q_S + 10$$
$$P = -5Q_D + 80$$

(a) Find the equilibrium price and quantity

$$2Q + 10 = 80 - 5Q \Longrightarrow Q_E = 10, \quad P_E = 30$$

(b) If the government deducts as tax, 15% of the market price of each product, determine the new equilibrium price and quantity.

The new supply equation is $P(1 - 0.15) = 2Q + 10$. We find the new equilibrium as follows

$$2Q + 10 = \frac{80 - 5Q}{0.85} \Longrightarrow \hat{Q}_E = 9.28, \quad \hat{P}_E = 33.6$$

4. Given the total revenue and total cost functions

$$TR = 4350Q - 13Q^2$$
$$TC = Q^3 - 5.5Q^2 + 150Q + 675$$

Using Maple, find the breakeven points, the value of $Q$ which maximises total revenue and the value of $Q$ which maximises the profit function. On the same graph plot the total revenue, total cost and profit functions

The breakeven points occur at the positive solutions of

$$Q^3 - 5.5Q^2 + 150Q + 25000 = 4350Q - 13Q^2$$

namely, $Q = 6.071$ and $Q = 57.741$ to 3 d.p.s. The maximum revenue occurs when $\frac{dTC}{dQ} = 0$, which is when $Q = 167.08$

The profit function is

$$\pi = 4200Q - 7.5Q^2 - Q^3 - 25000$$

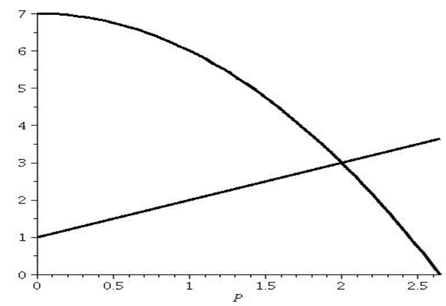
The profit is maximised when $\frac{d\pi}{dQ} = 0$ which is when $Q = 35$, since the negative root has no meaning for us. We can check that this is a maximum since $\left.\frac{d^2\pi}{dQ^2}\right|_{Q=35} = -225$

**More Challenging Questions**

5. The supply ($Q_S$) and demand ($Q_D$) for a product are related to its price ($P$) by

$$\begin{aligned} Q_D &= 7 - P^2 \\ Q_S &= 1 + P. \end{aligned}$$

(a) Sketch the supply and demand curves.




(b) Find the equilibrium price $P_E$ and quantity $E$.

$$7 - P^2 = 1 + P \Longrightarrow (P+3)(P-2) = 0 \Longrightarrow P_E = 2, \quad Q_E = 3$$

(c) An excise tax $t$ is applied to this product.
State the new supply equation which includes the tax $t$.

$$Q_S = 1 + P - t$$

Find the equilibrium price $P_E(t)$ and quantity $E(t)$ as functions of $t$.

$$1 + P - t = 7 - P^2 \Longrightarrow P'_E = -\frac{1}{2}\left(1 - \sqrt{25+4t}\right), \quad Q'_E = t + \frac{1}{2}\left(1 + \sqrt{25+4t}\right)$$

(d) Find the value of the tax $t$ which will result in the maximum tax yield being generated. (You will need to use Maple to find this value).

$$Y(t) = tQ'_E = t\left(t + \frac{1}{2}\left(1 + \sqrt{25+4t}\right)\right)$$

$$Y'(t) = -\frac{\sqrt{25+4t}(4t-1) - 25 - 6t}{2\sqrt{25+4t}}$$

The maximum is when $Y'(t) = 0$, $t = 1.83$ to 2 d.p.s. We can check that this is indeed a maximum either by plotting the graph of $Y(t)$ or having Maple calculate the value of $\left.\frac{d^2Y}{dt^2}\right|_{t=1.83} = -1.67$ to 2 d.p.s.