CM 1021 Mathematical Methods for Computing I

Supplementary Exercise Sheet A - SOLUTIONS

1. Find the domains of the following functions:

   (a) (i) $y = \sqrt{x-3}$ $\quad x \geq 3$ (ii) $y = \sqrt{1-2x^2}$ $\quad |x| \leq \dfrac{1}{\sqrt{2}}$

   (iii) $y = \dfrac{1}{1-x}$ $\quad x \in \mathbb{R} \setminus x = 1$

   (b) (i) $y = \dfrac{1}{1-x^2}$ $\quad x \in \mathbb{R} \setminus x = \pm 1$ (ii) $y = \dfrac{1}{\sqrt{x^2-3}}$ $\quad |x| > 3$

   (iii) $y = \dfrac{1}{\sqrt{x^2+3x+1}}$ $\quad -\dfrac{11}{4} < x < -\dfrac{1}{4}$

   (c) (i) $y = \dfrac{1}{\sqrt{x^2-4x+3}}$ $\quad 1 < x < 3$ (ii) $y = \ln(x-4)$ $\quad x > 4$

   (iii) $y = \ln(3-x^2)$ $\quad |x| < \sqrt{3}$.

2. Find the range of the following functions, where $x \in \mathbb{R}$ unless otherwise specified:

   (a) (i) $y = x^2+2x+5$ $\quad$ range $= [4, \infty)$ (ii) $y = x^2+7x+13$ $\quad$ range $= \left[\dfrac{3}{4}, \infty\right)$

   (iii) $y = |3x-1|$ $\quad$ range $= [0, \infty)$

   (b) (i) $y = e^{3x-1}$ $\quad$ range $= [0, \infty)$ (ii) $y = 3+2x-x^2$ $\quad$ range $= (-\infty, 4]$

   (iii) $y = 1-e^{-x}$, $\quad x \geq 0$ $\quad$ range $= [0, 1)$

   (c) (i) $y = 2^x$, $\quad x \leq 5$ $\quad x \leq 5$.

3. The functions $f(x), g(x)$ and $h(x)$ are defined as follows:

$$f(x) = 4+3x \qquad g(x) - = 7-2x \qquad h(x) = 1+x^2$$

   Find the following

   (a) (i) $f^{-1}(x) = \dfrac{x-4}{3}$ (ii) $g^{-1}(x) = \dfrac{7-x}{2}$ (iii) $f(g(x)) = 25-6x$

   (b) (i) $h(g(x)) = 50-28x+4x^2$ (ii) $f^{-1}(g^{-1}(x)) = -\dfrac{1+x}{6}$ (iii) $h(h(x)) = x^4+2x^2+2$

   (c) (i) $h(h(h(x)) = x^8+4x^6+8x^4+8x^2+5$ (ii) $f(f(x)) = 16+9x$

   (iii) $f(f(f(x))52+27x$ (iv) $f^{[n]}(x) = 2(3^n-1)+3^n x$

4. Sketch the following graphs, showing clearly the values of any points of intersection with the x and y axes as well as any local maxima and minima.

   (a) $3x + 2y - 6 = 0$

   (b) $y = 8 - x$

   (c) $y = x^2 + 6x + 5$

   (d) $y = x^2 + 6x + 10$

   (e) $y = 5 - 6x - x^2$

   (f) $y = \dfrac{3\,x^2 - 5}{x^2 - 4\,x + 3}$.

5. Find the equations of the straight lines between each of the following pairs of points:

   (a) (i) $(9, 3)$, $(-1, 8)$ $2y + x - 15 = 0$ (ii) $(4, 3)$, $(-12, 19)$ $y + x - 7 = 0$ (iii) $(-1, -3)$, $(-5, 1)$ $x + y + 4 = 0$.

6. Find the coordinates of the centre and the radius of each of the following circles:

   (a) $x^2 - 4x + y^2 - 4y - 28 = 0$ center $(2, -2)$ radius $= 6$

   (b) $x^2 + 6x + y^2 + 10y - 25 = 0$ center $(-3, -5)$ radius $= 3$

   (c) $9x^2 + 12x + y^2 - 10y - 20 = 0$ center $(-\frac{2}{3}, 5)$ radius $= 7$

   (d) $16x^2 - 24x + 9y^2 - 6y - 15 = 0$ center $(\frac{3}{4}, \frac{1}{3})$ radius $= 5$.

7. Find the inverses of the following functions and in each case sketch the graph of the function and its inverse on the same plot:

   (a) (i) $f(x) = \dfrac{2 - 3x}{5x + 6}$, $f^{-1}(x) = \dfrac{2(1 - 3x)}{5x + 3}$ (ii) $g(x) = \dfrac{3x - 7}{1 - 2x}$, $g^{-1}(x) = \dfrac{x - 7}{2x + 3}$.

8. Find the inverses of the following functions:

   (a) (i) $h(x) = e^{2x-3}$, $h^{-1} = \frac{1}{2}(\ln x + 3)$ (ii) $p(x) = \ln(2 - 3x)$, $p^{-1}(x) = \frac{1}{3}(e^x - 2)$ (iii) $q(x) = 10^{3x}$, $q^{-1}(x) = \frac{1}{3}\log_{10} x$

9. Calculate:

   (a) (i) $4^5 \div 4^{-3} = 4^8 = 65536$ (ii) $5^4 - 4^5 = -399$ (iii) $3^{2^3} = 6561$ (iv) $(3^2)^3 = 729$.

10. Simplify as far as possible:

    (a) (i) $\dfrac{x^2y^3z^4}{x^3y^2z^5} = \dfrac{y}{xz}$ (ii) $\dfrac{x^{\frac{1}{2}}y^{\frac{3}{2}}x^{\frac{1}{3}}}{xy^{\frac{1}{2}}z^2} = \dfrac{y}{x^{\frac{1}{2}}z^{\frac{3}{2}}}$ (iii) $\ln 64 = 6\ln 2$

    (b) (i) $\dfrac{1}{2}\ln 3 - \ln 9 + 3\ln 3 = \frac{3}{2}\ln 3$ (ii) $\ln x^6 - 4\ln x + 3\ln x^3 = 11\ln x$

11. State whether (or where) the following functions are one to one, odd or even or neither, increasing or decreasing or periodic (state the period):

(a) (i) x^3 is odd, one to one and strictly increasing
(ii) $x^2 + 4$ is even, decreasing for $x < 0$ and increasing for $x > 0$
(iii) $x^4 + 3x^2 + 2$ is even, decreasing for $x < 0$ and increasing for $x > 0$
(iv) $x^3 - x^5$ is odd, one to one and strictly decreasing
(v) $\cos \dfrac{2x}{3}$ is even and periodic with period 3π

(b) (i) $\tan \dfrac{2x}{3}$ is odd and periodic with period $\frac{3\pi}{2}$
(ii) $\sin(4x - 1)$ is odd with period $\frac{\pi}{2}$
(iii) $x^2 \sin x$ is odd, not periodic
(iv) $x \sin 3x$ is even, not periodic
(v) e^{-3x} is decreasing and one to one

(c) (i) $4 - e^{-3x}$ is increasing and one to one
(ii) $e^x - \ln x$ is increasing and one to one

12. Find the limit of each of the following expressions as $n \to \infty$:

(a) (i) $\lim_{n\to\infty} \dfrac{2n^3 - 1}{n^3 + 64n^2} = 2$ (ii) $\lim_{n\to\infty} \dfrac{1 - n}{1 + n} = -1$
(iii) $\lim_{n\to\infty} \dfrac{1 - 3n^2}{1 - n^3} = 0$ (iv) $\lim_{n\to\infty} 1 + 3e^{-n} = 1$