UNIVERSITY OF SURREY

MAT 1015
Calculus I

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Contents

Calculus I
Autumn Semester 2010-2011

The calculus module runs for both Autumn and Spring semesters. These are the notes for Autumn semester. The assessment for the Autumn semester is based on two class tests, the first counting for 15% and the second 25% and an examination which counts for the remaining 60%. The Autumn semseter counts for 50% of the total module marks.

There will be three courseworks which will be marked but not assessed.

Contacting the lecturer

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Notes

These notes are issued with numerous gaps which will be completed during lectures. Supplementary material will also be distributed from time to time.

Lecture attendance is therefore essential to gain a full understanding of the material.

Exercises

There are exercises at the end of each chapter and solutions will be distributed progressively. Slightly more challenging exercises are marked with an asterisk.

Background material

While these notes contain all the material you will need to cover during the Autumn semester there are numerous excellent calculus textbooks in the library. Although they contain far more material than will be covered during the Autumn semester they provide interesting and useful background and you are encouraged to look at the early chapters of some of them.

In the University Library, the section coded **51** contains books on ‘General Mathematics’. Books specifically on Calculus can be found in the section coded **517**.

- Guide^2 Mathematical methods, by John Gilbert and Camilla Jordan, Palgrave
- Calculus - a complete course, by Robert A. Adams, Addison Wesley
- Calculus - late transcendentals, by Howard Anton et.al, John Wiley and Sons
- Calculus, by Ron Larsen and Bruce Edwards, Brooks/Cole.

The latter three books are very comprehensive and cover all the calculus you are likely to need for your degree.

Chapter 1

Complex numbers

1.1 The square root of -1

A quadratic equation $x^2 + ax + b = 0$ may or may not have solutions in $\mathbb{R}$. Consider the equation $x^2 + 4x + 5 = 0$. If its roots are α and β then by the theory of quadratic equations, $\alpha + \beta = -4, \alpha\beta = 5$.

Solving the equation gives $x = -2 \pm \sqrt{-1}$. We know that $\sqrt{-1}$ does not exist in $\mathbb{R}$. However, if $\alpha = -2 + \sqrt{-1}$ and $\beta = -2 - \sqrt{-1}$, then $\alpha + \beta = -4$ and $\alpha\beta = (-2 + \sqrt{-1})(-2 - \sqrt{-1}) = 4 + 2\sqrt{-1} - 2\sqrt{-1} - (\sqrt{-1})^2 = 4 - (-1) = 5$.

We define i such that $i^2 = -1$. An expression of the form $z = x + yi$, where $x, y \in \mathbb{R}$, is called a **complex number**. If $x = 0$ then $z = yi$ is called a **purely imaginary** number. The set of all complex numbers is denoted by $\mathbb{C}$. Clearly $\mathbb{R} \subset \mathbb{C}$.
x is called the **real part** of z, $\mathrm{Re}(z)$. y is called the **imaginary part** of z, $\mathrm{Im}(z)$.

1.2 Algebra of complex numbers

Two complex numbers are defined to be **equal** if their real parts are equal and their imaginary parts are equal,

Example 1

$$x + yi = 3 - 5i$$

We do arithmetic in $\mathbb{C}$ by treating a complex number as a linear function of i, where $i^2 = -1$. Thus

Example 2

$$(a+bi)+(c+di) =$$
$$(a+bi)(c+di) =$$

It can be shown that all the usual rules of arithmetic apply to complex numbers, e.g. if $z_1, z_2, z_3 \in \mathbb{C}$ then

$z_1 + z_2 = z_2 + z_1$	addition is commutative
$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$	multiplication is distributive over addition
$(z_1 z_2)z_3 = z_1(z_2 z_3)$	multiplication is associative

1.2.1 Complex conjugates

$\overline{z} = x - yi$ is called the **conjugate** of $z = x + yi$. It is sometimes denoted by $z^\star$. Note that $z\overline{z} = x^2 + y^2$, which is a real number.

Division of complex numbers is carried out by making the denominator real:

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}}.$$

Example 3

$$z = \frac{a+bi}{c+di}$$

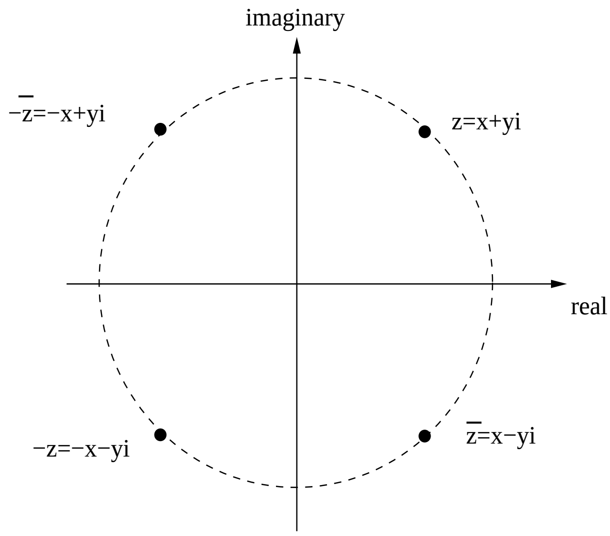
1.2.2 Some useful properties of complex numbers

It is straightforward to prove the following:

$$
\begin{array}{lcl}
\mathrm{Re}(z+w) & = & \mathrm{Re}(z)+\mathrm{Re}(w) \\
\mathrm{Im}(z+w) & = & \mathrm{Im}(z)+\mathrm{Im}(w) \\
|\mathrm{R}(z)| & \leq & |z| \\
|\mathrm{Im}(z)| & \leq & |z| \\
z+\bar{z} & = & 2\mathrm{Re}(z) \\
z-\bar{z} & = & 2\mathrm{Im}(z) \\
|z|=|\bar{z}| & = & |-z|=|\overline{-z}| \\
\overline{z \pm w} & = & \bar{z} \pm \bar{w} \\
\bar{\bar{z}} & = & z \\
\overline{zw} & = & \bar{z}\bar{w} \\
\overline{\left(\frac{z}{w}\right)} & = & \frac{\bar{z}}{\bar{w}} \\
|zw| & = & |z||w| \\
\left|\frac{z}{w}\right| & = & \frac{|z|}{|w|} \\
|z+w| & \leq & |z|+|w| \quad \text{(the triangle inequality)}
\end{array}
$$

1.3 The Argand diagram

The complex number $z = x + yi$ can be represented by the point (x, y) in an **Argand diagram** or **complex plane**. Then addition and subtraction correspond to the same operations with vectors. The graph shows that $z, -z, \bar{z}, \overline{-z}$ lie on a circle.



Example 4

Find $\sqrt{i}$

This method can be used to find the square roots of any complex number.

1.4 Polar form of complex numbers

A point in two-dimensions can be specified by its **cartesian** coordinates (x, y) or its **polar** coordinates (r, θ). If $z = x + yi$, then r and θ determine the **polar form** of z.

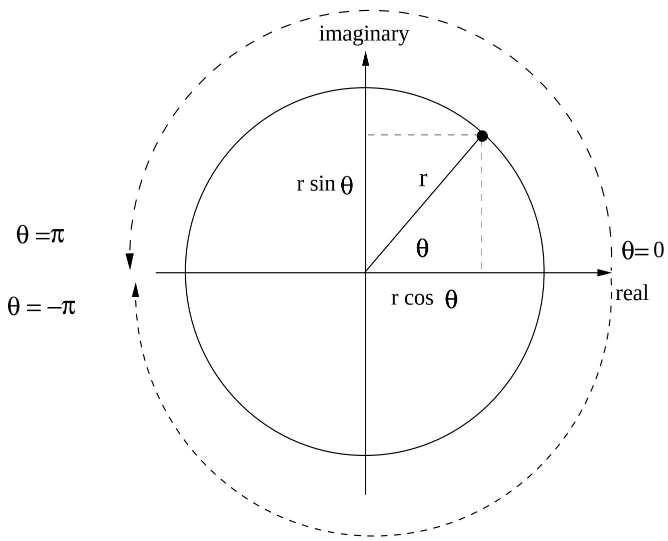
The **modulus** of $z = x + yi$ is $|z| = r = \sqrt{x^2 + y^2}$. Thus $z\overline{z} = |z|^2$.

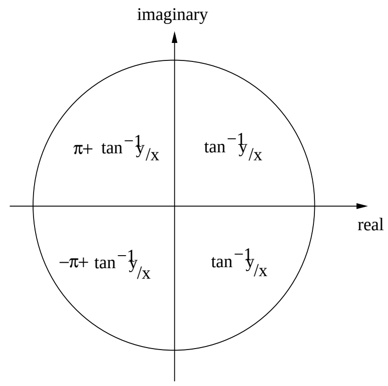
The **argument** of $z = x + yi$ is $\arg(z) = \theta$ where $\tan\theta = \frac{y}{x}$. In order to obtain the correct value of the argument it is always a good idea to sketch the Argand diagram.

The **principal value** of the argument is in the interval $(-\pi, \pi]$, so we take $0 < \theta \leq \pi$ if $y > 0$ and $-\pi < \theta < 0$ if $y < 0$. Note that the argument is a many-valued function so that we could write

$$\arg(1 + i) = \frac{\pi}{4} + 2n\pi, \quad n = 0, \pm 1, \pm 2 \ldots$$

In polar form, the complex number with modulus r and argument θ is $r(\cos\theta + i\sin\theta)$.





Example 5

Express $z = 2 + 2i$ and $z = 4 - 4i$ in polar form.

1.5 De Moivre's Theorem

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then, using the trigonometric addition formulae, we have $z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$. Thus z_1z_2 has modulus r_1r_2 (multiply the moduli) and argument $(\theta_1 + \theta_2)$. (add the arguments)

Dividing z_1 by z_2 gives $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$. Hence $\dfrac{z_1}{z_2}$ has modulus $\dfrac{r_1}{r_2}$ (divide the moduli) and argument $(\theta_1 - \theta_2)$.(subtract the arguments)

It follows by induction that if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ then for $n \in \mathbb{N}, z^n = r^n(\cos n\theta + i\sin n\theta)$; this is called **De Moivre's Theorem** and in fact it is true for all $n \in \mathbb{R}$.

If $z = r(\cos\theta + i\sin\theta)$, then $\dfrac{1}{z} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$ and $\dfrac{1}{z^n} = \dfrac{1}{r^n}(\cos n\theta - i\sin n\theta)$.

De Moivre's Theorem allows us to calculate many useful trigonometrical identities in a straightforward manner

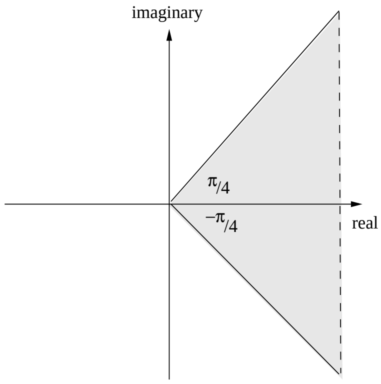
Example 6

Find $\cos 3\theta$ and $\sin 3\theta$ in terms of powers of cosines and sines respectively.

1.6 Sets of complex numbers

We can describe lines and curves in the complex plane by defining conditions on z: if these conditions are inequalities we have a description of a region of the complex plane.

- The distance between two points in the complex plane, z and w is $|w - z|$, thus the set $\{z : |z| = a\}$ is a circle centre at the origin and radius a.
- The set $\{z : |z - b| = a\}$ is a circle centre $(b, 0)$ and radius a. The set $\{z : |z - (b + ci)| = a\}$ is a circle centre (b, c) and radius a.
- The set $\{z : |z - ai| = |z - bi|\}$ is the set of points that are the same distance from ai as they are from bi - they lie on the perpendicular bisector of ai and bi.
- $\arg(z - (a + bi)) = \psi$ is a line passing through the point (a, b) and making an angle of ψ with the positive x-axis.
- The set $\{z : |z| < 3\}$ is all of the points that lie inside a circle centre the origin and radius 3 - excluding the boundary. (This is called an open set).
- The set $\{z : a \leq |z| \leq b\}$ comprises points that lie in an annulus bounded by two circles, centre the origin, of radii a and b, including the boundary of the circles themselves (a closed set).
- The set $\{z : -\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}\}$ is all the points that lie in the triangular sector shown below. (The diagonal boundaries are included but the set is open.)



- The set $\{z : |z-(3+i)| < |z+1|\}$ is the set of points that are nearer to $3+i$ than they are to -1. This is the set of points on the right hand side of the perpendicular divisor of the line joining $3+i$ and -1. The mid point of this line is $(1, \frac{1}{2}i)$ and the gradient of the line is $\frac{1}{4}$. Thus the points lie on the right hand side of

$$y - \frac{1}{2} = -4(x-1) \longrightarrow 2y + 8x = 9$$

Roots of polynomial equations

The **Fundamental Theorem of Algebra** states that every polynomial equation with coefficients in $\mathbb{C}$ has all its solutions in $\mathbb{C}$

Complex roots of **real** polynomials occur in conjugate pairs. Suppose that $P(z) = a_0 + \sum_{r=1}^{n} a_r z^r$. Then it is straightforward to show that $\overline{P(z)} = P(\bar{z})$. Now, if α is a root of $P(z)$ then $P(\alpha) = 0$ so $\overline{P(\alpha)} = 0$ and thus $P(\bar{\alpha}) = 0$. hence if α is a complex root of a polynomial with real coefficients then so is $\bar{\alpha}$.

Example 7

Given that $x = 1 + \sqrt{3}i$ is a root of $x^4 - 5\,x^3 + 12\,x^2 - 16\,x + 8 = 0$, find the other three roots.

Exercises

1. Express in the form $a + ib$, where $a, b \in \mathbb{R}$,

   (a) $(2 - 3\text{i}) - (4 - 5\text{i})$, (b) $(3 + 4\text{i})(2 - 3\text{i})$, (c) $(5 - \text{i})^2$, (d) $\dfrac{6 - 2i}{3 + 4i}$.

2. For each of the following, find (i) its modulus, (ii) its argument in radians between $-\pi$ and π, in terms of π or as a decimal.

   (a) $1 + \text{i}$, (b) $3 - 4\text{i}$, (c) $-2 + 5\text{i}$, (d) $-\sqrt{3} - \text{i}$, (e) -7i (f) 7.

3. Find all the solutions in $\mathbb{C}$ of the equations (a) $4x^2 + 1 = 0$, (b) $x^2 + 2x + 5 = 0$.

4. Prove from the definitions that for complex numbers w and z,

   (a) $\overline{w + z} = \overline{w} + \overline{z}$, (b) $\overline{wz} = \overline{w}\,\overline{z}$.

5. Show that $z \div \overline{z}$ has modulus 1. Express $\arg(z \div \overline{z})$ in terms of $\arg(z)$.

6. Find the quadratic equation which has $2 + 3\text{i}$ as one of its roots.

7. Suppose $(a + b\text{i})^2 = 5 + 12\text{i}$. By expanding the left-hand side and equating the real and imaginary parts, find the possible values of the real numbers a and b. Hence write down the two square roots of $5 + 12\text{i}$.
   Deduce the value of $\tan\phi$, if $\tan 2\phi = 12/5$ and $0 < \phi < \pi$.

8. If $z = \cos\theta + i\sin\theta$, expand z^4 by the binomial theorem. Hence express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

9. Find $\cos 7\theta$ in terms of powers of $\cos\theta$

10. *Find (i) $\tan 3\theta$ and (ii) $\tan 5\theta$ terms of powers of $\tan\theta$

11. If $z = \cos\theta + i\sin\theta$, show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

    Deduce that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

12. *Express (i) $\cos^6\theta$ (ii) $\cos^7\theta$ in terms of cosines of multiples of θ.

13. Given that $z = 2 + 3i$ is a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$, find the other two roots.

14. Sketch the following curves in the Argand diagram and give the Cartesian form of the equation.

    (a) $|z + 2 - i| = |z - 1 + 2i|$
    (b) $\text{Re}(z) = \text{Im}(z)$
    (c) $|z - 2 + 3i| = 4$
    (d) *$\arg\dfrac{z - 2}{z + 5} = \dfrac{\pi}{4}$

15. Prove de Moivre's Theorem by induction for $n \in \mathbb{N}$.

16. Sketch the following regions in the Argand diagram, labelling clearly the boundaries that are included

    (a) $\text{Im}(z) \geq (\text{Re}(z))^2$

    (b) $\{z : -\frac{\pi}{2} < \arg(z) < -\frac{\pi}{6}\}$

    (c) $\{z : |z| > 5|z+6|\}$

    (d) $*\{z : 3 \leq |z-2+3i| \leq 4\}$

17. Show, geometrically or otherwise that

$$||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1 + z_2|$$

for any two complex numbers z_1, z_2.

18. Show that $x = 2 + \sqrt{3}i$ is a root of

$$x^4 - 8x^3 + 26x^2 - 40x + 21$$

and find the others.

Chapter 2

Functions

2.1 Domain and range

Let X and Y be sets. A **function** f : $X \to Y$ is a rule which assigns to each x in the **domain** X *exactly one* element f(x) in the **codomain** Y. We call x the argument of the function f. Note that a function is **single valued** - in other words any vertical line cuts its graph no more than once.

If $Y \subseteq \mathbb{R}$, f is a **real-valued** function. If $Y \subseteq \mathbb{C}$, f is a **complex-valued** function. In this course, X and Y will be subsets of $\mathbb{R}$ unless otherwise stated.

We write $f : x \mapsto f(x)$, where $x \in X$, and read this as f *maps* x to its *image* $f(x)$.

If no domain is specified, we take f to have its **maximal domain**, e.g. $f : x \mapsto \sqrt{1-x^2}$ can only be defined for $-1 \leq x \leq 1$. The complete definition of a function requires that the domain is specified, thus $f(x) = \sin x, x \in [0, \pi]$ is not the same function as $f(x) = \sin x, x \in \mathbb{R}$.

The subset of Y given by $\{f(x) \ : \ x \in X\}$, or $f(X)$, is called the **range** of f. Thus the range of a function is the set of y-coordinates at all the points on the graph of the function.

2.1.1 Finding the range of a function

If f is quadratic, we can find its range by finding its minimum or maximum point, either using calculus or by completing the square. $f : x \mapsto x^2 + 4x - 3 \equiv (x+2)^2 - 7$ has range $f(x) \geq -7$, or equivalently $[-7, \infty)$. The minimum point on the graph of $f(x)$ is at $(-2, -7)$.

For more general functions with domain $\mathbb{R}$, the range can often be found by writing $f(x) = y$ and finding a condition for this to have real roots for x.

Example 8

Find the range of $f(x) = \dfrac{3x+2}{x^2+4}$,

2.2 Types of functions

In this section we consider a real-valued function f with domain $X \subseteq \mathbb{R}$.

2.2.1 Odd and even functions

If $f(-x) = f(x)$ for all $x \in X$ we say that f is an even function. The graph of an even function is symmetric about the y-axis.

If $f(-x) = -f(x)$ for all $x \in X$ then f is an odd function. The graph of an odd function is the same when rotated by 180° about the origin.

Example 9

If $f(x)$ is even and $g(x)$ is odd, then:

$$f(x) \times g(x) =$$

We have the following rules for functions

- odd $\pm$ odd = odd
- even $\pm$ even = even
- odd $\pm$ even = neither odd nor even
- odd $\times$ odd = even
- even $\times$ even = even
- even $\times$ odd=odd.

2.2.2 Periodic functions

We describe a function as periodic if, for some $k > 0$, $f(x + k) = f(x)$ for all $x \in X$ and $k \in \mathbb{Z}$. The smallest such positive k is called the *minimal* period. The function is said to exhibit translational symmetry.

2.2.3 One to One functions

A one-to-one or **injective** function is such that if $f(a) = f(b)$ then $a = b$, i.e. no two elements of X have the same image under f.

Example 10

$f(x) = x^3$ and $g(x) = x^4$

Any horizontal line cuts the graph of a one to one function **no more than once**,

2.2.4 Onto functions

An onto or **surjective** function is one where the range of f is the whole codomain of f. Thus for every element in the codomain there exists an element in the domain which maps to it.

Example 11 f : $\mathbb{R} \to \mathbb{R}$ *where* $f(x) = 5x + 2$ *and* $g : \mathbb{R} \to \mathbb{R}$ *where* $g(x) = x^2 + 2x$

It is often simpler to show that f is not onto by finding a counter example.

Any horizontal line cuts the graph of an onto function **at least once**.

2.2.5 Bijective functions

A function is bijective if it is both injective and surjective. **Every** horizontal line cuts the graph of a bijective function **exactly once**.

2.2.6 Polynomials

A polynomial is a function of x of the form

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n,$$

where $a_0, ...a_n$ are constants. n is the **degree** of the polynomial.

2.2.7 Monotonicity

We say that $f(x)$ is monotone increasing in some interval $[a, b]$ if $f(x_1) \geq f(x_2) \Leftrightarrow x_1 \geq x_2$ for all $x \in [a, b]$. If the relationship is $f(x_1) > f(x_2) \Leftrightarrow x_1 > x_2$ then f is strictly monotone increasing. If $f(x_1) \geq f(x_2) \Leftrightarrow x_1 \leq x_2$ then we say that f is monotone decreasing while if $f(x_1) > f(x_2) \Leftrightarrow x_1 < x_2$, then f is strictly monotone decreasing.

An equivalent definition is that if $f(x)$ is monotone increasing in some interval $[a, b]$ then $f'(x) \geq 0$ for all $x \in [a, b]$.

Theorem

If $f : A \to B$ is **strictly** monotone, then f is one to one.

2.3 Some important functions

The identity function

The identity function id is defined by

$$\mathrm{id}(x) = x \text{ for all } x.$$

2.3.1 The absolute value function

The absolute value or **modulus** abs(x) or $|x|$ is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Note that $\sqrt{x}$ always means the *positive* square root of x, and so $|x| = \sqrt{x^2}$.

2.3.2 The signum function

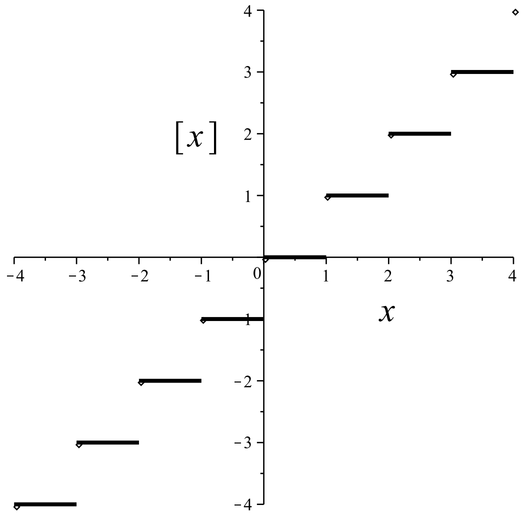
The **signum function**, sgn is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Thus for $x \neq 0$, $\operatorname{sgn}(x) = \dfrac{x}{|x|}$ and $\operatorname{sgn}(x) = \dfrac{\sqrt{x^2}}{x}$.

2.3.3 Integer part

The integer part or **floor** of x, denoted by $\lfloor x \rfloor$ or $[x]$, is defined to be the largest integer less than or equal to x. For example, $\lfloor 7 \rfloor = 7$, $\lfloor -2.3 \rfloor = -3$ and $\lfloor \pi \rfloor = 3$.



2.3.4 Heaviside function

We define the Heaviside function as

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} .$$

The Heaviside function is a "switch" which turns a function on at a particular value thus the graph of $f(x) = H(x - \pi)\cos x$ starts at $(\pi, -1)$.

2.4 Composition of functions

For functions f and g, the **composite function** g ∘ f, or gf, is defined by $(\mathrm{g} \circ \mathrm{f})(x) \equiv \mathrm{g}(\mathrm{f}(x))$. To find this, substitute $\mathrm{f}(x)$ in place of x as the argument of $\mathrm{g}(x)$, and simplify.

If f has domain X and g has domain Y, the domain of g ∘ f is $\{x \in X \ : \ \mathrm{f}(x) \in Y\}$.

Similarly $(\mathrm{f} \circ \mathrm{g})(x) = \mathrm{f}(\mathrm{g}(x))$. In general $\mathrm{g} \circ \mathrm{f} \neq \mathrm{g} \circ \mathrm{f}$. However, composition *is* associative, i.e. $(\mathrm{f} \circ \mathrm{g}) \circ \mathrm{h} = \mathrm{f} \circ (\mathrm{g} \circ \mathrm{h})$. Note that $\mathrm{f} \circ \mathrm{id} = \mathrm{id} \circ \mathrm{f} = \mathrm{f}$.

Example 12

$f(x) = \sqrt{1+x}$. and $g(x) = \cos x$

It is useful to be able to identify functions as compositions,

Example 13

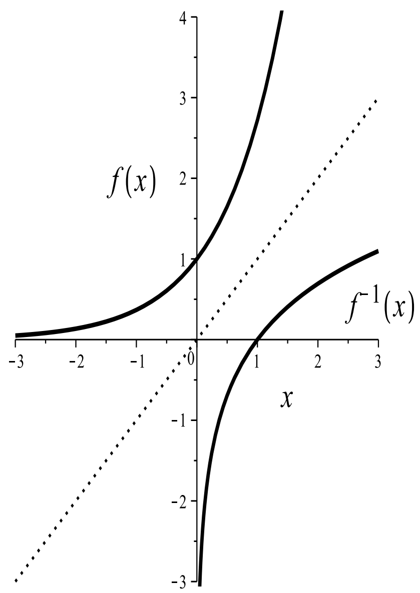
$\sin(\sqrt{x+3})$

2.5 Inverse functions

If f is bijective, with domain X and codomain Y, then there exists an **inverse function** f^{-1} with domain Y and range X. The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$.

f^{-1} is defined by the property

$$f \circ f^{-1} = f^{-1} \circ f = id, \quad \text{i.e.} \quad f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x.$$



Sometimes the inverse can be found by inspection. Otherwise, make x the subject of $y =$ f(x) and then swap x and y in the answer. It is normal to use x in defining the inverse function, but y or any other symbol is not wrong if it is used consistently.

Example 14

Find the inverse of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 + 3x$.

A function whose graph is symmetric about $y = x$ is **self-inverse**, i.e. $f = f^{-1}$, such as $f(x) = \dfrac{x}{x+1}$.

If f is not bijective, we may obtain an invertible function by **restricting the domain**.

Example 15

$f(x) = x^2 - 8x + 11 = (x - 4)^2 - 5$

Theorem

If $f : A \to B$ and f is invertible then if f is strictly decreasing (increasing) f^{-1} is strictly decreasing (increasing).

2.6 Partial fractions

A **rational function** of x has the form $\dfrac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials.

Many rational functions can be expressed in **partial fractions**. The principles are:

(i) A polynomial of degree n in the denominator, which does not factorise, requires a polynomial of degree $n-1$ in its numerator, i.e.

$$\frac{ax^2+bx+c}{(x+d)(x^2+ex+f)} \equiv \frac{p}{x+d} + \frac{qx+r}{x^2+ex+f}.$$

(ii) If there is a linear factor to the power n in the denominator, there may be partial fractions with all powers of this linear factor up to the nth in their denominators, i.e.

$$\frac{ax^2+bx+c}{(x+d)^2(x^2+ex+f)} \equiv \frac{p}{x+d} + \frac{q}{(x+d)^2} + \frac{rx+s}{x^2+ex+f}.$$

(iii) If the degree of numerator $\geq$ degree of denominator, there will be some non-fractional terms in the answer. In this case, do a long division first:.

Example 16

$$\frac{2x^4}{x^3+4x^2+3x+12}$$

We usually find partial fractions over $\mathbb{Q}$. However, we can also have partial fractions over $\mathbb{R}$ or $\mathbb{C}$, e.g.

Example 17

$$\frac{1}{x^2-3}$$

Example 18

$$\frac{1}{x^2 + 4}$$

Exercises 2(a)

1. State the maximal domains of the functions

   (a) $\mathrm{f} : x \mapsto \sqrt{x^2 - 9}$, (b) $\mathrm{g} : x \mapsto \dfrac{3x+1}{x^2 - 2x - 3}$.

2. Find the range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 4x - 3$. Why does f not have an inverse? How can the domain and codomain of $f$be restricted so that the resulting function *does* have an inverse?

3. State the values of $\mathrm{sgn}(-3)$, $|-\pi|$, $\lfloor\sqrt{2}\rfloor$, $\sqrt{(-1)^2}$.

4. Sketch graphs of (i) $\mathrm{sgn}(1-x)$, (ii) $x - \lfloor x \rfloor$, (iii) $\mathrm{sgn}(\sin x)$ (iv) $\mathrm{H}(x-2)x^2$

5. If $\mathrm{f} : (-\infty, 0) \to (1, \infty)$ by $x \mapsto 1 - 5x$ and $\mathrm{g} : [1, \infty) \to [0, \infty)$ by $x \mapsto \sqrt{x-1}$, define (if they exist) the functions f^{-1}, g^{-1}, $\mathrm{f} \circ \mathrm{g}$ and $\mathrm{g} \circ \mathrm{f}$.

6. For each of the following functions with maximal domain, and codomain $\mathbb{R}$, state whether it is even, odd, periodic, one-to-one, onto. Find and simplify the composite functions $\mathrm{f} \circ \mathrm{f}$, $\mathrm{f} \circ \mathrm{g}$, $\mathrm{g} \circ \mathrm{f}$, $\mathrm{g} \circ \mathrm{h}$, and state their domains.

   (a) $\mathrm{f} : x \mapsto x^2 + 1$, (b) $\mathrm{g} : x \mapsto \dfrac{2x}{x-2}$, (c) $\mathrm{h} : x \mapsto \tan 2x$,

7. If $f(x) = \dfrac{2x+1}{x^2+2}$ for $x \in \mathbb{R}$, find the set of values of y for which $\mathrm{f}(x) = y$ has real roots for x. Hence state the range of f.

8. Sketch a graph of

$$f(x) = \begin{cases} 2x, & -1 \le x < 0 \\ x^2, & 0 \le x \le 1 \end{cases}$$

   .

   Define the inverse function.

9. Show that if $f \circ g$ is invertible, then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

10. Express each of the following in partial fractions:

    (a) $\dfrac{4x-3}{(x+1)(x^2+x+1)}$ (b) $\dfrac{2x}{(x-5)^2(x+1)}$

    (c) $\dfrac{x^3+1}{x^2+7x+12}$ (d) $\dfrac{3x+7}{(x+1)^2(x+3)}$

11. Express each of the following in partial fractions:

    (a) $\dfrac{x^2+2}{4x^5+4x^3+x}$ (b) $\dfrac{3x^4+6x^3-2x^2+4}{x^3+2x^2}$

12. *The functions f and g are defined as follows

$$f(x) = \begin{cases} x^2+4 & x \ge 1 \\ x & x < 1 \end{cases} \qquad g(x) = \begin{cases} 3x & x \ge 0 \\ -2x & x < 0 \end{cases}$$

    Find $f \circ g$ and $g \circ f$.

13. Assume that f is an odd function and g is an even function, both defined on the real line. Is each of the following functions odd, even or neither?

$$f \circ g, \quad g \circ f, \quad f \circ f, \quad g \circ g.$$

14. Find all the real values of the constants A and B such that $f(x) = Ax + B$ satisfies (i) $f \circ f(x) = f(x)$ for all x, (ii) $f \circ f(x) = x$ for all x.

15. *For what values of a, b and c is the function $f(x) = \dfrac{x-a}{bx-c}$ self inverse?

16. Show that if $f(x)$ is an odd function, its derivative is an even function.

17. *Express the following in partial fractions over $\mathbb{C}$

(i) $\dfrac{3x-4}{x^2+16}$ (ii) $\dfrac{-7\sqrt{3}-ix}{x^2+3}$ (iii) $\dfrac{x^2-2}{x^3-1}$

2.7 Trigonometric functions

In order that we can extend the trigonometric functions which are first encountered defined in a right angle triangle to the whole real line, let P (x, y) be a point on the circle with centre at the origin and radius 1, and let θ be the angle measured anti-clockwise from the positive x-axis to OP. Then we define:

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}.$$

The sine and cosine functions have range $[-1, 1]$, so a function of the form $r\sin(nx + \alpha)$ has range $[-r, r]$. r is the **amplitude** and the **period** is $\dfrac{2\pi}{n}$. The **phase** is α

If $f(x) = \sin x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ then f is one-to-one. With this restriction of the domain, the **inverse trigonometric function** $\arcsin x$ or $\sin^{-1} x$ is defined for $-1 \leq x \leq 1$. Similarly, restricting the domain of $\cos x$ to $[0, \pi]$, the inverse function is $\arccos x$ or $\cos^{-1} x$, for $-1 \leq x \leq 1$. Also, restricting the domain of $\tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the inverse function is $\arctan x$ or $\tan^{-1} x$, for $x \in \mathbb{R}$.

Note that $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$.

2.8 The exponential and logarithmic functions

The **exponential function** $\exp(x)$ can be **defined** as the sum of the convergent series:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$\exp(1)$ is an irrational (cannot be expressed as p/q where $p, q \in \mathbb{Z}$), transcendental (is not the root of a polynomial equation with rational coefficients) number 2.718 2818 2845 9045 2354..., which we call e. It can be shown that $\exp(x)$ is this number to the power x, so $\exp(x)$ is denoted by e^x.

e^x can also be defined in this way:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

The exponential function is not bijective: for instance, there is no $x \in \mathbb{R}$ such that $e^x = -1$, showing that it is not surjective. However if the codomain is restricted to the positive real numbers then the exponential function becomes bijective; its inverse is the natural logarithm function $\ln x$. Note that $\ln x$ is defined only for $x > 0$. Thus if $y = e^x$ then $x = \ln y$.

Other exponential functions can be defined on $\mathbb{R}$, e.g. $a^x = \exp(x \ln a)$.

2.9 Hyperbolic functions

Any function f, defined on a domain which is symmetrical about the origin, can be expressed as the sum of an odd function and an even function as follows :

$$f(x) \equiv \frac{1}{2}(f(x) - f(-x)) + \frac{1}{2}(f(x) + f(-x)).$$

Taking $f(x) = e^x$, the odd and even components are the **hyperbolic functions**

$$\sinh x \equiv \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

We also define

$$\text{cosech } x \equiv \frac{1}{\sinh x}, \quad \tanh x \equiv \frac{\sinh x}{\cosh x},$$

and

$$\text{sech } x \equiv \frac{1}{\cosh x}, \quad \coth x \equiv \frac{\cosh x}{\sinh x} \equiv \frac{1}{\tanh x}.$$

The hyperbolic functions have many properties similar to those of the trigonometric functions, but **they are not periodic**.

For example, $\cosh^2 x - \sinh^2 x \equiv 1$; compare this with $\cos^2 x + \sin^2 x \equiv 1$.

Similarly, $1 - \tanh^2 x \equiv \text{sech}^2 x$, $\coth^2 x - 1 \equiv \text{cosech}^2 x$.

Note the important identity

$$\cosh x + \sinh x = e^x.$$

2.9.1 Osborne's rule

To convert a trigonometric identity into a hyperbolic one, replace cos by cosh and sin by sinh *but* whenever $\sin^2$ occurs either explicitly or implicitly (e.g. in $\tan^2$), change the sign.

1. $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ becomes
   $\sinh(A + B) \equiv \sinh A \cosh B + \cosh A \sinh B$;

2. $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ becomes
   $\cosh(A + B) \equiv \cosh A \cosh B + \sinh A \sinh B$;

3. $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ becomes $\tanh(A - B) \equiv \dfrac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$
   (Note the implicit product of sines in $\tan A \tan B$).

Example 19

Prove that

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

To solve hyperbolic equations and establish identities we can use the definitions in terms of exponentials, *or* any of the standard identities.

Example 20

Solve $\cosh x + 2\sinh x = 6$

The following **inverse hyperbolic functions** are defined on the given domains:

$$\begin{aligned} \sinh^{-1} x &= \ln(x + \sqrt{x^2+1}) && x \in \mathbb{R} \\ \cosh^{-1} x &= \ln(x + \sqrt{x^2-1}) && x \in [1, \infty) \\ \tanh^{-1} x &= \frac{1}{2}\ln\frac{1+x}{1-x} && x \in (-1, 1). \end{aligned}$$

Exercises 2(b)

1. Find the amplitude and the period and the phase where appropriate of

   (a) $3\sin 3x$ (b) $8\sin 3x\cos 3x$

   (c) $2\cos(2x+4)$ (d) $6\cos x + 8\sin x$

2. Starting from the definitions, prove that $\cosh^2 x - \sinh^2 x \equiv 1$.

3. Solve, in terms of natural logarithms, (a) $4\sinh^2 x = \cosh^2 x$, (b) $7\sinh x = 24$.

4. Find the exact value of arcosh $\frac{13}{12}$.

5. Find the coordinates of any points of intersection of the curves $y = \cosh 2x$ and $y = 3 - 2\cosh x$.

6. Find an identity relating $\coth^2 x$ and $\mathrm{cosech}^2 x$. Hence solve $\coth^2 x = 2\mathrm{cosech}\, x$.

7. Prove the expressions for the inverse hyperbolic functions given in the preceding section

8. Prove that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

9. Find an expression for $\mathrm{sech}\, x$ as a function of x.

10. Prove the following identities:

    (a) $\tan^{-1} x + \tan^{-1}\dfrac{1}{x} = \dfrac{\pi}{2} \quad x > 0$

    (b) $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad |x| \le 1$

    (c) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy} \quad xy \ne 1$

2.10 Functions of a complex variable

In this chapter so far we have been considering functions with a real argument which give a real output. We now brief consider some of the properties of complex functions $f : W \to \mathbb{C}$, where $W \subseteq \mathbb{C}$.

2.10.1 The complex exponential function

Many power series are valid for complex numbers also. In particular, if $z \in \mathbb{C}$ then the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

converges for all z. We call its limit $\exp(z)$ or e^z.

e^z has the following properties

- $e^z e^w \equiv e^{z+w}, \quad e^{-z} \equiv \dfrac{1}{e^z}$.
- $e^{i\theta} = \cos\theta + i\sin\theta$. The unit circle in the complex plane can thus be described by the set $\{e^{i\theta} : -\pi < \theta \leq \pi\}$.
- $e^{i\pi} + 1 = 0$. This is Euler's famous identity.
- $e^z = e^{z+2n\pi i}, n \in \mathbb{Z}$. The complex exponential function is **periodic**.
- $|e^z| = e^{Re(z)}$
- $\overline{e^z} = e^x(\cos y - i\sin y) = e^{\bar{z}}$ (The complex conjugate of the exponential is the exponential of the complex conjugate).

We can thus express $z = x + yi$ in the form $re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.

Example 21

Express $z = 1 + \sqrt{3}i$ in exponential form.

Since $e^{iz} = \cos z + i\sin z$ and $e^{-iz} = \cos z - i\sin z$ we have the following Euler relations:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$

which give the following relationships between trigonometric and hyperbolic functions:

$$\cosh z = \cos iz, \quad \cosh iz = \cos z, \quad \sinh iz = i\sin z, \quad i\sinh z = \sin iz$$

Note that these relationships, which can be shown to hold for $z \in \mathbb{C}$ of course hold for $\theta \in \mathbb{R}$.

De Moivre's theorem in exponential form

De Moivre's Theorem can be stated as

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

This is valid for any $n \in \mathbb{R}$.

2.10.2 The cube roots of unity

The cube roots of unity are the three roots of $z^3 = 1$; if we factorise $z^3 - 1$ we obtain $(z-1)(z^2+z+1) = 0$. We solve this equation to find the three cubes roots of 1, these are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. They are usually denoted by $1, \omega, \omega^2$. They have some interesting properties,

$$1 + \omega + \omega^2 = 0, \qquad \bar{\omega} = \omega^2, \qquad \overline{\omega^2} = \omega, \qquad \omega = \frac{1}{\omega^2}.$$

We can write 1 as $\cos 0 + i \sin 0$, or e^{0i}, as well as $\cos 2\pi + i \sin 2\pi = e^{2\pi i}$ and $\cos 4\pi + i \sin 4\pi = e^{4\pi i}$. More generally we have

$$1 = e^{0i+2\pi ki}, k \in \mathbb{Z}$$

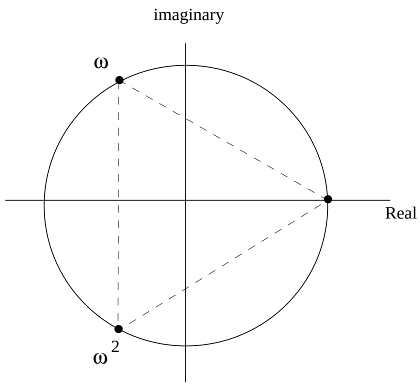
Then from de Moivre's theorem $\sqrt[3]{1} = e^{\frac{0i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$. The last expression is of course $e^{-2\pi i/3}$. Thus

$$e^{2\pi i/3} = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \omega,$$

while

$$e^{-2\pi i/3} = \cos\frac{2\pi}{3} - i \sin\frac{2\pi}{3} = \omega^2.$$

The cube roots of unity divide the unit circle into three equal parts, they form the apices of an equilateral triangle.



The nth roots of unity

In an analogous manner, the nth roots of unity are the vertices of a regular n-sided polygon with one vertex at 1.

The seventh roots of 1 are $1, \mathrm{e}^{2\pi \mathrm{i}/7}, \mathrm{e}^{4\pi \mathrm{i}/7}, \mathrm{e}^{6\pi \mathrm{i}/7}, \mathrm{e}^{-6\pi \mathrm{i}/7}, \mathrm{e}^{-4\pi \mathrm{i}/7}, \mathrm{e}^{-2\pi \mathrm{i}/7}$.

Example 22

Find the fourth root of $z = \sqrt{3} - i$.

2.10.3 The logarithm of a complex number

We define the logarithm of a complex number in the obvious way, if $z = e^w$ then $w = \ln z$. To find what we mean by $\ln z$ suppose that

$$z = r(\cos\theta + i\sin\theta) = e^{u+iv} = e^u(\cos v + i\sin v).$$

Then $r = e^u$ and $v = \theta + 2k\pi, k = 0, \pm 1, \pm 2 \ldots$, thus

$$\mathrm{Log}(z) = u + iv = \ln|z| + i(\theta + 2k\pi), k = 0, \pm 1, \pm 2 \ldots.$$

The complex logarithm is many valued, which is why we use the symbol Log, the principal value has $\theta \in (-\pi, \pi]$ and is thus $\ln z = \ln r + i\theta$.

Example 23

Find $\ln(4 + 5i)$.

The complex logarithm has the same properties as the real logarithm.

2.10.4 Real and imaginary parts of complex valued functions

We frequently want to find the real and imaginary parts of complex functions, in the form $f(z) = u(x, y) + iv(x, y)$. Here are some examples,

Example 24

Find the real and imaginary parts of e^z, $\ln z$, e^{-z^2}, $\sin z$, $\tan z$.

2.10.5 Solving equations in complex functions

Example 25

Solve $\sin z = 4$

Exercises 2(c)

1. Find the real and imaginary parts of the following functions
   $z + \dfrac{1}{z}$ (ii) $z^2 - \dfrac{1}{z^2}$

2. Find the real and imaginary parts of the following functions:
   (i) $\cos z$ (ii) $e^{\frac{1}{z}}$ (iii) z^z (iv) $\arctan(z)$

3. Solve the following equations:
   (i) $\dfrac{z+1}{z-1} = 2 + 3i$
   (ii) $z^2 + 2z = 2 + 4i$
   (iii) $\cos z = 5$
   (iv) $\cosh z = -2$

4. Prove that $\bar{\omega} = \omega^2$

5. Find the principal values of the logarithms of (i) -5 (ii) $2 + 7i$

6. Find the real and imaginary parts of $(1+i)^(1+i)$.

7. *Prove that i^i is real.

8. *Prove that $\ln(z^2) = 2\ln(z)$

Chapter 3

Limits, graphs and equations

3.1 Limits

The **limit** of $f(x)$ as x tends to a, written $\lim\limits_{x \to a} \mathrm{f}(x)$, is a number ℓ such that we can make $f(x)$ as close as we like to ℓ by taking x very close (but not equal) to a.

For a continuous function

$$\lim_{x \to a} f(x) = f(a).$$

Some functions have different limits as $x \to a$ from below (from the left) and from above (from the right). In such a case, $\lim\limits_{x \to a} \mathrm{f}(x)$ does not exist.

Example 26

Consider the function defined as follows

$$f(x) = \begin{cases} x, & x < 1 \\ x - 1, & x \geq 1 \end{cases}$$

3.1.1 A note about infinity

Strictly speaking, $\tan \frac{\pi}{2} = \infty$ is incorrect, since ∞ is not a number. A correct way to write this would be

$$\lim_{x \to \frac{\pi}{2}} \tan x = \infty.$$

$\lim_{x \to \infty} \frac{1}{x} = 0$ means that as x gets larger and larger, $\frac{1}{x}$ tends to 0. Do NOT write $\frac{1}{0} = \infty$ or $\frac{1}{\infty} = 0$ as ∞ is not a number.

We *can* write $\lim_{x\to 0+} \frac{1}{x} = \infty$, but this is just another was to state that the limit does not exist.

3.1.2 Rules for limits

If $f(x)$ and $g(x)$ both have finite limits as $x \to a$, then

$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$ for any $c \in \mathbb{C}$

$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$(the limit of the sum is the sum of the limits)

$\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$,(the limit of the product is the product of the limits)

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.(The limit of the quotient is the quotient of the limits)

Example 27

$$\lim_{x\to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}.$$

$$\lim_{x\to \infty} \frac{ax^n + (a_0 + a_1 x + a_2 x^2 \ldots + a_{n-1} x^{n-1})}{bx^m + (b_0 + b_1 x + b_2 x^2 \ldots + b_{m-1} x^{m-1}))}$$

3.1.3 Some important limits for trigonometric functions

Geometrically (as well as by other methods) it can be shown that for any acute angle x, $\sin x < x < \tan x$. Dividing through by $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}, \quad \text{so} \quad \cos x < \frac{\sin x}{x} < 1.$$

Now as $x \to 0, \cos x \to 1$, so also $\frac{\sin x}{x} \to 1$. It can be deduced that:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1 - \cos x}{x} = 0, \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1.$$

3.1.4 Limits for exponential and logarithmic functions

We have seen that the exponential function can be defined in terms of a limit as follows:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Letting $n = \frac{1}{p}$, we also have $e^x = \lim_{p\to 0}(1+px)^{1/p}$.

We can easily show that the inverse function of $(1+px)^{1/p}$ is $\frac{x^p-1}{p}$, so the inverse function of e^x can also be expressed as a limit:

$$\ln x = \lim_{p\to 0}\left(\frac{x^p-1}{p}\right).$$

Many of the properties of logarithms can be deduced directly from this limit.

3.1.5 The Squeeze Theorem

Example 28

Find $\lim_{x\to 0}\sqrt{x}\cos\left(x+\frac{1}{x}\right)$.

3.2 Curve Sketching

Sketching graphs of functions can provide a great deal of information about their behaviour. This is the process that should be followed when sketching the graph of $y = f(x)$.

1. Intersections with the axes

Put $y = 0$, $x = 0$ to find the intersections with the x axis (the roots of $f(x)$) and the y axis $f(0)$.

2. Vertical asymptotes

When the denominator of a rational function approaches zero then $y \to \pm\infty$. At such a value of x there is a vertical line which the graph approaches but never meets.

Example 29

$$y = \frac{x-3}{(x-1)(x-4)}$$

3. Horizontal asymptotes

Find $\lim_{x \to \pm\infty} \mathrm{f}(x)$ to see what the graph is like when $|x|$ is very large.

Example 30

$$y = 3 + e^{-x}$$

4. Oblique asymptotes If the numerator is of higher degree than the denominator, there may be **oblique asymptotes**. To find these, divide through.

Example 31

$$y = \frac{x^2 + 2x + 3}{x - 1}$$

A graph may sometimes cross an oblique asymptote, but never a vertical nor a horizontal one.

In some cases the asymptote may be a curve.

Example 32

$$y = \frac{x^3 - x^2 - 8}{x - 1}.$$

5. (Obvious) symmetries

If the function is even then there is symmetry about the y-axis. If the function is odd then there is 180° rotational symmetry about the origin. If the function is periodic the function will show translational symmetry. If the function is self-inverse then there is symmetry about $y = x$.

6. Stationary points

Turning points can be found by the usual calculus methods: solve $\mathrm{f}'(x) = 0$. The second derivative $\mathrm{f}''(x)$ is positive at a minimum, negative at a maximum and 0 at a point of inflexion.

3.2.1 Inequalities

Sketching the graph can often help with the solution of inequalities. To find where $f(x) > 0$, sketch $f(x)$ and see where the graph lies above the x-axis.

To solve $f(x) > g(x)$, either sketch both graphs and see where $f(x)$ is above $g(x)$, or sketch $f(x) - g(x)$ and find the values of x where this is positive.

Example 33 Solve $\cos x > 2 \sin x$.

3.2.2 Related graphs

Related graphs can be obtained in various standard ways. For example, to obtain $y = |\mathrm{f}(x)|$, simply reflect in the x-axis those parts of the graph that lie below it.

To get $y = \dfrac{1}{\mathrm{f}(x)}$ note that $\dfrac{1}{f(x)}$ has asymptotes where $f(x) = 0$, and vice-versa.

3.3 Standard transformations

There are a number of standard (linear) transformations:

- $y = \mathrm{f}(x) + a$: translate $y = \mathrm{f}(x)$ by a units parallel to the y-axis (upwards).
- $y = \mathrm{f}(x + a)$: translate $y = \mathrm{f}(x)$ by $-a$ units parallel to the x-axis (a to the left).
- $y = \mathrm{f}(-x)$: reflect $y = \mathrm{f}(x)$ in the y-axis.
- $y = -\mathrm{f}(x)$: reflect $y = \mathrm{f}(x)$ in the x-axis.
- $y = k\mathrm{f}(x)$: stretch $y = \mathrm{f}(x)$ by a factor of k parallel to the y-axis.
- $y = \mathrm{f}(kx)$: 'stretch' $y = \mathrm{f}(x)$ by a factor of $\frac{1}{k}$ parallel to the x-axis.

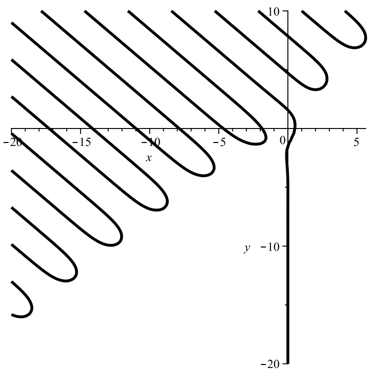
Implicit Equations

Up to now we have looked at functions defined *explicitly* in the form $y = f(x)$. However, sometimes functions are given *implicitly* in the form $f(x, y) = g(x, y)$.

Example 34

$3y^2 + 4x^5y - 2x^2 - 4 = 0.$

We can try and spot forms that suggest standard curves such as conic sections, but in many cases, e.g $\cos(x+y) = xe^{y-x}$, we cannot obtain an explicit form nor easily find the graph of the function. In fact the graph of this function is as below:



Sometimes a closed curve, which does not represent a true function but does express a relationship between two variables, is best described implicitly.

The **circle** with centre (a, b) and radius r has equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

Example 35

$$x^2 + y^2 + 4x - 8y - 5 = 0.$$

3.4 Parametric Equations

It is often convenient to describe a curve (not necessarily representing a well-defined function) by two equations giving each of x and y in terms of a variable **parameter**, say t, in the form $x = f(t), y = g(t)$.

Example 36 The circle $x^2 + y^2 = a^2$.

Other common parametrisations are

- The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has parametric form $x = a\cos\theta, y = b\sin\theta, \quad \theta \in [0, 2\pi)$
- The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has parametric form $x = a\cosh t, y = b\sinh t$ or $x = a\sec\theta, y = b\tan\theta)$
- The straight line passing through x_0, y_0 and x_1, y_1 can be parametrised as

$$x = x_0 + t(x_1 - x_0), y = y_0 + t(y_1 - y_0), \quad t \in [0, 1].$$

Many other types of curve can be described parametrically and this is a particularly useful technique when the Cartesian equations of the curve would be extremely complicated.

Sketching a graph defined by parametric equations

The simplest way to do this is to eliminate the parameter if this is possible

Example 37 Sketch the curve defined by $x = t^2 - 1, y = t + 1$.

If the parameter cannot be eliminated, an alternative way to sketch the curve is to work out x and y for a suitable range of values of the parameter.

Example 38 $x = t - 3\sin t, y = 4 - 3\cos t$

3.5 Polar coordinates

Plane polar coordinates are an alternative set of coordinates to the usual Cartesian or rectangular coordinates in two dimensions. We define the coordinates of a point as the distance r from the pole (origin) and the angle θ (in radians) made with the initial line, taken to be the positive x-axis. Then, in general if the polar coordinates of a point are (r, θ), the points $(r, \theta \pm 2k\pi, \quad k \in \mathbb{Z})$ also represent the same point. We allow $r < 0$ by considering negative r to be measured from the pole opposite to the direction of positive r. Thus the points $(2, \frac{4\pi}{3})$ and $(-2, \frac{\pi}{3})$ are the same.

3.5.1 Relationship with rectangular coordinates

We have $x = r\cos\theta, y = r\sin\theta$, alternatively $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\dfrac{y}{x}$.

Example 39

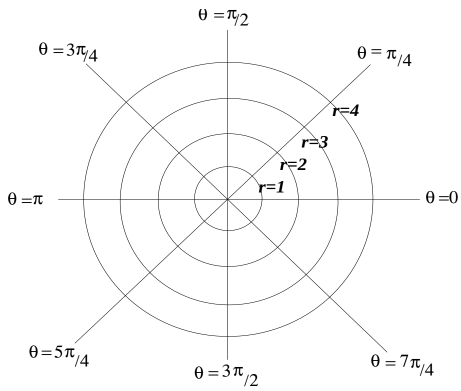
The straight line

Example 40

The curve with polar form $r = 2a\cos\theta$

3.5.2 Drawing polar curves

As with parametric curves, if polar curves cannot be transformed into rectangular coordinates it is often necessary to plot some points on polar graph paper, which can be easily constructed as below.



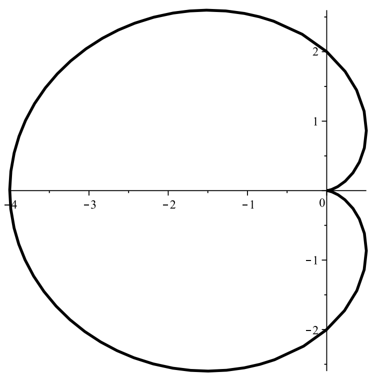
However, for simpler functions the sketching process can be accomplished as follows:

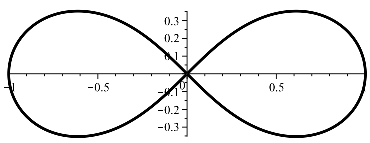
1. Plot a graph of θ against r as if they were rectangular coordinates. This will show clearly how θ varies with r.
2. Find the values of θ for which $r = 0$ and indicate them with rays, these are the directions in which the curve approaches the origin.
3. Find maxima and minima of r to show where the curve is furthest away from and nearest to the origin.

Example 41 Sketch the polar graph of $r = 3\cos 2\theta$

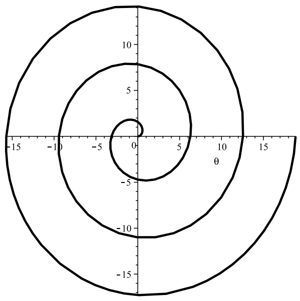
Some common polar curves

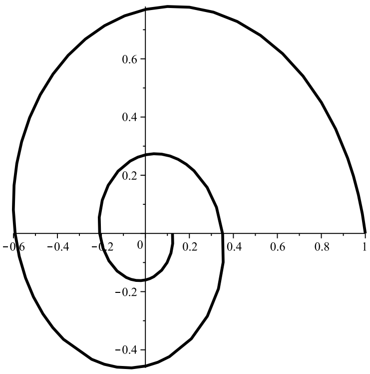
- The equation $r = c$ is a circle radius c, centre the origin.
- $\theta = c$ is a straight line from the origin at c radians to the initial line.
- A circle with centre at polar coordinates (a, ψ) has equation $r = 2a\cos(\theta - \psi)$. In general the polar graph of equation
  $r = f(\theta - \psi)$ is the graph of $r = f(\theta)$ rotated **anticlockwise** by ψ.
- $r = 2(1 - \cos\theta)$ is known as a cardioid, $r^2 = \cos 2\theta$ is a lemniscate





- $r = \theta$ is an equiangular spiral, $r = e^{-\theta}$ is an exponential spiral





Note that whereas the curves in the previous item are 2π periodic, these spirals are not periodic.

Exercises 3

1. Find the following limits:

   (a) $\lim_{x\to 0} \frac{2x^2+3x}{x}$, (b) $\lim_{x\to\infty} \frac{12x+6}{3x-4}$, (c) $\lim_{x\to 3} \frac{x-3}{x^2-9}$.

2. The diameter of the pupil of an animal's eye is given by $\mathrm{f}(x)$ mm, where x is the intensity of the light on the pupil,
   where $\mathrm{f}(x) = \frac{80x^{-0.3}+60}{8x^{-0.3}+15}$,
   find (i) the diameter of the pupil when there is no light,
   (ii) the limit to which the diameter tends as the amount of light becomes large.

3. Find the following limits or explain why they do not exist
   (i) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$, (i) $\lim_{x\to\pi} \frac{(x-\pi)^2}{\pi x}$, (i) $\lim_{x\to 0} \frac{|x-2|}{x-2}$
   (i) $\lim_{x\to 2} \frac{|x-2|}{x-2}$, (i) $\lim_{x\to 2} \frac{x^4-16}{x^3-8}$.

4. If $\lim_{x\to 0} \frac{f(x)}{x^2} = -2$, find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} \frac{f(x)}{x}$

5. if $2 - x^2 \le f(x) \le 2\cos 2x$ for all x, find $\lim_{x\to 0} f(x)$

6. *Find $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

7. Sketch the following curves, and state the range of each function.

   (a) $y = \frac{x}{1+x}$, (b) $y = \frac{3x+2}{3-2x}$, (c) $y = \frac{2}{1-x^2}$, (d) $y = \frac{3x}{1-x^2}$,

   (e) $y = \frac{(x+1)(x-2)}{(x-1)(x+2)}$, (f) $y = \frac{x^2+x-1}{x-1}$, (g) $y = \frac{x^2+1}{x^2+2}$.

8. Sketch the following curves

   (a) $y = \frac{\ln x}{1+|x|}$ (b) $y = \frac{1-x^2}{1+x^2}$

   (c) $y = \frac{1+x^2}{x(x+1)(x-1)}$ (d) $y = \frac{x\ln x}{x+e^x}$

9. Solve $\frac{x-1}{x^2+x-12} \ge \frac{x^2-9}{x^2-x-2}$.

10. $\mathrm{f}(x)$ is defined to be the larger solution (for y) of $2y^2 - 4xy - 3x^4 = 0$.
    Find an explicit formula for $\mathrm{f}(x)$.

11. The circle C has equation $x^2 + y^2 - 6x + 8y - 144 = 0$. Find the centre and radius of C. Show that the point A $(8, 8)$ lies on C and find the coordinates of B such that AB is a diameter of C.

12. The parametric equations of a curve C are $x = 1 + \sinh t$, $y = 5 - 4\cosh t$. Sketch C for $-1 \leq t \leq 1$. Show that C meets the x-axis at two points, and state their coordinates.

13. Convert the following polar equations to equations in rectangular coordinates:

    (a) $r = 3\sec\theta$

    (b) $r = \sin\theta + \cos\theta$

    (c) $r = \dfrac{2}{2 - \cos\theta}$

    (d) $r = \sec\theta\tan\theta$

14. Convert the following equations into polar form:

    (a) $y^2 = x(a - x)$

    (b) $x^2 + y^2 = 2a^2xy$

15. Sketch the following polar curves

    (a) $r = 3(1 + \sin\theta)$

    (b) $r^2 = 16\sin 2\theta$

    (c) $r = 4\theta, \quad \theta \leq 0$ and $r = 4\theta, \quad \theta \geq 0$

    (d) $*r = \frac{1}{\theta}$ and $r = \frac{1}{\theta - 2}$. Can you explain your results?

16. Show that the distance between the two points (r_1, θ_1) and (r_2, θ_2) is given by

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

17. Sketch the graph of $y = (x - 4)^{3/2}$.

Chapter 4

Differentiation

The gradient of the graph of a function at a given point describes the rate at which the function is changing at that point. For example, if the function describes the displacement of an object, then the gradient of a tangent to the graph gives its velocity.

Geometrically, the gradient of a graph at P $(x, f(x))$ is found by drawing a chord from P to a nearby point $Q = (x + \delta x,\ f(x + \delta x))$, where δx means a small increment in the x direction.

The gradient of this chord is $\dfrac{f(x + \delta x) - f(x)}{\delta x}$.

As $\delta x \to 0$, the gradient of the chord tends to the gradient of the tangent.

Definition

Let f be a function defined in some neighbourhood of a point $(x, f(x))$. If the limit

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

exists then f is said to be *differentiable* at $(x, f(x))$. The value of the limit is the *derivative* or *derived function* or *differential coefficient* of $f(x)$ at this point, which we can write as $f\,'(x)$ or $\dfrac{d}{dx}(f(x))$.

f is a **differentiable function** if it is differentiable (i.e. the above limit exists) at every point where it is defined.

If $f(x) = y$, the derivative is denoted by $\dfrac{dy}{dx}$, which can be understood as $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$.

When δy and δx are corresponding small increments in y and x, $\delta y \approx \frac{dy}{dx}\ \delta x$. This formula is used to find the effect on y of known small changes in x.

An alternative version of the definition, giving the derivative where $x = a$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

These formulae are used to differentiate functions from **first principles**.

Example 42

Differentiate $f(x) = x^3$ from first principles.

4.1 Derivatives of Hyperbolic Functions

From the definitions in terms of exponentials, it follows that

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \text{and} \quad \frac{d}{dx}(\cosh x) = \sinh x.$$

Using these results, the quotient and composite function rules give the following standard derivatives of hyperbolic functions:

$f(x)$	$f'(x)$
$\sinh(ax + b)$	$a \cosh(ax + b)$
$\cosh(ax + b)$	$a \sinh(ax + b)$
$\tanh(ax + b)$	$a \operatorname{sech}^2(ax + b)$
$\coth(ax + b)$	$-a \operatorname{cosech}^2(ax + b)$
$\operatorname{sech}(ax + b)$	$-a \operatorname{sech}(ax + b) \tanh(ax + b)$
$\operatorname{cosech}(ax + b)$	$-a \operatorname{cosech}(ax + b) \coth(ax + b)$

4.2 Differentiating Inverse Functions

If y is a function of u which is a function of x, the rule for composite functions states that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Putting $u = x$ and $x = y$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{so} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = 1.$$

If $y = \mathrm{f}(x)$ is an invertible function of x, and $f^{-1}(x)$ is easier to differentiate than $f(x)$, this technique can be used: express x in terms of y, differentiate with respect to y to get $\frac{\mathrm{d}x}{\mathrm{d}y}$ and then use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$.

(**First** derivatives obey the laws of fractions, higher derivatives do not.)

Example 43

$$y = \sin^{-1}(2x - 3)$$

This method gives the following standard results for the inverse trigonometric and hyperbolic functions.:

$\mathrm{f}(x)$	$\mathrm{f}'(x)$	
$\arcsin x \equiv \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$,	$x \in (-1, 1)$
$\arccos x \equiv \cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$,	$x \in (-1, 1)$
$\arctan x \equiv \tan^{-1} x$	$\frac{1}{1+x^2}$	
$\operatorname{arsinh} x \equiv \sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$	
$\operatorname{arcosh} x \equiv \cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$x \in (1, \infty)$
$\operatorname{artanh} x \equiv \tanh^{-1} x$	$\frac{1}{1-x^2}$,	$x \in (-1, 1)$

4.3 Parametric Differentiation

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$.

Example 44 Find y' if $x = a \sec x$ and $y = b \tan x$

For second derivatives, use

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{f'(t)g''(t) - f''(t)g'(t)}{(f'(t))^3}$$

Example 45 Find y' if $x = t^3 - 3t, y = t^2$

.

4.4 Implicit Differentiation

An **implicit equation** $f(x, y) = g(x, y)$ can be differentiated term-by-term with respect to x. Whenever y occurs, it must be differentiated to get $\frac{dy}{dx}$, which can be abbreviated as y'.

Example 46

$y \sin x = x^3 - \cos y$

To obtain second derivatives we differentiate through the equation again; y' differentiates to y''.

Example 47

$xy + y^2 = 2x$

4.5 Logarithmic Differentiation

Taking logarithms of both sides of an equation sometimes helps in differentiation.
If $y > 0$, $y = f(x)$ becomes $\ln y = \ln f(x)$, which differentiates to $\frac{1}{y}\frac{dy}{dx} = \frac{1}{f(x)}f'(x)$.

Example 48

$f(x) = a^x$

Example 49

$y = \sqrt{x}\sin x e^x$,

4.6 Leibniz's Rule

This is a method for finding the nth derivative of a *product* of two functions.

Let f and g be n-times differentiable functions. Let h be defined by $h(x) = f(x)\ g(x)$. Then h is also n times differentiable and it can be proved that

$$h^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} f^{(r)}(x)\, g^{(n-r)}(x),$$

where $h^{(n)}(x)$ means the nth derivative of $h(x)$ with respect to x. We can also use D the differential operator to express this, where $\frac{d^n}{dx^n} f(x) = D^n(f(x))$.

Example 50

$$\frac{d^3(x^3 e^{2x})}{dx^3}$$

4.7 Derivatives of functions in polar coordinates

Consider a function $r = f(\theta)$ in plane polar coordinates. Then $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Thus we have

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$$

and

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta.$$

Solving for the values of θ at which $\frac{dy}{d\theta} = 0$ will give the points at which the tangent to the curve is horizontal and at which $\frac{dx}{d\theta} = 0$ the points at which it is vertical. In general we have

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 51 Find the points where the tangent lines to the cardioid $r = 1 + \cos\theta$ are horizontal and vertical.

4.8 Applications of differentials

You should be familiar with the application of differentiation to gradients, tangents, normals, turning points, maxima and minima and rates of change.

4.8.1 Radius of Curvature

Another feature of a graph that can be found by differentiation is the radius of curvature of $y = f(x)$, defined as

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}.$$

The **curvature** is defined to be $\kappa = \frac{1}{\rho}$.

Example 52

Find the radius of curvature of $y = \cosh x$ at $x = \ln 2$

4.8.2 l'Hôpital's Rule

Suppose f and g are differentiable real-valued functions with $\mathrm{f}(a) = \mathrm{g}(a) = 0$. Then

$$\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \lim_{x\to a} \frac{\mathrm{f}(x) - \mathrm{f}(a)}{\mathrm{g}(x) - \mathrm{g}(a)} = \lim_{x\to a} \left(\frac{\mathrm{f}(x) - \mathrm{f}(a)}{x - a} . \frac{x - a}{\mathrm{g}(x) - \mathrm{g}(a)} \right)$$

for $x \neq a$

$$= \frac{\lim\limits_{x\to a} \frac{\mathrm{f}(x) - \mathrm{f}(a)}{x - a}}{\lim\limits_{x\to a} \frac{\mathrm{g}(x) - \mathrm{g}(a)}{x - a}} = \frac{f'(a)}{g'(a)}$$

provided $\mathrm{g}'(a) \neq 0$.

Example 53 $\lim\limits_{x\to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$

If $\mathrm{f}'(a) = \mathrm{g}'(a) = 0$ we can repeat the process to get

$$\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{f''(a)}{g''(a)},$$

provided $\mathrm{g}''(a) \neq 0$.

Example 54 $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ then l'Hôpital's rule is expressed in the form:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Example 55 $\lim_{x \to \infty} \frac{x}{e^x}$

4.8.3 Related rates of change

If x and y are both functions of t then if $z = f(x, y)$ we have

$$\frac{dz}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

Example 56

If the sides, and of a rectangle are 25 cms and 40 cms and are increasing at 2 cm per second and decreasing at 3 cm per second respectively find the rate of change in the area,

4.8.4 Small changes in the dependant variable

We use the equation $\Delta y \approx \dfrac{\mathrm{d}y}{\mathrm{d}x}\ \Delta x$.

Example 57

Find the change in the period of a simple pendulum relative to a change in its length

Exercises 4

1. Differentiate from first principles : (a) x^5, (b) $\dfrac{1}{x}$, (c) $\sin 2x$.
2. Differentiate with respect to x:

   (a) $(x^2+3)^{5/2}$, (b) $\sinh(\cosh x)$, (c) $\arcsin\left(\frac{x+3}{2}\right)$

   (d) $\operatorname{arctanh}\,(1-x^2)$,(e) $\coth^2(\mathrm{e}^x)$,(f) $\operatorname{arcsech}\, x$ (i.e. $\operatorname{sech}^{-1}x$)
3. *Differentiate with respect to x:

   (a) $\sin^{-1}\sqrt{\dfrac{1-2x^2}{1+2x^2}}$ (b) $\cot\left(\dfrac{\operatorname{cosec} x}{x^3+5}\right)$ (c) $\dfrac{(2x+3)^3}{(4x^2-1)^8}$

   Hint: use logarithmic differentiation.
4. Use logarithmic differentiation to find the derivative with respect to x of

   (a) $(\sin x)^x$, (b) $3^{\cosh x}$, (c) $\dfrac{x\tan^2 x}{x^3-1}$, where $x > 1$.
5. Use logarithmic differentiation to compute the value of y' where $y = \dfrac{\sqrt{1+x}(1-x)^{\frac{1}{3}}}{(1+5x)^{\frac{4}{5}}}$
6. Express $\dfrac{d^2x}{dy^2}$ in terms of $\dfrac{dy}{dx}$.
7. Find the maximum and minimum values of $\sin^{-1}(x^2-1)$ for $-1 \le x \le 1$.
8. A curve has parametric equations $x = a(1-\cos 2t), y = a(2t+\sin 2t)$, where a is a non-zero real constant and $0 \le t \le \frac{\pi}{2}$. Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $t = \frac{\pi}{4}$.

9. A circle has equation $x^2 + y^2 - 2x + 6y - 15 = 0$. Find the gradient of the tangent to this circle at the point where $x = 5$ and $y < 0$,

   (a) by implicit differentiation,

   (b) by finding the centre and radius and using coordinate geometry.

10. The implicit equation of a curve is $2y^2 - 3xy + x = 6$. Find equations of the tangent and the normal to the curve at the point where $x = 1$ and $y > 0$.

    Also find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x, y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

11. Given that $f(x) = \frac{1}{3}\sinh x(2 + \cosh^2 x)$, show that $f''(x) = \cosh^3 x$.

    Find, to the nearest integer, the radius of curvature of the curve $y = \mathrm{f}(x)$ at the point where $x = \ln 2$.

12. Use Leibniz's Rule to find

    (a) the third derivative of $x^4 \cos 2x$,

    (b) an expression for the nth derivative of $x^3 \sinh x$.

13. Use l'Hôpital's rule to find the following limits:

    (a) $\lim_{x\to a} \dfrac{x^2 - a^2}{x^3 - a^3}$, (b) $\lim_{x\to 2} \dfrac{\sin \pi x}{x^2 - 4}$, (c) $\lim_{x\to 0} \dfrac{\arctan x}{\ln(x+1)}$, (d) $\lim_{x\to 1} \dfrac{\ln x}{\mathrm{e}^x - \mathrm{e}}$.

14. The equation of curve S is $x^3 + y^3 - 6y - 9x = 0$

    (a) Find the co-ordinates of the points where S intersects the x axis.

    (b) Find the equation of the normals to S when $x = 3$.

    (c) Find the x co-ordinates of the turning points of S.

    (d) Find the values of y at which the curve is vertical.

    (e) What is the equation of the oblique asymptote of S?

    (f) Sketch the curve.

15. Prove Leibnitz's rule. hint: use induction.

16. *Find the points at which the tangents to the following polar curves are horizontal or vertical. Sketch the curves

    (i) $r = \sin 2\theta$ (ii) $r = e^{\theta}$ (iii) $r = \sin\theta \cos^2\theta$

17. Determine the equation (in rectangular coordinates) of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

Chapter 5

Series

5.1 Power series

A series of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots = \sum_{r=0}^{\infty} a_r (x-c)^r$$

is called a **power series** in $x - c$ or a power series about the point $x = c$. c is the **centre of convergence** of the series. For the values of x for which the series converges the sum defines a function of x. The series converges provided that $|x - c| < R$, i.e. $c - R \leq x \leq c + R$ where R is the **radius of convergence**,given by

$$R = \frac{1}{L} \quad \text{and} \quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Note that if $L = 0, R = \infty$ and if $L = \infty, R = 0$. The series obviously converges for $x = c$ if nowhere else.

Example 58 Find the centre, radius and interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(2nx+5)^n}{3^n(n^2+1)}$

.

5.1.1 Operations on power series

A convergent power series in x (i.e. with $c = 0$) or $x - c$ is simply a function of x. Thus if $f(x)$ is represented by the convergent power series $P(x)$ then $f(g(x))$ is represented by $P(g(x))$. The interval of convergence of the original series was $|x| < R$, of the new series it is the interval such that $|g(x)| < R$.

Example 59

Find the binomial expansion for $\dfrac{1}{1 + x^2}$

The Cauchy product

We can add and subtract convergent series to obtain another series with radius of convergence at least as large as the smaller of the radii of convergence of the original series. We can multiply two convergent series together, to obtain the Cauchy product thus

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0.$$

Example 60

Find the series expansion for $\dfrac{1}{(1-x^2)^2}$ by direct multiplication.

5.1.2 Differentiating power series

We treat a convergent power series as an infinite polynomial in x and differentiate it term by term to obtain another series with same radius of convergence.

Example 61

find the series expansion for $\dfrac{1}{(1-x^2)^2}$ by differentiation.

5.1.3 Integrating power series

We can integrate a convergent power series term by term, making sure to evaluate the constant of integration, and obtain another series with same radius of convergence.

Example 62

Find a power series expansion for $\tan^{-1} x$ about $x = 0$.

5.2 Taylor series

It is straightforward to show that, provided $f^{[n]}(c)$ exists for all $n \in \mathbb{N}$, we can represent $f(x)$ as a power series in $x - c$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots.$$

This is called the Taylor series of f about c. If $c = 0$ it is known as a Maclaurin series. The series will converge provided that $|x| < R$, where R is the radius of convergence:

$$R = \lim_{n\to\infty} \left| \frac{(n+1)f^{[n]}(c)}{f^{[n+1]}(c)} \right|.$$

Remark

We can also write the Taylor series in the form

$$f(x+h) = f(x) + hf'(x) + h^2\frac{f''(x)}{2!} + h^3\frac{f'''(x)}{3!}\ldots$$

This form has important applications in approximating functions and in the numerical solution of differential equations.

5.2.1 Some common Maclaurin series

Here are some standard Maclaurin series. They are convergent for all real x unless otherwise stated. Note that in trigonometric series, x must be in radians.

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}$

- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}$

- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{r=0}^{\infty} \frac{x^{2r}}{(2r)!}$

- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{r} \quad (-1 < x \leq 1)$

- $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots = \sum_{r=1}^{\infty} \frac{x^r}{r} \quad (-1 \leq x < 1)$

Note that $\ln x$ itself has no Maclaurin series, as it is not defined at $x = 0$. $\sqrt[3]{x}$ has no Maclaurin series as its first derivative is not defined at $x = 0$.

5.2.2 Obtaining Taylor and Maclaurin series

While directly evaluating the derivatives of the function whose Taylor expansion is required will always give the series, it is often much simpler to manipulate other, known series, by multiplication, division, differentiation and integration of those series as well as simply by substituting for the argument. The following series are convergent for all real x unless otherwise stated.

Example 63

Find the Maclaurin series for $e^{-\frac{x^2}{2}}$.

Example 64 Find the Maclaurin series for $\tan x$.

Example 65 Find the Maclaurin series for $\sin^2 x$.

Example 66 Find the Taylor series for $\ln x$ about $x = 3$.

5.2.3 The remainder term

Truncating the Taylor series for $f(x)$ around c at the term in $(x-c)^n$ will leave an error term. This term can be written as the Lagrange remainder, E_n:

$$E_n = \frac{f^{[n+1]}(\xi)(x-c)^{n+1}}{(n+1)!}$$

where $\xi \in [c, x]$.

Example 67

Find the error in calculating $e^{-0.5}$ if the Maclaurin series is truncated at the term in x^4

Exercises 5

1. Find the sums of the following series
   (i) $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$ (ii) $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$ (iii) $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$.

2. Find the centre, radius of convergence and interval of convergence of the following power series:

   (i) $\sum_{n=0}^{\infty} 3n(2x-1)^n$ (ii) $\sum_{n=0}^{\infty} n^3(2x-3)^n$.

3. Use multiplication of series to find a power series representation of
   $\frac{1}{(1-x)^3}$ for $|x| < 1$.

4. Find the radius of convergence and the sum of the following infinite series

   - $1 - 4x + 16x^2 - 64x^3 \ldots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$
   - $3 + 4x + 5x^2 + 6x^3 \ldots = \sum_{n=0}^{\infty} (n+3)x^n$
   - $2 + 4x^2 + 6x^4 + 8x^6 = \sum_{n=0}^{\infty} 2(n+1)x^{2n}$.

5. Find Maclaurin series for the following functions and state the radius of convergence.

   (i) $\cos 3x^3$ (ii) $\sin x \cos x$ (iii) $\ln(2+x^2)$ (iv) $\cosh x - \cos x$
   (v) $\sec x$

6. *Find Maclaurin series for the following functions and state the radius of convergence

   (i) $e^{\sin x}$ (ii) $\frac{1}{1+x+x^2}$ (iii) $\frac{1}{1+\tan x}$.

7. By direct multiplication of the series for e^x and e^y show that $e^{x+y} = e^x e^y$.

8. Find the Taylor series for the following functions and state the interval of convergence:
   (i) e^x about $x = 4$ (ii) $\sin x$ about $x = \frac{\pi}{4}$ (iii) $\frac{1}{x^2}$ about $x = -2$.

9. Estimate the error if the Maclaurin series for the following functions are terminated as indicated:

   (i) $\sin 0.2$, terminated with the term in x^5
   (ii) $\ln(\cosh 2)$, terminated with the term in x^4 .

10. *Find the following

(i) $\displaystyle\sum_{n=0}^{\infty}\frac{\sin n\theta}{n!}$,

(ii) *$\displaystyle\sum_{n=1}^{\infty}\frac{\cos n\theta}{n}$

Chapter 6

Integration

6.1 The anti-derivative

There are two (equivalent) ways of defining integration. The first is as the inverse of differentiation. Given a function f, we look for a function F such that $F'(x) = f(x)$. If this can be found, we denote $F(x)$ by $\int f(x)dx$. This is an *indefinite integral.*

Example 68

If $F(x) = \tan^{-1} e^x$ find $f(x)$

$F(x)$ is sometimes called an **anti-derivative** or **primitive** of $f(x)$.

Clearly if $F(x)$ is an anti-derivative of $f(x)$ then so is $F(x) + c$ for any real constant c.

6.2 The definite integral

The second approach uses area. If f is defined on the interval (α, β) and the area enclosed by its graph and the x-axis between $x = \alpha$ and $x = \beta$ is finite, this area is called the definite integral of $f(x)$ from α to β, denoted by $\int_\alpha^\beta f(x)\, dx$ and we then say that f(x) is **integrable** on (α, β).

This yields the definition of a definite integral as

$$\int_{\alpha}^{\beta} f(x)\ dx = \lim_{N\to\infty} \sum_{i=0}^{N} f(\alpha + i\delta x)\delta x$$

with $\delta x = \dfrac{\beta - \alpha}{N}$. Note that area below the x-axis is regarded as negative.

6.3 The Fundamental Theorem of Calculus

The relation between definite and indefinite integration is given by this important theorem.

Let $f : (\alpha, \beta) \to \mathbb{R}$ be an integrable function. (as defined in the previous section)
Then there exists a function $F : (\alpha, \beta) \to \mathbb{R}$ such that $F' = f$, and

$$\int_{\alpha}^{\beta} f(x)\ dx = F(\beta) - F(\alpha).$$

It follows that

$$\frac{d}{dx}\int_{\alpha}^{x} f(t)\ dt = \frac{d}{dx}\left(F(x) - F(\alpha)\right) = f(x).$$

Differentiating an integral

Example 69

Find $\dfrac{d}{dx}\displaystyle\int_{a}^{x} e^{-t} dt$

More generally

$$\frac{d}{dx}\int_{g(x)}^{f(x)} h(t)\ dt = h(f(x))f'(x) - h(g(x))g'(x)$$

Example 70

Find $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^4} \ln t \; dt$

6.4 Properties of Definite Integrals

- $\displaystyle\int_{\alpha}^{\beta} af(x)\; dx = a\int_{\alpha}^{\beta} f(x)\; dx$, for $a \in \mathbb{R}$
- $\displaystyle\int_{\alpha}^{\beta} (f(x) \pm g(x))\; dx = \int_{\alpha}^{\beta} f(x)\; dx \pm \int_{\alpha}^{\beta} g(x)\; dx$
- $\displaystyle\int_{\beta}^{\alpha} f(x)\; dx = -\int_{\alpha}^{\beta} f(x)\; dx$
- If $\alpha \leq \gamma \leq \beta$ then $\displaystyle\int_{\alpha}^{\beta} f(x)\; dx = \int_{\alpha}^{\gamma} f(x)\; dx + \int_{\gamma}^{\beta} f(x)\; dx$
- $\displaystyle\int_{0}^{\alpha} f(x)\; dx = \int_{0}^{\alpha} f(\alpha - x)dx$
- If f is odd, $\displaystyle\int_{-\alpha}^{\alpha} f(x)\; dx = 0.$
- If f is even, $\displaystyle\int_{-\alpha}^{\alpha} f(x)\; dx = 2\int_{0}^{\alpha} f(x)\; dx$
- If f is periodic with period k and $N \in \mathbb{Z}$ then $\displaystyle\int_{0}^{\alpha} f(x)\; dx = \int_{Nk}^{Nk+\alpha} f(x)\; dx$
- If $f(x) \leq g(x)$ and $\alpha < \beta$ then $\displaystyle\int_{\alpha}^{\beta} f(x)\; dx \leq \int_{\alpha}^{\beta} g(x)\; dx$

The natural logarithm function can be **defined** as $\ln x = \displaystyle\int_{1}^{x} \frac{1}{t}\; dt.$

Sometimes relationships between functions can be deduced from different forms of the same indefinite integral.

Example 71

Show $\arcsin x + \arccos x = \frac{\pi}{2}$ by integration.

6.5 Integration techniques

An essential part of integration techniques is to be able to process integrands so that they resemble functions whose integrals are known. Often we cannot integrate directly, but have to transform an integral into a recognisable standard form. In this respect, integration is much less routine than differentiation which is a very straightforward process. Indeed many quite simple functions cannot be integrated to give closed form functions. They are however typically integrable - i.e. the definite integral can be evaluated numerically.

You should always verify your answers by differentiation to obtain the integrand.

6.5.1 Standard Integrals

Here a and b are real constants. In each case a constant of integration, c should be added for indefinite integrals.

f(x)	$\int$ f(x) dx	
$\sinh(ax+b)$	$\frac{1}{a}\cosh(ax+b)$	
$\cosh(ax+b)$	$\frac{1}{a}\sinh(ax+b)$	
a^x $(a>0)$	$\frac{a^x}{\ln a}$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$	$x\in(-a,a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\frac{x}{a}$	
$\frac{1}{\sqrt{x^2-a^2}}$,	$\cosh^{-1}\frac{x}{a}$	$x\in(a,\infty), a>0$

6.5.2 Integration by Substitution

In the substitution method we choose a suitable variable $u=g(x)$, so $du=g'(x)\ dx$, and obtain the integral of a function of u with respect to u.
Thus

$$\int f(x)dx=\int f\left(g^{-1}u\right)\frac{du}{g'\left(g^{-1}u\right)}.$$

Having integrated, the answer must then be expressed in terms of x.

The limits on a definite integral can be converted into values of the new variable.

Example 72 $\int_{\pi/2}^{\pi}\cos^3 x\ dx$

Rational functions containing square roots of quadratic functions can often be converted (e.g. by completing the square) to one of the following types:

- If $\sqrt{a^2-x^2}$ occurs, try substituting
  $x=a\sin u$, so $dx=a\cos u\ du$.
  Simplify using $a^2(1-\sin^2 u)\equiv a^2\cos^2 u$.

- If $\sqrt{x^2 - a^2}$ occurs, try substituting
  $x = a\cosh u$, so $dx = a\sinh u\ du$, since $\cosh^2 u - 1 \equiv \sinh^2 u$,
  or $x = a\sec u$, so $dx = a\sec u\tan u\ du$, since $\sec^2 u - 1 \equiv \tan^2 u$.
- If $\sqrt{x^2 + a^2}$ occurs, try substituting
  $x = a\sinh u$, so $dx = a\cosh u\ du$, since $\sinh^2 u + 1 \equiv \cosh^2 u$,
  or $x = a\tan u$, so $dx = a\sec^2 u\ du$, since $\tan^2 u + 1 \equiv \sec^2 u$.

(Sometimes one alternative will work better than the other.)

For integrals of the form $\displaystyle\int \frac{1}{x\sqrt{ax^2+bx+c}}\ dx$, first put $x = \dfrac{1}{u}$, so $dx = -\dfrac{1}{u^2}\ \mathrm{d}u$, etc.

Also be aware of: $\displaystyle\int \frac{1}{\sqrt{x+a}}\ dx = 2\sqrt{x+a} + c,\quad \int \frac{x}{\sqrt{x^2+a}}\ dx = \sqrt{x^2+a} + c.$

The substitution $t = \tan\frac{1}{2}x$

If $t = \tan\dfrac{x}{2}$, then $\tan x = \dfrac{2t}{1-t^2}$, from which we get

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad dx = \frac{2\ dt}{1+t^2}.$$

This is useful for some integrals of the form $\displaystyle\int \frac{1}{a\sin x + b\cos x + c}\ dx$, which are transformed into polynomials.

Example 73

Evaluate $\displaystyle\int \frac{dx}{\cos x + 2\sin x + 1}$

Forms to recognise

These forms are important to be able to recognise, doing so greatly simplifies the computation of the integral.

- $\displaystyle\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$
- $\displaystyle\int f'(x)f(x) = \frac{1}{2}\left(f(x)\right)^2 + c$
- $\displaystyle\int f'(x)e^{f(x)} = e^{f(x)} + c$
- $\displaystyle\int g'(x)f'(g(x)) = f(g(x)) + c$
- $\displaystyle\int \frac{f'(x)}{1+\left(f(x)\right)^2} = \tan^{-1}(f(x)) + c$

6.5.3 Integration by Parts

This method is obtained from the product rule for differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \text{hence} \qquad u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

Integration of this formula gives

$$\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx, \quad \text{or, equivalently} \quad \int u\ dv = uv - \int v\ du.$$

Integration by parts is normally used for a product in which one factor, such as a power of x, becomes simpler or disappears when differentiated. More than one application of the method may be needed.

Sometimes 'multiplication by 1' is used to express a single function as a product, i.e. $\mathrm{f}(x) = \mathrm{f}(x) \times 1$. when we know the differential but not the integral of a function.

Example 74

Find $\displaystyle\int \sin^{-1} x\ dx$

Miscellaneous examples

Example 75 $\displaystyle\int \frac{x^3+2}{x^2+2}\ dx$

Example 76 *Find* $\displaystyle\int \frac{1}{x(x^2+1)^2}\ dx.$

Example 77 *Find* $\displaystyle\int \frac{1}{\sqrt{-3x^2+12x-8}}\mathrm{d}x.$

Example 78 $\int \frac{1}{x^2\sqrt{x^2+1}}\,\mathrm{d}x$

Example 79 $\int \frac{\sin^2 x}{1+\cos x}\,\mathrm{d}x$

Example 80 $\int \sqrt{4-x^2}\ dx$

Example 81 $\int \frac{1}{\cos^4 x}\,\mathrm{d}x$

6.6 Reduction Formulae

There are many cases in which it is useful to reduce an integral involving a power of some function to one involving a lower power, e.g. if $I_n = \int \sin^n x \; dx$ we can express I_n in terms of $I_{n-2} = \int \sin^{n-2} x \; dx$. Such a **reduction formula** is commonly, but not always, found using integration by parts.

Example 82 Find $\displaystyle\int \sin^5 x \, dx$

6.7 Expressions that cannot be integrated to closed form functions

There are many expressions which cannot be integrated to simple closed form functions. The include such simple looking integrals as

$$\int e^{-x^2}\ dx, \quad \int \frac{\ln x}{x}\ dx, \int \frac{\sin x}{x}\ dx, \int \ln(\sin x))\ dx.$$

Numerical techniques can be used to evaluate definite integrals of these expressions. We may be able to use a Taylor or Maclaurin series to do this.

Example 83 Evaluate $\int_0^{0.2} \cos(x^2)$ to ten decimal places.

6.8 Integration and polar curves

6.8.1 Length of a polar curve

It is straightforward to show that the length of a continuous polar curve $r = f(\theta)$ as θ increases from a to b (provided no segment is traced more than once) is given by

$$L = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta.$$

Example 84 Find the total length L of the cardioid $r = 1 + \cos\theta$ is traced out as θ moves from zero to 2π

6.8.2 Areas bounded by polar curves

For a continuous curve $r = f(\theta)$, the infinitesimal area δA bounded by the rays $\theta = a$ and $\theta = b$ is a segment of a circle radius r with angle $\delta\theta = b - a$, thus

$$\delta A = \frac{\delta\theta}{2\pi}\pi r^2 \longrightarrow A = \int_a^b \frac{r^2}{2}\, d\theta = \int_a^b \frac{(f(\theta))^2}{2}\, d\theta.$$

Example 85 Find the area within the first quadrant of the cardioid $r = 1 - \cos\theta$.

Sketching the curve will frequently show how symmetry may be used to simplify the calculation of areas.

Example 86

(A method of finding the intersection of polar curves)

Find the points of intersection of the two cardioids $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$.

6.8.3 Volumes bounded by polar curves

For completeness we note that the volume of the solid obtained by rotation of the area of the continuous curve $r = f(\theta)$ lying between $\theta = a$ and $\theta = b$ around the initial line is

$$V = \frac{2\pi}{3}\int_a^b (f(\theta))^3 \sin\theta d\theta.$$

In practice such volumes are more easily obtained by double integration (covered in the Spring module)

6.9 Applications of integration

6.9.1 Mean value

The mean value of $f(x)$ over the interval $[\alpha, \beta]$ is equal to

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x)\ dx.$$

6.9.2 Length of arc

The length of the arc joining the points on $y = f(x)$ at which $x = \alpha$ and $x = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

6.9.3 Surface Area of a solid of rotation

If this arc is rotated once about the x-axis, the curved surface area is

$$A = 2\pi \int_{\alpha}^{\beta} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

6.10 Improper Integrals

It can happen that the definite integral of a function is finite even though x or y tends to ∞ or $-\infty$ at one of the limits of integration or at some value between them.

A limit of integration is infinite

We define $\displaystyle\int_{\alpha}^{\infty} f(x)\ dx$ to be $\displaystyle\lim_{\beta \to \infty} \int_{\alpha}^{\beta} f(x)\ dx$, if this limit exists and is finite. If the limit does not exist or is infinite then the integral *diverges*.

Example 87 $\displaystyle\int_{1}^{\infty} \frac{dx}{x^3}$

The function does not have a finite value at one of the limits of integration

If $f(x)$ does not have a finite value at $x = \alpha$ we define $\int_\alpha^\beta f(x)\ dx$ to be $\lim_{k\to\alpha} \int_k^\beta f(x)\ dx$, if this limit exists. If the limit does not exist or is infinite then the integral *diverges*.

Example 88 $\int_0^1 \frac{1}{\sqrt{1-x}}$

The function is not defined at some value within the interval of integration

Example 89 Note that the integral $\int_1^4 \frac{dx}{(x-2)^{2/3}}$ has a discontinuity at $x = 2$ and must therefore be evaluated as the sum of two improper integrals.

Exercises 6

1. Find the integrals

   (a) $\int \sinh(3x-4)\,\mathrm{d}x$ (b) $\int_0^1 \frac{5}{\sqrt{4-x^2}}\,\mathrm{d}x$ (c) $\int 3^x + \frac{\sin x}{\cos x + 1}\,\mathrm{d}x$

   (d) $\int \frac{1}{(x^2+2x+2)}\,\mathrm{d}x$ (e) $\int_3^4 \frac{1}{4x^2-20x+25}\,\mathrm{d}x$

2. Using suitable substitutions, or otherwise, find the integrals

   (a) $\int \frac{1}{\sqrt{(x-1)(x-3)}}\,\mathrm{d}x$ (b) $\int \frac{x-2}{\sqrt{x^2-1}}\,\mathrm{d}x$

   (c) $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ (d) $\int \frac{x+1}{\sqrt{x^2-x+1}}\,\mathrm{d}x$

   (e) $\int \frac{\sqrt{x^2-9}}{x}\,\mathrm{d}x$ (f) $\int \sqrt{2x-x^2}\,\mathrm{d}x$

3. Find the integrals:

   (a) $\int \frac{\cos x}{\sin x + \cos x}\,\mathrm{d}x$ (b) $\int \frac{1}{2-\sin x}\,\mathrm{d}x$ (c) $\int_{-1}^1 \sqrt{(1-x^2)}\,\mathrm{d}x$

4. Using partial fractions or otherwise, find the integrals

   (a) $\int \frac{x^2}{(x-1)(x-3)}\,\mathrm{d}x$ (b) $\int \frac{x+1}{x(x+5)^2}\,\mathrm{d}x$

   (c) $\int \frac{x^3}{(x+1)(x^2-4)}\,\mathrm{d}x$ (d) $\int \frac{x^3+4x^2-x+3}{(x-2)(x^2+1)}\,\mathrm{d}x$.

5. Prove that, for a, $b > 0$, $\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}\,\mathrm{d}x = \frac{\pi}{2ab}$.

6. Find the mean value of $y = \frac{1}{x^2+5x+6}$ over the interval $0 \le x \le 6$. By reference to a sketch, explain why the mean value you have found is reasonable.

7. A curve is given by the parametric equations $x = 2\sinh^3 t, y = 3\cosh^2 t$, for $0 \le t \le \ln 3$. Find the total length of the curve.

8. The arc of the curve $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated once about the x-axis. Find the area of the curved surface formed.

9. Given that $I_n = \int_0^1 x^n \cosh x\ \mathrm{d}x$, where n is a positive integer, prove that for $n \ge 2$,

   $$I_n = \sinh 1 - n\cosh 1 + n(n-1)I_{n-2}.$$

   Hence find, in terms of hyperbolic functions, the value of I_4.

10. Prove that if $I_n = \int \sec^n x dx$, then

- $I_0(c) = x + c$
- $I_1(x) = \ln\left(\dfrac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right)$
- $(n-1)I_n(x) = \tan x \sec^{n-2} x + (n-2)I_{n-2}(x), \quad x \geq 2.$

Using this, calculate $\int \sec^5 x$

11. Evaluate the following, if they exist.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,\mathrm{d}x$ (b) $\displaystyle\int_0^2 x^{-1/3}\,\mathrm{d}x$ (c) $\displaystyle\int_{-\infty}^{-2} \frac{-1}{x^2+4}\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi/2} \tan x\,\mathrm{d}x$ (e) $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-4x+3}}\,\mathrm{d}x.$

12. *The Gamma function $\Gamma(x)$ is defined as the improper integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt \quad x \in \mathbb{R}, x > 0$$

(a) Find $\Gamma(1)$.
(b) Prove that $\Gamma(x+1) = x\Gamma(x)$
(c) Using the results in the previous two parts find $\Gamma(2), \Gamma(3), \Gamma(4)$
(d) Given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, find $\Gamma(\frac{3}{2}), \Gamma(\frac{5}{2})$
(e) What is $\Gamma(n)$ for $n \in \mathbb{N}$?

13. Sketch the region whose area is $\int_0^{+\infty} \frac{dx}{1+x^2}$ and show that this area can also be expressed as $\int_0^1 \sqrt{\frac{1-y}{y}}dy$

14. *Integrate the following:

(i) $\displaystyle\int \sqrt{x^2-4x}dx$ (ii) $\displaystyle\int \frac{1}{x^6+x}dx$ (iii) $\displaystyle\int \sin(\ln(x))dx$ (iv) $\displaystyle\int e^{\sqrt{x}}dx$

(v) $\displaystyle\int \frac{dx}{1+x+x^2+x^3}$ $\displaystyle\int \ln(1+x^2)dx$ (vi) $\displaystyle\int \tan^{-1}\sqrt{x}dx$ (vii) $\displaystyle\int \frac{dx}{x\ln x}$

15. Evaluate the following:

(i) $\displaystyle\int_{1/e}^{e} |\ln(x)|dx$ (ii) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}+\sqrt{1+x}}$

16. Find the lengths of the following polar curves:
(i) $r = e^{4\theta}$, $\theta \in [0, 2]$ (ii) *$r = \frac{1}{\theta}$, $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$
In (ii) what can you say about the arc length of the portion of the curve that lies inside the circle $r = 1$?

17. Find the areas enclosed by the following polar curves:
(i) the cardioid $r = 2 + 2\cos\theta$
(ii) the inner loop of the limaçon $r = 1 + 2\cos\theta$ Hint: $r \leq 0$ over the interval of integration.
(iii) the area enclosed by the intersection of the circles $r = 4\cos\theta$ and $r = 4\sqrt{3}\sin\theta$
(iv) the lemniscate $r^2 = \sin 2\theta$. Hint:use the symmetry of the curve.

18. Find the volume of the solid produced when the cardioid $r = 1 + \cos\theta$ is rotated about the initial line.

19. The polar curve $r = f(\theta)$ can be parametrised as $x = f(\theta)\cos(\theta), y = f(\theta)\sin(\theta)$. Derive the formula for arc length of a polar curve.

Chapter 7

First order ordinary differential equations

7.1 Types of first order o.d.e.s

First order ordinary differential equations (o.d.e.s) contain only first derivatives, y'. The degree of the differential equation is the highest power of the derivative that appears. Thus $(y')^2 = x^2y$ is a first order o.d.e. of second degree. Where we are given the values of the variables at some point we refer to the o.d.e. as an initial value problem (IVP). We call the differential equations ordinary because there are no partial derivatives.

7.1.1 Solving a first order o.d.e

Solving the first order o.d.e. $y' = 2y$ gives $y(x) = Ce^{2x}$ and thus gives rise to a **family of solutions** depending on the parameter C. In this case y is defined for all $x \in \mathbb{R}$ and $y \in (0, \infty)$. Thus every point (x, y) lies on a solution curve, moreover it lies on only one solution curve. To find a specific, unique solution we need only specify a particular value of (x, y) such as $y(x_0) = y_0$. Thus $y' = 2y, \quad y(0) = 4$ has the solution $y(x) = 4e^{2x}$.

Always check that your solution is correct by verifying by differentiation and rearrangement that it satisfies the original differential equation

Many o.d.e.s will also have $y =$ constant as a solution (e.g. $y' = 2y$ has $y \equiv 0$). This is known as the trivial solution, you should therefore always check to see if there is a trivial solution to the o.d.e. you are solving.

7.2 Variables separable equations

These have the general form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

If we rearrange the equation to give $g(y)\dfrac{dy}{dx} = f(x)$ and then integrate both sides with respect to x we have

$$\int g(y)dy = \int f(x)dx.$$

Then we make y the subject of the equation. In principle this type of o.d.e. can always be solved, provided that f and g can be integrated to give closed form functions.

Example 90

$\dfrac{dy}{dt} = 5t^4y^2 \quad y(0) = 1$

Example 91

$\dfrac{dy}{dx} = \cos x \tan y.$

7.3 Linear equations - the integrating factor method

The equation must be in the following **specific form** (it is said to be linear in y and y');

$$\frac{dy}{dx} + yP(x) = Q(x).$$

We first transform the equation into one where the variables are separable and then apply the method of the previous section. The steps are as follows

1. Make sure the the o.d.e. is in the specific form above (by rearranging or substitution if necessary).
2. Work out the integrating factor $e^{\int P(x)dx}$ (Note that there is no constant of integration at this stage).
3. Multiply both sides of the equation by the integrating factor,

$$e^{\int P(x)dx}\frac{dy}{dx} + ye^{\int P(x)dx}P(x) = Q(x)e^{\int P(x)dx},$$

   the left hand side can be written as $\frac{d}{dx}\left(ye^{\int P(x)dx}\right)$ (check this by using the rule for differentiating products).
4. Now the equation is

$$\frac{d}{dx}ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

5. Integrate both sides with respect to x

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + c.$$

6. Make y the subject of the equation

$$y(x) = \frac{\int Q(x)e^{\int P(x)dx}dx + c}{e^{\int P(x)dx}}.$$

Example 92

$ty' = y + t^3$

Example 93

Solve the IVP $y' + 2xy = 4x \quad y(0) = 3$.

7.4 Homogeneous equations

These are equations where each of the polynomial expressions have the same order (e.g. x^2y^3 and xy^4 are both of order 5). They can be rearranged into the form $\frac{dy}{dx} = H\left(\frac{y}{x}\right)$. With the substitution $u = \frac{y}{x}$ or $y = ux$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$ by the chain rule. Thus we can rewrite the equation as

$$u + x\frac{du}{dx} = H(u)$$

which we can rearrange to give us a variables separable equation

$$x\frac{du}{dx} = H(u) - u \quad \Rightarrow \quad \int \frac{du}{H(u) - u} = \int \frac{dx}{x}$$

which we now integrate and complete the solution by substituting for u.

Example 94 $2xyy' - y^2 + x^2 = 0$

7.5 Bernoulli equations

These take the **specific** form:

$$\frac{dy}{dx} + yR(x) = y^n S(x).$$

The right hand side contains y^n so that the integrating factor method will not work without further transformation of the equation. We now make the substitution $u = y^{1-n}$ with

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}.$$

We can now transform the original equation in x and y into an equation in x and u as follows;

$$\frac{y^n}{1-n}\frac{du}{dx} + R(x)y = S(x)y^n$$

$$\frac{1}{1-n}\frac{du}{dx} + R(x)y^{1-n} = S(x) \quad \Rightarrow \quad \frac{1}{1-n}\frac{du}{dx} + R(x)u = S(x)$$

$$\frac{du}{dx} + u(1-n)R(x) = (1-n)S(x).$$

The o.d.e. is now in the correct form to be solved by the integrating factor method.

Example 95 $x\dfrac{dx}{dy} + y = x^2y^2\ln x$

7.6 Equations that can be transformed to one of the types above

Just as the substitution $u = \dfrac{y}{x}$ can be used to transform a homogeneous equation into one that can, in principle, be solved, other substitutions may also be used to transform what appear to be intractable equations, as the following miscellaneous examples show;

Example 96

$$y' = \frac{y}{t} + \frac{t-1}{2y}$$

Example 97

$$y' = y - 4t + y^2 - 8yt + 16t^2 + 4$$

Example 98

$$\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$$

Example 99

$\frac{dy}{dx} = \frac{1}{x+y}$

Exercise

As an exercise, check the four solutions above.

7.7 Second order equations which can be solved as first order equations

Provided that if y'' and y' appear in the equation, y does not, we can make the substitution $y' = p(x)$ then $y'' = p'$ and solve in the normal manner.

Example 100

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x$$

Example 101

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

7.8 First order o.d.e.s with degree greater than 1

First order o.d.e.s of second and third degree cannot in general be solved by any of the methods in this chapter. However, we can use the substitution $y' = p$. and attempt to solve for p in some cases:

Example 102

$$\left(\frac{dy}{dx}\right)^2 + (x+y)\frac{dy}{dx} + xy = 0.$$

We can sometimes solve for x

Example 103 $y = 3x\dfrac{dy}{dx} + 6y^2\left(\dfrac{dy}{dx}\right)^2$

On the other hand we can solve for y.

Example 104 $16x^2 + 2y\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right)^3 = 0$

7.9 Using Taylor series to obtain a series solution

Consider the differential equation (IVP) where we cannot obtain a solution by any of the known methods, but we do know the value of the solution at some value of x.

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

The Taylor series for y about $x = x_0$ is

$$y = y_0 + (x - x_0)\frac{dy}{dx}\bigg|_{x_0} + \frac{(x - x_0)^2}{2!}\frac{d^2y}{dx^2}\bigg|_{x_0} + \frac{(x - x_0)^3}{3!}\frac{d^3y}{dx^3}\bigg|_{x_0} \cdots$$

and we use this to establish a series solution.

Example 105

$$\frac{dy}{dx} = x^2 + y^2$$

7.10 Modeling with first order odes

We consider some examples of the use of first order o.d.e.s to build simple but useful models of the real world.

7.10.1 Population modeling

If the population is $N(t)$ the simplest model is $\frac{dN}{dt} = \rho N$, where ρ is the difference between the per capita birth and death rates. If the population at some time t_0 is N_0 the model gives us

$$N(t) = N_0 e^{\rho(t-t_0)}.$$

This very simple model is a useful way to study the early stage of the growth of cancer cells, if $r < 0$ the cells die out. If $r > 0$ the cancer grows exponentially.

For animal and insect populations a better model recognises that resources are finite and the o.d.e is now

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Here r is the birth rate and K is the so-called carrying constant for the environment. We see that $\frac{dN}{dt} = 0$ (i.e. the system is at equilibrium) when $N = 0$ (no population) and $N = K$, the maximum population that the environment can support.

We can solve this o.d.e (left as an exercise) to give the **logistic** model:

$$N(t) = \frac{N_0 K}{(K - N_0)e^{-r(t-t_0)} + N_0}.$$

Note that, for $N_0 \neq 0, \lim N_{t\to\infty} = K$ and that in theory it takes an infinite time to reach this equilibrium.

7.10.2 Carbon dating

Radioactive decay is modelled (quite precisely) by $\frac{dM}{dt} = -kM$ where $M(t)$ is the amount of radioactive material and k is a known decay constant specific to the isotope concerned. If the amount present at $t = t_0$ is M_0 the model is

$$M(t) = M_0 e^{-k(t-t_0)}.$$

Knowing the amount of radioactivity (e.g. ^{14}C) that would have been present in the wood in a tree when it was cut down, the present level of radioactivity and the rate of decay of ^{14}C it is a simple matter to calculate the age of a wooden object.

7.10.3 Forensics - time of death

Newton's law of cooling for a body at a temperature of $\theta(t)$ gives $\frac{d\theta}{dt} = k(\theta - \theta_c)$, where θ_c is the ambient temperature and k is a constant specific to the material of which the body is made. This gives

$$\theta(t) = \theta_c + \theta_0 e^{-kt}$$

where θ_0 is the temperature at t_0. Knowing the temperature of the body, that of the environment and the normal body temperature of 37.4C allows an estimation of the time of death.

7.10.4 Finance - the continuous compounding of money

If the interest rate paid on the amount of money in an account $A(t)$ is r per year, compounded over n periods each year then the amount of money in the account after m years is

$$A_m = A_0 \left(1 + e^{\frac{r}{n}}\right)^{mn}.$$

As $n \to \infty$ we know that this is $A_m = e^{rm}$. From this we have that

$$\frac{dA}{dt} = -rA.$$

We can say that an amount A_0 invested at r continuously compounded will be worth $A(t) = A_0 e^{rt}$ at time t. Conversely an amount of money A received at time t is worth Ae^{-rt} at today's value (the present value). The present value P of an annuity of A_0 per year paid for n years will be the solution of

$$\frac{dP}{dt} = A_0 e^{-rt} \longrightarrow P(t) = \frac{A_0}{r}\left(1 - e^{-rt}\right).$$

7.10.5 Free fall with air resistance

Th downward force on a falling body is mg, where g is gravity and m is mass. The air resistance is proportional to the velocity v so it is kv. Then from Newton, the downward force on the body is

$$m\frac{dv}{dt} = mg - kv \longrightarrow \frac{dv}{dt} + \frac{k}{m}v = g.$$

Then

$$v(t) = \frac{g}{k}\left(1 - e^{-kt}\right),$$

so that terminal velocity occurs when $v = \frac{g}{k}$. We find the distance fallen by solving

$$\frac{dx}{dt} = \frac{g}{k}\left(1 - e^{-kt}\right) \longrightarrow x(t) = \frac{gt}{k} - \frac{g}{k^2}\left(1 - e^{-kt}\right).$$

Exercises 7

Solve the following differential equations. Verify that your solutions do indeed solve the given o.d.e.s

1. (a) $y' = \dfrac{e^{-2x}}{y^2}$
   (b) $xy' = (x-1)y \quad y(1) = 1$
   (c) $\sqrt{1+t^2}y' = ty^3 \quad y(0) = 2$
   (d) $y' = y^2 - 1 \quad y(0) = 2$
   (e) $\sin xy' = 2y \cos x$
   (f) $yy' = 2(xy + x)$
   (g) $ye^{x+y}y' = 1$
   (h) $y' = \dfrac{2(y^2+y-2)}{x^2+4x+3}$
   (i) $y'' + (y')^2 + 1 = 0$ (Hint; substitute $y' = u$)
   (j) $xy'' = y'$
   (k) $2xy' + y = 0, \quad y(4) = 1$
   (l) $(1-x^2)y' + 4xy = 0, \quad y(0) = 2$

2. (a) $y' + y = 5e^x$
   (b) $xy' + y = x^4 - x \quad y(1) = 2$
   (c) $y' + \dfrac{2y}{x+1} = 1$
   (d) $ty' + 2y = e^t$
   (e) $xy' = 2y + x^2$
   (f) $y' + 2xy + x = e^{-x^2}$
   (g) $y' + y\tan x = \sec x$
   (h) $x^2y' + 2xy - x + 1 = 0$
   (i) $(1-x^2)y' + xy = 2x$
   (j) $y' + \dfrac{y}{1-x} = x^2 - x$
   (k) $xy' + (1+x)y = e^{-x}$
   (l) $y + y' = e^x, \quad y(0) = 2$
   (m) $(1+x^2)y' = 1 + xy, \quad y(1) = 0$
   (n) $xy' + x^2 - 3y = 0$ with (a) $y(1) = -1$ and (b) $y(-1) = 1$
   (o) $y^2 + (3xy - 4y^3)y' = 0$ (Hint; consider x as the dependent variable.)

3. (a) $x^2yy' = x^3 - y^3 \quad y(1) = 1$
   (b) $xy' = y + \sqrt{x^2+y^2}, \quad y(4) = 3$

(c) $y' = \dfrac{2x - y}{x - 2y}$

(d) $y' = \dfrac{x + y}{x - y}$

(e) $y' = \dfrac{x - y + 5}{x + y - 1}$ (Hint; substitute $x = p + A, y = q + B$ where A and B are suitable constants to transform this into a homogeneous equation.)

(f) $y' = \dfrac{2x + 2y - 1}{3x + y - 2}$

4. Use the substitutions given to solve the following equations for y

(a) $y' = (y - t)^2 - (y - t) - 1 \quad \text{let} \quad u = y - t$

(b) $y' = \dfrac{yt}{2} + \dfrac{e^{\frac{t^2}{2}}}{2y} \quad \text{let} \quad u = y^2$

(c) $y' = \dfrac{y}{1+t} - \dfrac{y}{t} + t^2(1 + t) \quad \text{let} \quad u = \dfrac{y}{1+t}$

5. Find the series solutions in ascending powers of x up to and including the term in x^3 of

(a) $\dfrac{dy}{dx} = y \cos x, \quad y(0) = 1$

(b) $\dfrac{dy}{dx} = e^{xy}, \quad y(0) = 1$

In the first case compare your solution with the result of solving the o.d.e. by direct integration and explain your conclusions.

6. Solve the following o.d.e.s. with the substitution $u = x + y$:

(a) $y' = \dfrac{1}{(x + y)^2}$

(b) $y' = \sin(x + y)$

(c) $y' = (1 - x - y) \cos x - 1$

(d) $y' = \sqrt{x + y}$

7. Solve the following o.d.e.s:

8. Solve the following o.d.e.s with the substitution $p = \dfrac{dy}{dx}$.

(a) $xy \left(\dfrac{dy}{dx}\right)^2 + (x^2 + xy + y^2)\dfrac{dy}{dx} + xy = 0$

(b) $\left(\dfrac{dy}{dx}\right)^2 - 2\dfrac{dy}{dx} \cosh x + 1 = 0$

(c) $y = x + \left(\dfrac{dy}{dx}\right)^3$

(d) $\dfrac{dy}{dx}\left(y + \dfrac{dy}{dx}\right) = x(x + y)$

(e) $y = x\dfrac{dy}{dx} + a\dfrac{dy}{dx}\left(1 - \dfrac{dy}{dx}\right)$

Chapter 8

Introduction to Fourier Series

8.1 Definition of a Fourier series

While Taylor and Maclaurin series are very useful polynomial approximations of functions, they do not approximate periodic functions effectively. For this we use Fourier series.

Consider $f(x)$ where f is 2π periodic, i.e. for all x we have $f(x) = f(x + 2\pi)$. We can approximate $f(x)$ by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx f(x)dx \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx f(x)dx.$$

(The derivation of the Fourier series will be covered in the lectures). If the function is $2L$ periodic then with a scale change we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} \cos \frac{n\pi x}{L} f(x)dx \quad b_n = \frac{1}{L} \int_{-L}^{L} \sin \frac{n\pi x}{L} f(x)dx.$$

a_0, a_n and b_n are known as Fourier coefficients.

Recap - Products of odd and even functions

Suppose that $f(x)$ is even and $g(x)$ is odd. Then $f(x) = f(-x)$ and $g(x) = -g(-x)$ so that $f(x)g(x) = f(-x)(-g(-x)) = -f(-x)g(-x)$. Thus $f(x)g(x)$ is odd.

If both $f(x)$ and $g(x)$ are even, then $f(x) = f(-x)$ and $g(x) = g(-x)$ so that $f(x)g(x) = f(-x)g(-x)$ and so $f(x)g(x)$ is even.

Lastly, if both $f(x)$ and $g(x)$ are odd then $f(x) = -f(-x)$ and $g(x) = -g(-x)$ so that $f(x)g(x) = (-f(-x))(-g(-x)) = f(-x)g(-x)$ and so $f(x)g(x)$ is even.

The rule is odd×odd=even, even×even=even and odd×even=odd. This rule helps considerably in reducing the work involved in computing Fourier series.

8.2 Integrating odd and even functions

If $f(x)$ is even then $\displaystyle\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.
If $g(x)$ is odd then $\displaystyle\int_{-L}^{L} g(x)dx = 0$ so $a_0 = 0$ in this case.
Now, if $f(x)$ is even, then $\cos\dfrac{n\pi x}{L} f(x)$ is even while $\sin\dfrac{n\pi x}{L} f(x)$ is odd.

Thus, for an even function:

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} f(x)dx = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} f(x)dx, \quad \text{and} \quad \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)dx = 0.$$

The Fourier series of an **even function contains only the cosine terms**.

If $g(x)$ is odd then $\cos\dfrac{n\pi x}{L} g(x)$ is odd while $\sin\dfrac{n\pi x}{L} g(x)$ is even, hence

$$\frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)dx = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} f(x)dx, \quad \text{and} \quad \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} g(x)dx = 0.$$

The Fourier series of an **odd function contains only the sine terms**.

In summary;

- if f is even then
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$
- if f is odd then
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Example 106

Compute the Fourier series for

$$f(x) = \begin{cases} 0 & : \quad x \in ((4j+1),(4j+3)] \\ 1 & : \quad x \in (4j-1, 4j+1] \end{cases}$$

for all $j \in \mathbb{Z}$.

Example 107

Find the Fourier series for the sawtooth wave function given by

$$g(x) = x - 2jL \quad \text{for each interval } \Big[(2j-1)L, (2j+1)L\Big], \text{ for all } j \in \mathbb{Z}.$$

Example 108

Find the Fourier series for

$$f(x) = x^2 + x, \quad -\pi < x < \pi \quad f(x) = f(x + 2\pi$$

8.3 Complex Fourier series

We have derived the identities $e^{\pm inx} \equiv \cos nx \pm i \sin nx$ and

$$\cos nx \equiv \frac{1}{2}\left(e^{inx} + e^{-inx}\right) \quad \sin nx \equiv \frac{1}{2i}\left(e^{inx} - e^{-inx}\right).$$

We can therefore express the general term of the Fourier series for a 2π periodic function $f(x)$ as

$$a_n \cos nx + b_n \sin nx = \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \frac{b_n}{2i}\left(e^{inx} - e^{-inx}\right).$$

and we can write the right hand side as

$$c_n e^{inx} + k_n e^{-inx} \quad \text{where} \quad c_n = \frac{1}{2}(a_n - ib_n) \quad \text{and} \quad k_n = \frac{1}{2}(a_n + ib_n).$$

Hence we can calculate

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x)\cos nx - if(x)\sin nx\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

$$k_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x)\cos nx + if(x)\sin nx\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx.$$

Since $k_{-n} = c_n$ we can therefore express the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

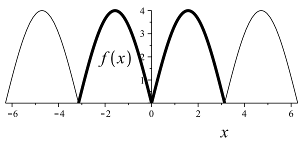
In some circumstances this form of the Fourier series is more convenient to work with than the sine and cosine series, as the following example shows.

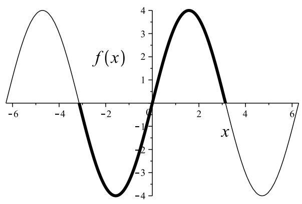
Example 109

We require the Fourier series of the function $f(x) = e^{x-2j\pi x}, \quad j \in \mathbb{Z}$ for each interval $x \in [(2j-1)\pi, (2j+1)\pi)$.

8.4 Fourier series for functions given on one interval only

Many functions are only physically interesting for a specific interval, thus the vibration of a violin string is only given over its length. The idea is to extend the function so that it becomes periodic. The extensions can be either odd (in which case we get the sine terms) or even (in which case we get the cosine terms) and are known as half-range extensions.This is illustrated in the following graph for the function $f(x) = 4\sin x, \quad x \in (0, \pi)$.





Example 110

Find the Fourier series for the odd and even half range extensions of the function

$$f(x) = \begin{cases} \frac{2x}{L} & 0 < x \leq \frac{L}{2} \\ \frac{2}{L} & \frac{L}{2} < L \end{cases}$$

8.5 Convergence of the Fourier series

We state the sufficient conditions for a function to have a convergent Fourier expansion, they are known as Dirichlet's conditions.

If $f(x)$ is a bounded periodic function that in any period has a finite number of isolated maxima and minima and a finite number of finite discontinuities (i.e. it is bounded) then the Fourier expansion of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous and to the average of the right and left hand limits of $f(x)$ at points where $f(x)$ is discontinuous.

Broadly speaking, the rate of convergence where the function has jump discontinuities will be more rapid the "smoother" the function and in particular the rate of convergence is slowest closest to a point of discontinuity. (This is known as the Gibbs phenomenon).

Exercises 8

1. Find the Fourier series of the following 2π periodic functions;

   (a)
   $$f(x) = x, \quad x \in (-\pi, \pi]$$

   (b)
   $$f(x) = x^2, \quad x \in (-\pi, \pi]$$

   (c)
   $$f(x) = \begin{cases} -1 & : \quad x \in (-\pi, 0] \\ 1 & : \quad x \in (0, \pi] \end{cases}$$

   (d)
   $$f(x) = \begin{cases} x & : \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 1 & : \quad x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases}$$

   (e) $f(x) = e^x, \quad x \in (-\pi, \pi]$ (Use the real form of the Fourier series and compare the difficulty with that using the complex form.)

2. Find the Fourier series of the following $2L$ periodic functions;

   (a) $f(x) = |x|, \quad x \in (-2, 2], \quad L = 2$

   (b) $f(x) = \dfrac{\pi x^3}{2}, \quad x \in (-1, 1], \quad L = 1$

   (c)
   $$f(x) = \begin{cases} \frac{1}{2} + x & : \quad x \in \left(-\frac{1}{2}, 0\right] \\ \frac{1}{2} - x & : \quad x \in \left(0, \frac{1}{2}\right] \end{cases} \qquad L = \frac{1}{2}$$

   (d) $f(x) = \pi \sin \pi x, \quad x \in (0, 1), \quad L = \frac{1}{2}$

3. Find the complex Fourier series for the following 2π periodic functions;

   (a)
   $$f(x) = |x|, \quad x = (-\pi, \pi]$$

   (b)
   $$f(x) = \begin{cases} -1 & : \quad x \in (-\pi, 0] \\ 1 & : \quad x \in (0, \pi] \end{cases}$$

4. $f(x) = \pi x - x^2, \quad x \in [0, \pi]$. Find the Fourier series for both the odd and the even half range extensions

5. $f(x) = \begin{cases} \pi^2 & x \in (-\pi, 0) \\ (x-\pi)^2 x \in (o, \pi) \end{cases}$ and $f(x + 2\pi) = f(x)$. Prove that the Fourier series expansion is

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \left[\frac{2}{n^2} \cos nx + \frac{(-1)^n}{n} \pi \sin nx \right] + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}.$$

Use this result to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

SAMPLE CLASS TEST 1

You should show sufficient working to demonstrate the method you have used.
You have 45 minutes.

1. Find the four complex fourth roots of $2 + 2i$ in surd form and plot them in the Argand diagram.

2. Sketch the following regions in the complex plane, indicating clearly which boundaries are included and which excluded.

$$\text{(i) } 4 < |z| \le 2 \qquad \text{(ii) } |z-1| < |z-i|.$$

3. Find the range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 6x - 4$. By restricting the range appropriately find the inverse function. Prove that your answer is correct.

4. Starting from the definitions prove

$$\text{(i) } \sinh^{-1} x = \ln(x + \sqrt{1 + x^2}) \qquad \text{(ii) } \sinh 2x = 2 \sinh x \cosh x.$$

5. (i) Find the real and imaginary parts of e^{-z^2}

   (ii) Solve the equation $\cosh z = -2$.

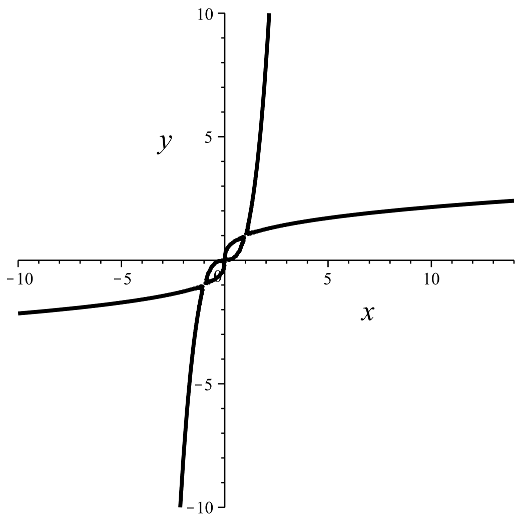
6. Find the following limits

$$\lim_{x \to 1} \frac{1-x}{1-\sqrt{x}} \qquad \text{(ii) } \lim_{x \to \infty} \frac{\sqrt{x^4 + 4x}}{\sqrt[3]{2x^6 + 1}}$$

SAMPLE CLASS TEST 2

You should show sufficient working to demonstrate the method you have used.
You have 45 minutes.

1. This is the graph of $y^3x^3 - x^4 - y^4 + yx = 0$.



   (a) Explain the shape of the graph.

   (b) Show that $\dfrac{dy}{dx} = \dfrac{3\,y^3x^2 - 4\,x^3 + y}{4\,y^3 - x - 3\,y^2x^3}$.

   (c) Find the gradients of the two tangents to the curve when $x = \frac{1}{2}$, expressing your answer in exact (i.e. surd) form.

   (d) Find the area enclosed by one of the loops. Hint: Find the equations of the upper and lower curves in a loop. [25]

2. Find the following limit: $\lim_{x\to\infty} \sqrt{x}^{\sqrt{x}}$ Hint: use l'Hôpital's rule on the logarithm.[10]

3. Show that the Maclaurin series for $\sec x$ is

$$\sec x = 1 + \frac{1}{2}\,x^2 + \frac{5}{24}\,x^4 \ldots.$$

   Hence find an approximation to $\int_0 0.1 \sec(x^2)dx$ correct to 4 decimal places. [15]

4. Evaluate the following integrals:

   (a) $\int dx(1 + x^2)\tan^{-1} x$.

   (b) $\int \sqrt{x^2 + 4\,x - 1}dx$

(c) $\displaystyle\int \frac{3\,x^2 + 4\,x - 1}{x^3 - 7\,x + 6} dx.$

(d) $\displaystyle\int \sin \ln x \, dx.$ [25]

5. Solve the following ordinary differential equations:

(a) $\sin x \dfrac{dy}{dx} = y.$

(b) $\dfrac{1}{x^2}\dfrac{dy}{dx} + xy = x.$

(c) $\dfrac{1}{xy}\dfrac{dy}{dx} + x^2 = x^2y^2.$ [15]

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Answer all five questions, they carry equal marks.

1. (a) Find the Maclaurin series for $\cos^2 2x$ up to and including the term in x^6. Hint: Start with an appropriate trigonometric formula. (7)

   (b) Explain why no Maclaurin series exists for (i) $\ln x$ and (ii) $\sqrt{x}$. (5)

   (c) By finding the Maclaurin series for $\sin(x^2)$, evaluate $\int_0^1 \sin(x^2)dx$ accurate to 3 decimal places. (8)

2. (a) Use de Moivre's Theorem to find trigonometric identities for $\cos 3x$ in terms of $\cos x$ and $\sin 3x$ in terms of $\sin x$. (14)

   (b) Hence, or otherwise evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\cos^3 x + \sin^3 x\right) dx$ exactly. (6)

3. Evaluate the following integrals

   (a) $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} dx$ Hint: substitute $u = \tan x$ (14)

   (b) $\int_0^1 \ln x dx$ Hint: Use l'Hopital's rule (6)

4. Solve the following ordinary differential equations

   (a) $(1 + x^2)\frac{dy}{dx} = x + xy^2$ (5)

   (b) $e^{-x}\frac{dy}{dx} + 2y = 1$ Hint: integrating factor method (5)

   (c) $\frac{dy}{dx} + y = y^3 e^x$ Hint: substitute $u = \frac{1}{y^2}$ (5)

   (d) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 4e^{-3x}$ Hint: substitute $u = \frac{dy}{dx}$ (5)

5. Find the Fourier series for the following period 4 function:

$$f(x) = \begin{cases} x & : \quad x \in [0, 2) \\ 1 & : \quad x \in [-2, 0) \end{cases}$$

(20)