Answers to Exercises in MS100 Notes (Part B)

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1. (a) $x^2 - 9$ must not be negative, so the maximal domain is $\{x \in \mathbf{R} \ : \ |x| \geq 3\}$, i.e. $(-\infty, -3] \cup [3, \infty)$.

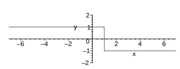
   (b) The denominator must not be zero, so the maximal domain is $\mathbf{R} \setminus \{-1, 3\}$.

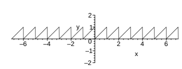
2. $\mathrm{f}(x) = (x-2)^2 - 7$ so the minimum value of $\mathrm{f}(x)$ is -7 (when $x = 2$). Thus the range of f is $\mathrm{f}(x) \geq -7$, or $[-7, \infty)$.

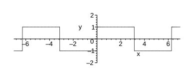
   f is not bijective so has no inverse, but if we restrict the domain to $[2, \infty)$and the codomain to $[-7, \infty)$ then the inverse is $\mathrm{f}^{-1} : [-7, \infty) \to [2, \infty)$ by $x \mapsto \sqrt{x+7} + 2$.

3. $-1, \quad \pi, \quad 1, \quad 1.$

4. (i) $y = \mathrm{sgn}\ (1-x)$, (ii) $y = x - \lfloor x \rfloor$, (iii) $y = \mathrm{sgn}\ (\sin x)$.







5. If $y = 1 - 5x$ then $x = \dfrac{1-y}{5}$, so $\mathrm{f}^{-1} : (1, \infty) \to (-\infty, 0)$ by $x \mapsto \dfrac{1-x}{5}$,

   $\mathrm{g}^{-1} : [0, \infty) \to [1, \infty)$ by $x \mapsto x^2 + 1$.

   Range of g is $\mathrm{g}(x) \geq 0$ and this does not intersect the stated domain of f, so $\mathrm{f} \circ \mathrm{g}$ cannot be defined.

   Range of f is $\mathrm{f}(x) > 1$ and we can apply g to this, so $\mathrm{g} \circ \mathrm{f} : (1, \infty) \to (0, \infty)$ by $x \mapsto \sqrt{-5x}$.

6. f is even; its maximal domain is $\mathbf{R}$.

   g is one-to-one; its maximal domain is $\{x \in \mathbf{R} \ : \ x \neq 2\}$.

   h is odd, periodic (with minimal period π) and onto; its maximal domain is

   $\{x \in \mathbf{R} \ : \ x \neq (2n+1)\pi/2\}$ where $n \in \mathbf{N}$.

   $(\mathrm{f} \circ \mathrm{f})(x) = (x^2 + 1)^2 + 1$, with domain $\mathbf{R}$.

   $(\mathrm{f} \circ \mathrm{g})(x) = \left(\dfrac{2x}{x-2}\right)^2 + 1$, with domain $\{x \in \mathbf{R} \ : \ x \neq 2\}$.

   $(\mathrm{g} \circ \mathrm{f})(x) = \dfrac{2x^2 + 2}{x^2 - 1}$, with domain $\{x \in \mathbf{R} \ : \ |x| \neq 1\}$.

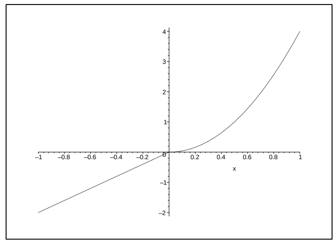
   $(\mathrm{g} \circ \mathrm{h})(x) = \dfrac{2 \tan 2x}{\tan 2x - 2}$, with domain $\{x \in \mathbf{R} \ : \ \tan 2x \neq 2\}$.

7. Let $y = \dfrac{2x+1}{x^2+2}$, so $yx^2 - 2x + (2y-1) = 0$.

   This has real roots for x when $4 - 4y(2y-1) \geq 0$, so $2y^2 - y - 1 \leq 0$, so $(2y+1)(y-1) \leq 0$.

   Hence $-\dfrac{1}{2} \leq y \leq 1$, so the range of f is $\left[-\dfrac{1}{2}, 1\right]$.

8. This is an example of a *piecewise-defined function.* The graph is:



Inverse is $\mathrm{f}^{-1} : x \mapsto \left\{ \begin{array}{ll} \frac{1}{2}x, & -2 \le x < 0 \\ \frac{1}{2}\sqrt{x}, & 0 \le x \le 4 \end{array} \right\}$.

9. We know that composition of functions is associative, so

$(\mathrm{f} \circ \mathrm{g}) \circ (\mathrm{g}^{-1} \circ \mathrm{f}^{-1}) = \mathrm{f} \circ (\mathrm{g} \circ \mathrm{g}^{-1}) \circ \mathrm{f}^{-1} = (\mathrm{f} \circ \mathrm{id}) \circ \mathrm{f}^{-1} = \mathrm{f} \circ \mathrm{f}^{-1} = \mathrm{id}$.

Similarly $(\mathrm{g}^{-1} \circ \mathrm{f}^{-1}) \circ (\mathrm{f} \circ \mathrm{g}) = \mathrm{id}$, showing that $(\mathrm{g}^{-1} \circ \mathrm{f}^{-1})$ is the inverse of $(\mathrm{f} \circ \mathrm{g})$.

10. (a) Let the partial fractions be $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+x+1}$.

Then $A(x^2+x+1) + (Bx+C)(x+1) \equiv 4x - 3$.

$x = -1$ gives $A = -7$. $A + B = 0$ so $B = 7$. $A + C = -3$ so $C = 4$.

Thus we get $\dfrac{-7}{x+1} + \dfrac{7x+4}{x^2+x+1}$.

(b) Let the partial fractions be $\dfrac{A}{x+1} + \dfrac{B}{x-5} + \dfrac{C}{(x-5)^2}$.

Then $A(x-5)^2 + B(x+1)(x-5) + C(x+1) \equiv 2x$. $x = -1$ gives $A = -\frac{1}{18}$.

$x = 5$ gives $C = \frac{5}{3}$. $A + B = 0$ so $B = \frac{1}{18}$.

Thus we get $-\dfrac{1}{18(x+1)} + \dfrac{1}{18(x-5)} + \dfrac{5}{3(x-5)^2}$.

(c) Divide first to get $x - 7 + \dfrac{37x+85}{(x+3)(x+4)}$.

Let the fractional term be $\dfrac{A}{x+3} + \dfrac{B}{x+4}$ so $A(x+4) + B(x+3) \equiv 37x + 85$.

Then $A = -26, B = 63$ and we get $x - 7 - \dfrac{26}{x+3} + \dfrac{63}{x+4}$.

(d) Let the partial fractions be $\dfrac{A}{x+3} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

Then $A(x+1)^2 + B(x+1)(x+3) + C(x+3) \equiv 3x + 7$.

$x = -1$ gives $C = 2$. $x = -3$ gives $A = -\frac{1}{2}$. $A + B = 0$ so $B = \frac{1}{2}$.

Thus we get $-\dfrac{1}{2(x+3)} + \dfrac{1}{2(x+1)} + \dfrac{2}{(x+1)^2}$.