

**Chapter 5 Exercises**  
**Solutions**

1. A company can produce six different types of fruit juice at its factory. Each type of juice requires time on three different types of machine (juicer, mixer and pasteuriser) but production is limited by the time available on each of the machines. The company wishes to maximise the profit made by producing these juices. The data is summarised below, in terms of minutes per litre required on each machine. Note that production of fractions of a litre of juice is possible.

	Juice						Availability
	Orange	Apple	Pineapple	Tomato	Grapefruit	Tropical	
Juicer	1	2	3	1	2	4	110 hrs/day
Mixer	1	1	2	2	1	2	80 hrs/day
Pasteuriser	1	1	2	2	3	3	150 hrs/day
Profit £	0.10	0.20	0.25	0.30	0.35	0.40	

- (a) Formulate this problem as a Linear Programming problem.

- The decision variables are  $x_i$  for  $i = 1 \dots 6$ , the numbers of litres of orange, apple, pineapple, tomato, grapefruit and tropical fruit juice produced daily respectively.

- The objective function to be maximised is the daily profit  $\mathcal{L}z$  which is given by

$$z = 0.1x_1 + 0.2x_2 + 0.25x_3 + 0.3x_4 + 0.35x_5 + 0.4x_6.$$

- The constraints are as follows, in units of minutes

$$x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 + 4x_6 \leq 6600$$

$$x_1 + x_2 + 2x_3 + 2x_4 + x_5 + 2x_6 \leq 4800$$

$$x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 + 3x_6 \leq 9000$$

- The non-negativity conditions are

$$x_i \geq 0, \quad i = 1, 2, 3, 4, 5, 6$$

The Linear Programming problem is now to maximise the objective function  $z$  subject to the constraints and the non-negativity conditions.

- (b) Solve this problem using Excel.

[Include a copy of the sheet that defines the problem as well as the corresponding Answer Report and Sensitivity Report.]

See attached sheets.

- (c) From your solution, state the number of litres of each type of juice that should be produced daily in order to maximise the profit. What is the corresponding profit that is made?

To maximise the profit, the number of litres of each type of juice that should be made each day is 0, 700, 0, 1000, 2100 and 0 respectively. The corresponding daily profit is £1175.

- (d) Explain why there is a limit on the number of different juices that can appear in the optimal solution. What is this limit?

The problem in standard form involves six decision variables, together with three slack variables arising from the constraints. The number of (non-zero) basic variables is the same as the number of constraints and so at most three of these eight variables will be non-zero. Thus, at least three of the decision variables must be zero at the optimal solution.

- (e) *The company can reorganise the machine shop workforce at zero extra cost in such a way as to provide 1000 minutes of additional machine time. On which machine/s should the additional time be used and how much additional profit can now be produced. Justify your answer.*

*[Your answers to parts (e), (f) and (g) should **only** use information from the Sensitivity Report.]*

From the sensitivity report, we note that the allowable increase in juicer minutes is 3000, in mixer minutes 4200 and in pasteuriser minutes 1400. Thus adding 1000 extra minutes on any machine will not change the corner point corresponding to the optimal solution.

The change in the profit corresponding to an increase in machine time is given by the shadow price times the increase in hours per day. For juicers, the shadow price is £0.0333 and so the increase in profit from an extra 1000 minutes per day machine time will be £33.3 per day. For mixers the shadow price is £0.05833 and so the increase in profit from an extra 1000 minutes per day machine time will be £58.3 per day. For pasteurisers the shadow price is £0.075 and so the increase in profit from an extra 1000 minutes per day machine time will be £75 per day.

Clearly the company should increase the number of hours available on the pasteurising machine, resulting in total profits of £1250 per day.

- (f) *The company wants to maintain the optimal production schedule determined in part (b). However, it wants to increase its profit further by increasing the profit on one of its juices. The profit can only be raised on one type of juice. On which juice should the profit be increased, and by how much, if the current solution is to remain optimal? Justify your answer.*

In the optimal production schedule, only apple, tomato and grapefruit juice are produced, therefore the company must raise the profit on one of these.

From the adjustable cells section of the sensitivity report, the allowable increase on apple juice profit is £0.15 per litre, which when multiplied by the 700 litres produced per day gives an additional profit of £105 per day. The allowable increase on tomato juice profit is £0.1 per litre, which when multiplied by the 1000 litres produced each day gives an additional profit of £100 per day. The allowable increase on grapefruit juice profit is £0.1167 per litre, which when multiplied by the 2100 litres produced each day gives an additional profit of £245 per day.

Raising the profit on grapefruit juice by £0.1167 gives the biggest increase in overall profit, so this is what should be done, provided the company can sell all the grapefruit juice it produces at this price.

- (g) *The company is having difficulty selling its apple juice and wishes to reduce its price and thus its profit on this product. To what level can the profit on apple juice be reduced while still maintaining the same optimum production schedule?*

The allowable decrease in the profit on apple juice is £0.05 per litre so that the profit can be reduced to £0.15 per litre without changing the production schedule.

*There is a strike of distribution workers so that only a total of 3000 litres of juice can now be shipped daily and there are no storage facilities. What should the new production schedule be and what is the corresponding profit?*

*[Include a copy of the sheet that defines the new problem as well as the corresponding Answer Report.]*

There is now an additional constraint:  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3000$ .

Adding this to the table and solving the new problem gives a new optimal production schedule of 0, 0, 0, 0, 2700, 300 litres per day of orange, apple, pineapple, tomato, grapefruit and tropical fruit juice respectively. The new profit is £1065 per day.

See the Excel worksheets.

2. A brewery produces three types of beer - light, regular and dark. A vat of light beer requires 6 bags of barley, 1 of sugar and 1 of hops, a vat of regular beer requires 4 bags of barley, 3 of sugar and 1 of hops while a vat of dark beer requires 2 bags of barley, 2 of sugar and 4 of hops. Each week the company has available 680 bags of barley, 1200 bags of sugar and 450 bags of hops. The profit per vat of light beer is £10, of regular beer £20 and of dark beer £30. Each vat is ready to be shipped after one week. The company's objective is to maximise its weekly profit.

Using Excel Solver find the optimum solution. (Your answer must **include the Answer and Sensitivity Reports relating to this and each of the subsequent questions**)

Using the **Answer and Sensitivity Reports only** answer the following questions, carefully explaining your reasoning.

- (a) *How much does profit on light beer have to increase before it the company should produce it?*

(See attached Answer and Sensitivity reports)

The profit on light beer would need to rise by £12 before it became profitable to produce it.

- (b) *More sugar has become available at £3 per bag, what should the company do?*

Sugar is abundant, the constraint is not binding (active). The company should not buy more sugar.

- (c) *The company is able to purchase **either** additional barley at £1 per bag **or** additional hops at £3 per bag . What should the company do and how much more profit could be made.*

The shadow prices of both barley and hops are the same (£5), so that the marginal profit per bag is  $£(5 - 1) = £4$  per bag for barley and £2 per bag for hops. The maximum additional amount of barley that can be used without other constraints being breached is 240.43 bags so the maximum additional profit from increasing the amount of barley used is  $4 \times 241.43 = £985.72$ , while the maximum additional amount of hops that can be used is 910 bags, so that the maximum additional profit from increasing the amount of hops is  $910 \times 2 = £1820$ . The company should thus purchase 910 additional bags of hops.

- (d) *There is a strike of delivery drivers which means that the company can only ship 160 vats per week and there are no storage facilities. What should then company do?*

We need to add an additional constraint to the original problem, such that the sum of the three optimum values is not greater than 160. The solution is to produce  $63\frac{1}{3}$  vats of regular beer and  $96\frac{2}{3}$  vats of dark beer per week and a weekly profit of £6166.67

- (e) *The marketing department have proposed producing a premium lager for which each vat requires 1 bag of barley, 1 bag of sugar and 6 bags of hops. What profit per vat would have to be obtained for the lager to be profitable for the company to produce and what would be the production schedule at that point. Comment on the significance of the level of profit.*

We modify the original problem to include premium lager and we try a high level of profit, say £40 per vat. The sensitivity analysis shows we can decrease the profit on premium lager by £5 per vat. At £35 per vat the profit is £5650 at the optimum solution, which is  $219\frac{9}{17}$  vats of regular beer and  $39\frac{7}{17}$  vats of lager. This is precisely the same as the profit in the original problem, thus any point on the line connecting the original and this solution would be optimal. Thus the profit on premium lager must be at least £35 for it to be profitable for the company to produce it.

- (f) *The finance department want at least 100 vats of dark beer to be produced each week since the profit margin is so high. The sales department insists that some regular beer is produced but the finance department insists that not more than 20 vats of this beer is made. What would the new optimal solution be and would it be profitable for the company to do this?*

Applying the two new constraints to the original problem and solving gives an optimum solution of 30 vats of light beer, 20 vats on regular beer and 100 vats of dark beer, with a weekly profit of £3790. As this is less than the original optimum profit, the company should not make this change.

3. *A factory produces paint in six colours: red, blue, green, yellow, purple and orange. The amount of paint it produces is limited by the cost of materials, and the time available for mixing and packing the paints. The factory owners wish to maximize the profit made by producing these paints. The data is summarised below.*

	Paint						Availability
	Red	Blue	Green	Yellow	Purple	Orange	
Cost of materials (£/litre)	1	2	3	2	4	2	Up to £400 /day
Mixing time (mins/litre)	1	1	2	1	2	2	Up to 8 hours/day
Packing time (mins/litre)	2	2	4	2	4	2	Up to 8 hours/day
Profit (£/litre)	3	3	5	3	6	4	

- (a) *Formulate this problem as a Linear Programming problem.*

**Maximise**

$$z = 4x_1 + 3x_2 + 5x_3 + 3x_4 + 6x_5 + 4x_6$$

**Subject to**

$$x_1 + 2x_2 + 3x_3 + 2x_4 + 4x_5 + 2x_6 + x_7 = 400$$

$$x_1 + x_2 + 2x_3 + x_4 + 2x_5 + 2x_6 + x_8 = 480$$

$$2x_1 + 2x_2 + 4x_3 + 2x_4 + 4x_5 + 2x_6 + x_9 = 480$$

and

$$x_j \geq 0 \quad \text{for } j \in [1, 9]$$

- (b) *Solve this problem using Excel.*

See the Excel worksheet

- (c) *From your solution, state the number of litres of each type of paint that should be produced each day in order to maximise the profit. What is the corresponding profit that is made?*

The optimal solution is to produce 80 liters of red paint and 160 litres of orange paint each day.

- (d) *Even if the numbers in the data table are changed, it will never be optimal to produce all the different colours of paint. What is the maximum number of different colours in an optimal solution? Explain your answer.*

In an LP problem the number of basic variables (some of which may be zero) must be equal to the number of constraints in the problem. Thus no more than three different varieties of paint can be produced.

- (e) *The factory owners can invest some money in order to raise their profits. Either they can increase the daily budget for materials or they can hire another mixing machine at a cost of £0.40 per minute or another packing machine at a cost of £0.50 per minute. Which should they do, and what is the maximum increase in profit that could be expected after costs are taken into account? Justify your answers.*

*[Your answer should only use information from the Sensitivity Report]*

We find that the mixing constraint is non-binding so there is no benefit in additional time available on the mixing machine.

The maximum allowable increase in the materials resource is £80 worth. The shadow price of materials is 1 which means that at this level the additional profit is  $£1 \times 80 = £80$  so there is no benefit in additional materials being available.

The maximum allowable increase in the time available on the packing machine is 320 minutes per day. The shadow price of time on the packing machine is 1 so the additional profit would be £320, less the cost of the extra time which is  $£0.04 \times 320 = £128$ . The additional profit is £192.

- (f) *The company wants to maintain the optimal production schedule determined in part (b). However, it wants to increase its profit further by increasing the profit on one of its paints. On which colour should the profit be increased, and by how much, if the current solution is to remain optimal? Justify your answer.*

*[Your answer should only use information from the Sensitivity Report.]* The profit on red paint can be increased by £1 and that on orange paint by £2. Thus the extra profit if the price of red paint is increased to the maximum is £80 while on orange paint it is £160. Thus the price of orange paint should be increased.

- (g) *Owing to a shortage of warehouse space, the factory can now only make 200 litres of paint a day. What should the new production schedule be and what is the corresponding profit?*

See the Excel worksheet.

**Note:** *the scenarios described in parts (e), (f) and (g) are independent of each other. They should be considered individually, not together.*