Chapter 7

Transportation Problems

7.1 Modelling the transportation problem

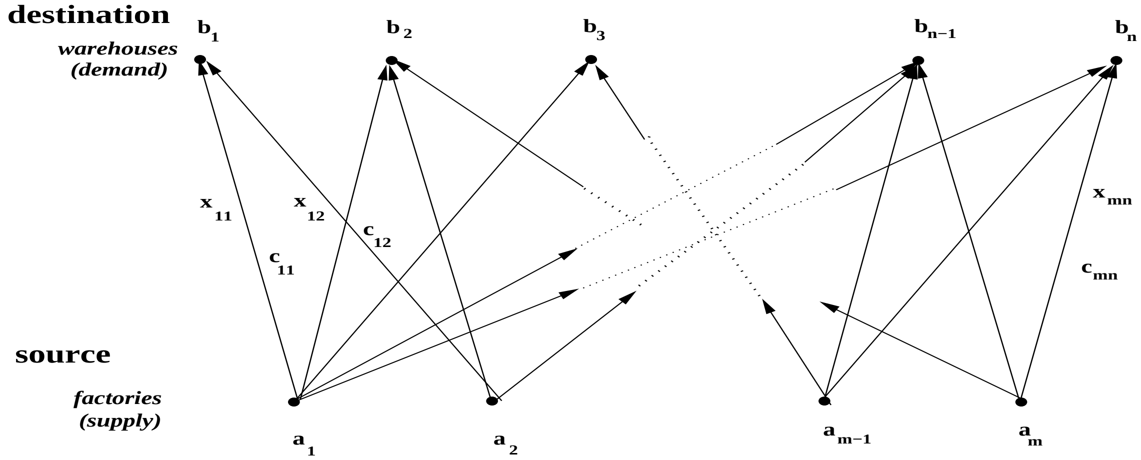
The transportation problem is concerned with finding the minimum cost of transporting a single commodity from a given number of sources (e.g. factories) to a given number of destinations (e.g. warehouses). These types of problems can be solved by general network methods, as in Chapter 9, but here we use a specific transportation algorithm.

The data of the model include

1. The level of supply at each source and the amount of demand at each destination.

2. The **unit** transportation cost of the commodity from each source to each destination.

Since there is only one commodity, a destination can receive its demand from more than one source. The objective is to determine how much should be shipped from each source to each destination so as to minimise the total transportation cost.



This figure represents a transportation model with m sources and n destinations. Each source or destination is represented by a node. The route between a source and destination is represented by an arc joining the two nodes. The amount of supply available at source i

is a_i, and the demand required at destination j is b_j. The cost of transporting one unit between source i and destination j is c_{ij}.

Let x_{ij} denote the quantity transported from source i to destination j. The cost associated with this movement is cost×quantity $= c_{ij}x_{ij}$. The cost of transporting the commodity from source i to all destinations is given by

$$\sum_{j=1}^{n} c_{ij}x_{ij} = c_{i1}x_{i1} + c_{i2}x_{i2} + \cdots + c_{in}x_{in}.$$

Thus, the total cost of transporting the commodity from all the sources to all the destinations is

$$\begin{aligned}\text{Total Cost} &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ &= c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{1n}x_{1n} + \\ &\quad c_{21}x_{21} + c_{22}x_{22} + \cdots + c_{2n}x_{2n} + \\ &\quad \vdots \\ &\quad c_{m1}x_{m1} + c_{m2}x_{m2} + \cdots + c_{mn}x_{mn}\end{aligned}$$

In order to minimise the transportation costs, the following problem must be solved:

$$\begin{aligned}\textbf{Minimise} &\quad z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \\ \textbf{subject to} &\quad \sum_{j=1}^{n} x_{ij} \leq a_i \text{ for } i = 1, \ldots, m \\ \textbf{and} &\quad \sum_{i=1}^{m} x_{ij} \geq b_j \text{ for } j = 1, \ldots, n \\ \textbf{where} &\quad x_{ij} \geq 0 \text{ for all } i \text{ and } j.\end{aligned}$$

The first constraint says that the sum of all shipments from a source cannot exceed the available supply. The second constraint specifies that the sum of all shipments to a destination must be at least as large as the demand.

The above implies that the total supply $\sum_{i=1}^{m} a_i$ is greater than or equal to the total demand $\sum_{j=1}^{n} b_j$. When the total supply is **equal** to the total demand (i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$) then the transportation model is said to be **balanced**. In a balanced transportation model, each of the constraints is an equation:

$$\begin{aligned}\sum_{j=1}^{n} x_{ij} &= a_i \text{ for } i = 1, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j \text{ for } j = 1, \ldots, n.\end{aligned}$$

A transportation model in which the total supply and total demand are **unequal** is called **unbalanced**. It is always possible to balance an unbalanced transportation problem.

Example 1 Balanced transportation model.

Consider the following problem with 2 factories and 3 warehouses:

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
Demand	7	10	13	

$$\begin{aligned} \text{Total supply} &= 20 + 10 = 30 \\ \text{Total demand} &= 7 + 10 + 13 = 30 \\ &= \text{Total supply} \end{aligned}$$

Since Total supply = Total demand, the problem is balanced.

Example 2 Unbalanced transportation model.

There are two cases to consider, namely excess demand and excess supply.

1. Suppose the demand at warehouse 1 above is 9 units. Then the total supply and total demand are unequal, and the problem is unbalanced. In this case it is not possible to satisfy all the demand at each destination simultaneously.

   We reformulate the model as follows: since demand exceeds supply by 2 units, we introduce a **dummy source** (i.e. a fictitious factory) which has a capacity of 2. The amount shipped from this dummy source to a destination represents the shortage quantity at that destination.

   It is necessary to specify the costs associated with the dummy source. There are two situations to consider.

   (a) Since the source does not exist, no shipping from the source will occur, so the unit transportation costs can be set to zero.

   (b) Alternatively, if a **penalty cost**, P, is incurred for every unit of unsatisfied demand, then the unit transportation costs should be set equal to the unit penalty costs.

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
dummy	P	P	P	2
Demand	7	10	13	

   In effect we are allocating the shortage to different destinations.

2. If supply exceeds demand then a dummy destination is added which absorbs the surplus units. Any units shipped from a source to a dummy destination represent a surplus at that source. Again, there are two cases to consider for how the unit transportation costs should be determined.

   (a) Since no shipping takes place, the unit transportation costs can be set to zero.

   (b) If there is a **cost for storing** , S, the surplus production then the unit transportation costs should be set equal to the unit storage costs.

	Warehouse 1	Warehouse 2	Warehouse 3	dummy	Supply
Factory 1	c_{11}	c_{12}	c_{13}	S	20
Factory 2	c_{21}	c_{22}	c_{23}	S	10
Demand	7	10	13	4	

   Here we are allocating the excess supply to the different destinations.

From now on, we will discuss balanced transportation problems only, as any unbalanced problem can always be balanced according to the above constructions.

7.2 Solution of the transportation problem

A balanced transportation problem has Total supply = Total demand which can be expressed as

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \tag{7.1}$$

A consequence of this is that the problem is defined by $n + m - 1$ supply and demand variables since, if a_i, $i = 2, 3, \ldots, m$ and b_j, $j = 1, 2, \ldots, n$ are specified, then a_1 can be found from (7.1). This means that one of the constraint equations is not required. Thus, a balanced transportation model has $n + m - 1$ independent constraint equations.

Since the number of basic variables in a basic solution is the same as the number of constraints, solutions of this problem should have $n + m - 1$ basic variables which are non-zero and all the remaining variables will be non-basic and thus have the value zero.

7.2.1 Starting the algorithm: finding an initial basic feasible solution

We consider two ways of constructing initial basic feasible solutions for a transportation problem, i.e. allocations with $n + m - 1$ basic variables which satisfy all the constraint equations.

Method 1: The North-West Corner Method

Consider the problem represented by the following **transportation tableau**. The number in the bottom right of cell (i, j) is c_{ij}, the cost of transporting 1 unit from source i to destination j. Values of x_{ij}, the quantity actually transported from source i to destination j, will be entered in the top left of each cell. Note that there are 3 factories and 4 warehouses and so $m = 3$, $n = 4$.

	W_1	W_2	W_3	W_4	Supply
F_1	10	0	20	11	20
F_2	12	7	9	20	25
F_3	0	14	16	18	15
Demand	10	15	15	20	

The **north-west corner method** generates an initial allocation according to the following procedure:

1. Allocate the maximum amount allowable by the supply and demand constraints to the variable x_{11} (i.e. the cell in the top left corner of the transportation tableau).
2. If a column (or row) is satisfied, cross it out. The remaining decision variables in that column (or row) are non-basic and are set equal to zero. If a row and column are satisfied simultaneously, cross only one out (it does not matter which).
3. Adjust supply and demand for the non-crossed out rows and columns.
4. Allocate the maximum feasible amount to the first available non-crossed out element in the next column (or row).
5. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

For the above example:

- $x_{11} = 10$. Cross out column 1. The amount left in row 1 is 10.
- $x_{12} = 10$. Cross out row 1. 5 units are left in column 2.
- $x_{22} = 5$. Cross out column 2. 20 units are left in row 2.
- $x_{23} = 15$. Cross out column 3. 5 units are left in row 2.
- Only column 4 is now left and so both the remaining variables x_{24} and x_{34} will be basic. The only feasible allocation of the 5 units in row 2 and the 15 units in row 3 is to allocate $x_{24} = 5$ and $x_{34} = 15$, which also ensures that the demand in column 4 is satisfied.

This provides the initial basic feasible solution $x_{11} = 10$, $x_{12} = 10$, $x_{22} = 5$, $x_{23} = 15$, $x_{24} = 5$, $x_{34} = 15$. The remaining variables are non-basic and therefore equal to zero. The solution has $m + n - 1 = 6$ basic variables as required.

The values of the basic variables x_{ij} are entered in the top left of each cell. There should always be $m + n - 1$ of these; in certain (degenerate) cases some of them may be zero. They must always add up to the total supply and demand in each row and column.

Note that some books position the data differently in the cells of the tableau.

Method 2: The Least-Cost Method

This method usually provides a better initial basic feasible solution than the North-West Corner method since it takes into account the cost variables in the problem.

1. Assign as much as possible to the cell with the smallest unit cost in the entire tableau. If there is a tie then choose arbitrarily.
2. Cross out the row or column which has satisfied supply or demand. If a row and column are both satisfied then cross out only one of them.
3. Adjust the supply and demand for those rows and columns which are not crossed out.
4. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

	W_1	W_2	W_3	W_4	Supply
F_1	10	0	20	11	20
F_2	12	7	9	20	25
F_3	0	14	16	18	15
Demand	10	15	15	20	

For the above example:

- Cells $(1, 2)$ and $(3, 1)$ both have zero cost so we arbitrarily choose the first and assign $x_{12} = 15$. Cross out column 2. The amount left in row 1 is 5.
- $x_{31} = 10$. Cross out column 1. The amount left in row 3 is 5.
- $x_{23} = 15$. Cross out column 3. The amount left in row 2 is 10.
- Only column 4 is now left and so all the variables in this column will be basic. The only feasible allocation is $x_{14} = 5$, $x_{24} = 10$ and $x_{34} = 5$.

This provides the initial basic feasible solution $x_{12} = 15$, $x_{31} = 10$, $x_{23} = 15$, $x_{14} = 5$, $x_{24} = 10$, $x_{34} = 5$. All the other variables are non-basic and are therefore equal to zero. Again, we have 6 basic variables as required.

7.2.2 Checking for optimality

So far we have only looked at ways of obtaining an initial basic feasible solution to the balanced transportation problem. We now develop a method for checking whether the current basic feasible solution is optimal. For illustrative purposes, we will start with the initial basic feasible solution that was provided by the North-West Corner method. Usually, initial basic feasible solutions obtained by the Least-Cost method (or other methods given in many text-books, such as Vogel's method) will give better starting configurations.

Suppose that the cost c_{ij} of transporting 1 unit from source i to destination j is made up of a **dispatch cost** λ_i and a **reception cost** μ_j so that

$$\lambda_i + \mu_j = c_{ij}$$

whenever x_{ij} is a basic variable.

Remarks

- The total number of λ_i and μ_j variables is $n+m$. However, there are only $n+m-1$ basic variables. Thus, we are free to choose *one* of the λ_i's or μ_j's arbitrarily. It is usual to set $\lambda_1 = 0$.
- These "costs" can take negative values if required.

Considering only these dispatch and reception costs, it would cost $\lambda_i + \mu_j$ to send 1 unit from source i to destination j. For (i, j) not corresponding to a basic variable, it will often be the case that $\lambda_i + \mu_j \neq c_{ij}$. In particular, if $\lambda_i + \mu_j > c_{ij}$ for a particular (i, j) not corresponding to a basic variable, then there would be a benefit from sending more goods that way.

So let $s_{ij} = c_{ij} - \lambda_i - \mu_j$. The s_{ij} values are entered in the top right of the cells. Then s_{ij} is the change in cost due to allocating 1 extra unit to cell (i, j) (in fact it is a shadow price). If any s_{ij} is negative (so that $\lambda_i + \mu_j > c_{ij}$), then the total cost can be reduced by allocating as many units as possible to cell (i, j). However, if *all* the s_{ij} are positive then it will be more expensive to change any of the allocations and so we have found a minimum cost.

Thus the procedure is as follows:

1. Assign values of λ_i and μ_j to the columns.
2. Enter the values $s_{ij} = c_{ij} - \lambda_i - \mu_j$ in every cell.
3. If all the s_{ij}'s are non-negative, we have an optimal solution.

Assigning values of λ_i and μ_j to our example with the initial basic feasible solution given by the North-West Corner method, gives the following transportation tableau:

	10	0	2	13
0	10 10	10 0	18 20	−2 11
7	−5 12	5 7	15 9	5 20
5	−15 0	9 14	9 16	15 18

Adding the s_{ij} variables to each cell, we find three negative values and so the solution is not optimal.

7.2.3 Iterating the algorithm

If the current solution is not optimal, we need a method for moving to a better basic feasible solution. As previously, this involves changing only one variable in the basis so again we must identify an entering variable and a departing variable in the basis.

Determining the entering variable

If the current solution is not optimal, choose the cell with the **most negative** value of s_{ij} as the entering variable, as the cost will be reduced most by using this route.

For our example, the most negative value is s_{31} and so the entering variable is x_{31}.

Determining the leaving variable

We construct a **closed loop** that starts and ends at the entering variable and comprises successive horizontal and vertical segments whose end points must be basic variables (except those associated with the entering variable). It does not matter whether the loop is clockwise or anticlockwise.

Starting Tableau

	10	0	2	13
0	10 10	10 0	20	11
7	12	5 7	15 9	5 20
5	0	14	16	15 18

We now see how large the entering variable can be made without violating the feasibility conditions. Suppose x_{31} increases from zero to some level $\varepsilon > 0$. Then x_{11} must change to $10 - \varepsilon$ to preserve the demand constraint in column 1. This has a knock on effect for x_{12} which must change to $10 + \varepsilon$. This process continues for *all* the corners of the loop.

The departing variable is chosen from among the corners of the loop which *decrease* when the entering variable increases above zero level. It is the one with the smallest current value, as this will be the first to reach zero as the entering variable increases. Any further increase in the entering variable past this value leads to infeasibility.

Clearly x_{22} is the departing variable in this case. The entering variable x_{31} can increase to 5 and feasibility will be preserved.

New values of the λ_i's and the μ_j's can now be assigned and the test for optimality applied. If the solution is still not optimal, new entering and departing variables must be determined and the process repeated.

Second Tableau

	10	0	17	28
0	5 10	15 0	3 20	−17 11
−8	10 12	15 7	15 9	10 20
−10	5 0	24 14	9 16	10 18

As before, we construct λ_i's and μ_j's which satisfy $\lambda_i + \mu_j = c_{ij}$ for the **basic** variables and enter the values of $s_{ij} = c_{ij} - \lambda_i - \mu_j$ for every cell. This tableau is not optimal as one of the s_{ij}'s is negative. The most negative value of s_{ij} occurs for x_{14} and so this is the entering variable.

Next we construct a loop which only involves the four corner cells in this case. The maximum that ε can be without one of the variables going negative is 5 which gives $x_{11} = 0$ and so this is therefore the departing variable.

Third Tableau

	−7	0	0	11
0	17 10	15 0	20 20	5 11
9	10 12	−2 7	15 9	10 20
7	10 0	7 14	9 16	5 18

We construct λ_i's, μ_j's and s_{ij}'s as before, and then check for optimality. The tableau is not optimal as x_{22} is negative and is therefore the entering variable. The loop construction shows that ε can be as large as 10, and that x_{24} is the departing variable.

Fourth Tableau

	-7	0	2	11
0	17 10	5 0	18 20	15 11
7	12 12	10 7	15 9	2 20
7	10 0	7 14	7 16	5 18

This is now optimal because $s_{ij} \geq 0$ in every cell.

Final Solution

The minimum cost is given by

$$5 \times 0 + 15 \times 11 + 10 \times 7 + 15 \times 9 + 10 \times 0 + 5 \times 18 = 460$$

which occurs when

$$x_{12} = 5, \quad x_{14} = 15, \quad x_{22} = 10, \quad x_{23} = 15, \quad x_{31} = 10, \quad x_{34} = 5$$

and all the other decision variables are equal to zero.

7.2.4 Solving the transportation problem with Excel Solver

We can use Excel Solver to solve the transportation problem. We set the problem out in the general form of a linear programming problem:

$$\begin{aligned}
\textbf{Minimise} \quad & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\
\textbf{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leq a_i \text{ for } i = 1, \ldots, m \\
\textbf{and} \quad & \sum_{i=1}^{m} x_{ij} \geq b_j \text{ for } j = 1, \ldots, n \\
\textbf{where} \quad & x_{ij} \geq 0 \text{ for all } i \text{ and } j.
\end{aligned}$$

The data is entered as two arrays - one of transportation costs and the other as decision variables. The Excel spreadhseet illustrating this is available on the module website.

Exercises

1. For the transportation problem given by the following tableau, find an initial basic feasible solution by the least-cost method and then find an optimal solution.

				Supply
	2	1	3	7
	4	5	6	8
Demand	5	6	4	

2. For the transportation problem given by the following tableau, find an initial basic feasible solution by the North-West corner method and then find an optimal solution.

						Supply
	10	15	10	12	20	8
	5	10	8	15	10	7
	15	10	12	12	10	10
Demand	5	9	2	4	5	

   The supply at Source 3 is now reduced from 10 to 6. There is a penalty of 5 for each unit required but not supplied. Find the new optimal solution.

3. Three refineries with maximum daily capacities of 6, 5, and 8 million gallons of oil supply three distribution areas with daily demands of 4, 8 and 7 million gallons. Oil is transported to the three distribution areas through a network of pipes. The transportation cost is 1 pence per 100 gallons per mile. The mileage table below shows that refinery 1 is not connected to distribution area 3. Formulate the problem as a transportation model and solve it. [Hint: Let the transportation cost for the non-connected route be equal to some large value M say and then proceed as normal.]

		Distribution Area		
		1	2	3
	1	120	180	—
Refinery	2	300	100	80
	3	200	250	120

4. In problem 4, suppose additionally that the capacity of refinery 3 is reduced to 6 million gallons. Also, distribution area 1 must receive all its demand, and any shortage at areas 2 and 3 will result in a penalty of 5 pence per gallon. Formulate the problem as a transportation model and solve it.

5. In problem 4, suppose the daily demand at area 3 drops to 4 million gallons. Any surplus production at refineries 1 and 2 must be diverted to other distribution areas by tanker. The resulting average transportation costs per 100 gallons are £1.50 from refinery 1 and £2.20 from refinery 2. Refinery 3 can divert its surplus oil to other chemical processes within the plant. Formulate the problem as a transportation model and solve it.

6. Five warehouses are supplied by four factories. The supply available from each factory, the demand at each warehouse and the cost per unit of transporting goods from the factories to the warehouses are summarised in the following table:

	W1	W2	W3	W4	W5	Supply
F1	13	9	15	10	12	40
F2	11	10	12	12	9	10
F3	12	9	11	12	9	20
F4	13	12	13	12	10	10
Demand	12	15	20	15	18	

(a) Use the North-West Corner method to find an initial basic feasible solution of this problem. (**Do NOT use the Least-Cost method.**)

(b) Find the optimal solution of this problem, i.e. the solution that minimises the transportation costs, clearly showing and explaining your working.

(HINT Recall that this problem will require that a basic solution contain 5+4-1=8 variables, one or more of which may be zero. You will need to make use of this fact at the final stage of your iteration of the algorithm.)

7. For the transportation problem given by the following tableau, find an initial basic feasible solution by the North-West corner method and then find an optimal solution.

				Supply
	9	15	12	10
	6	8	13	23
	9	3	11	27
Demand	21	14	25	