Chapter 3 Exercises
Solutions

1. Solve the following LP problems:

   (a) determine a suitable entering variable,

   (b) determine the departing variable

   (c) find the new form of the objective function and decide whether it has reached is optimum value. If it has not, then continue iterating until the optimum is reached.

   Case 1

$$\begin{array}{rrcrcrcr} \textbf{maximise} & z & = & 57 & - & 4x_1 & + & 3x_5 \\ \textbf{subject to} & x_2 & = & 21 & + & 2x_1 & - & 3x_5 \\ & x_3 & = & 30 & - & 3x_1 & - & 4x_5 \\ & x_4 & = & 12 & + & x_1 & - & 2x_5 \\ \textbf{and} & x_j & \geq & 0, & j = & 1, & \ldots, & 5. \end{array}$$

<u>Solution</u>

(a) Entering variable must be x_5, as increasing x_5 from 0 increases z while increasing x_1 decreases z.

(b) If we set $x_1 = 0$ in the three equations for the optimal basic variables we see that

- *$x_2 \geq 0$ implies $x_5 \leq 7$;*
- *$x_3 \geq 0$ implies $x_5 \leq 7.5$;*
- *$x_4 \geq 0$ implies $x_5 \leq 6$.*

The most stringent of these is $x_5 \leq 6$ so x_4 is the departing variable.

(c) We need an expression for x_5 in terms of the non-basic variables, x_1 and x_4. Rearranging the x_4 equation gives $x_5 = 6 + \frac{1}{2}x_1 - \frac{1}{2}x_4$. We then substitute for x_5 in the objective function to obtain

$$z = 75 - \frac{5}{2}x_1 - \frac{3}{2}x_4.$$

(d) Both x_1 and x_4 have negative coefficients in the equation for z, so putting $x_1 = x_4 = 0$ gives the optimal solution $z = 75$ at $(0, 3, 6, 0, 6)$.

Case 2

$$\begin{array}{rrcrcrcrcr} \textbf{maximise} & z & = & 20 & - & 9x_1 & + & 4x_3 & + & x_5 \\ \textbf{subject to} & x_2 & = & 12 & + & 2x_1 & - & x_3 & - & 3x_5 \\ & x_4 & = & 10 & - & 5x_1 & + & x_3 & + & 2x_5 \\ \textbf{and} & x_j & \geq & 0, & j = & 1, & \ldots, & 5. & & \end{array}$$

Solution

(a) The entering variable must be x_3 as this variable has the largest positive coefficient in the objective function.
(b) If we set $x_1 = 0$ and $x_2 = 0$ in the two equations for the optimal basic variables we see that

- $x_2 \geq 0$ **implies** $x_3 \leq 12$**;**
- $x_4 \geq 0$ **for all** $x_3 \geq 0$**.**

Thus x_2 is the departing variable.

(c) We need an expression for x_3 in terms of the non-basic variables. From the x_2 equation, we obtain $x_3 = 12 + 2x_1 - x_2 - 3x_5$. Substituting into the objective function gives

$$z = 68 - x_1 - 4x_2 - 11x_5$$

and into the second constraint gives

$$x_4 = 22 - 3x_1 - x_2 - x_5.$$

Thus, the optimal value is $z = 68$ at $(0, 0, 12, 22, 0)$.

2. Which of the variables should be chosen as the entering variable in the following to give the maximum increase in z at the first iteration? ($x_j \geq 0$ for each j.)

$$z = 6x_1 + 9x_2 + 8x_3$$

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 3 & - & x_1 & - & 2x_2 & - & 3x_3 \end{array}$$

Solution

Consider the maximum amount by which x_1, x_2 and x_3 can be increased in turn while the other two remain zero, such that x_4 and x_5 remain non-negative.

The maximum allowable increases in x_1, x_2 and x_3 respectively are $2.5, 1.5$ *and* 1. *These increases applied to z increase z from* 0 *to* $15, 13.5$ *and* 8 *respectively. Thus x_1 is the entering variable.*

Note that x_1 is the entering variable despite it having the lowest coefficient in the objective function.

3. *Use the tableau method to solve*

$$\textbf{Maximise} \qquad z = 4x_1 + 8x_2$$

$$\textbf{subject to} \quad \begin{cases} 5x_1 + x_2 \le 8 \\ 3x_1 + 2x_2 \le 4 \end{cases}$$

$$\textbf{and} \qquad x_j \ge 0, \quad j = 1, 2.$$

In standard form the problem is :

$$\textbf{Maximise} \qquad z = 4x_1 + 8x_2$$

$$\textbf{subject to} \quad \begin{cases} 5x_1 + x_2 + x_3 = 8 \\ 3x_1 + 2x_2 + x_4 = 4 \end{cases}$$

$$\textbf{and} \qquad x_j \ge 0, \; j = 1, ..., 4.$$

The initial tableau is

Basic	z	x_1	x_2	x_3	x_4	Solution
x_3	0	5	1	1	0	8
x_4	0	3	2	0	1	4
z	1	-4	-8	0	0	0

The entering variable is x_3, the row quotients are 8 for row 1 and 2 for row 2 so the departing variable must be x_4. Carrying out row operations we obtain the next tableau.

Basic	z	x_1	x_2	x_3	x_4	Solution	
x_3	0	$7/2$	0	1	$-1/2$	6	$r_1 := r_1 - r_2$
x_2	0	$3/2$	1	0	$1/2$	2	$r_2 := \frac{1}{2}r_2$
z	1	8	0	0	4	16	$r_3 := r_3 + 8r_2$

Since all the entries in the z row are non-negative the optimal value is $z = 16$, at the point $(0, 2, 6, 0)$.

4. *Use the tableau method to solve*

$$\textbf{Maximise} \qquad z = 5x_1 + 4x_2 + 3x_3$$

$$\textbf{subject to} \quad \begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \end{cases}$$

$$\textbf{and} \qquad x_j \geq 0, \quad j = 1, 2, 3.$$

In standard form the problem is:

$$\textbf{Maximise} \qquad z = 5x_1 + 4x_2 + 3x_3$$

$$\textbf{subject to} \quad \begin{cases} 2x_1 + 3x_2 + x_3 + x_4 = 5 \\ 4x_1 + x_2 + 2x_3 + x_5 = 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 = 8 \end{cases}$$

$$\textbf{and} \qquad x_j \geq 0, \ j = 1, ..., 6.$$

The initial tableau is

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	Solution
x_4	0	2	3	1	1	0	0	5
x_5	0	4	1	2	0	1	0	11
x_6	0	3	4	2	0	0	1	8
z	1	-5	-4	-3	0	0	0	0

The entering variable is x_1, the row quotients are $\frac{5}{2}, \frac{11}{4}, \frac{8}{3}$ for rows 1, 2 and 3 respectively so that the departing variable must be x_4. Carrying out row operations we obtain the next tableau.

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
x_1	0	1	$3/2$	$1/2$	$1/2$	0	0	$5/2$	$r_1 := \frac{1}{2}r_1$
x_5	0	0	-5	0	-2	1	0	1	$r_2 := r_2 - 4r_1$
x_6	0	0	$-1/2$	$1/2$	$-3/2$	0	1	$1/2$	$r_3 := r_3 - 3r_1$
z	1	0	$7/2$	$-1/2$	$5/2$	0	0	$25/2$	$r_4 := r_4 + 5r_1$

Now we have still a negative entry in the z row so x_3 must be the entering variable. The row quotients are 5 and 1 for row 1 and row 3 respectively (ignoring row 2 with its zero entry since this would imply an infinite row quotient). Thus x_6 is the departing variable and row operations produce the following tableau

Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	Solution	
x_1	0	1	$\frac{5}{2}$	0	$\frac{5}{2}$	0	-1	2	$r_1 := r_1 - \frac{1}{2}r_3$
x_5	0	0	-5	0	-2	1	0	1	$r_2 := r_2$
x_3	0	0	-1	1	-3	0	2	1	$r_3 := r_3 \div \frac{1}{2}$
z	1	0	3	0	1	0	1	13	$r_4 := r_4 + \frac{1}{2}r_3$

There are now no non-positive entries in the z row so the optimal value is $z = 13$, at $(2, 0, 1, 0, 1, 0)$.