# Operations Research for Computing, Spring 2010, Coursework 1

**solutions**

1. (a) The decision variables are

$$\begin{aligned} R &= \text{number of rag dolls made this week} \\ T &= \text{number of teddy bears made this week} \end{aligned}$$

[2]

(b) The objective function to be maximised is profit per week and is given by

$$z = 2T + 3R.$$

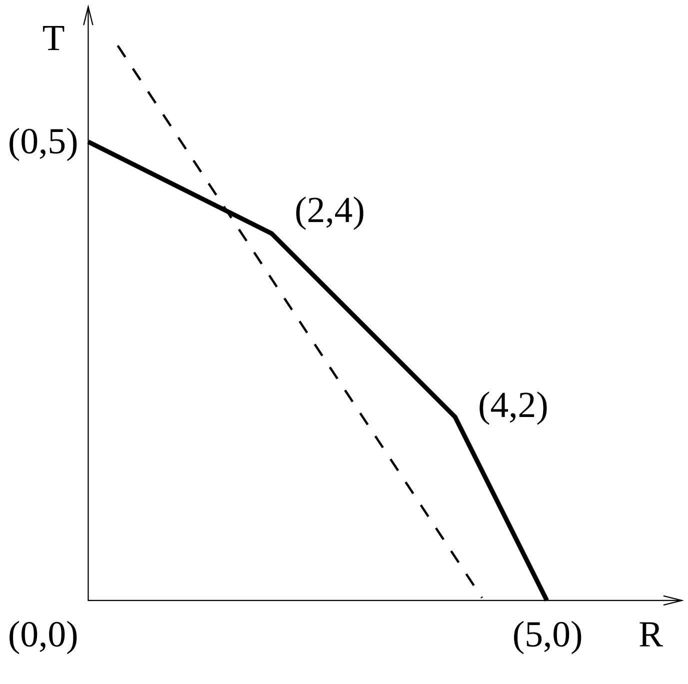
[2]

(c) The constraints are

$$\begin{aligned} 2T + 2R &\leq 12 \\ 2T + R &\leq 10 \\ 2T + 4R &\leq 20 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

[5]

(d) The feasible region is shown below together with the contour of $z = 13$. The corner points are $(0,0)$, $(0,5)$, $(2,4)$, $(4,2)$, $(5,0)$. [5]



(e) As the contours of $z$ move away from the origin, $z$ is increasing. The last corner point of the feasible region that a contour will touch is the corner at $(T, R) = (4, 2)$ and so this will give the maximum of $z$. Evaluating $z$ at this point gives the maximum value of $z = 16$.

Alternatively, the objective function could be determined at every corner point and the corner point giving the largest value of $z$ found, giving the same answer as above. [4]

(f) Angela should make 2 teddy bears and 4 rag dolls each week. [2]

(g) Suppose that the profit on a teddy bear is £$\alpha$, so that the objective function is $z = \alpha T + 3R$. Then the optimal corner point will no longer be optimal when the slope of a contour of the objective function is shallower than the slope of the constraint $2T + 2R \leq 12$. The two have the same slope when $\alpha = 3$ and so the profit must increase to more than £3 before the optimal solution changes. [3]

The new optimal solution occurs at $(T, R) = (4, 2)$ and so then Angela should make 4 teddy bears and 2 rag dolls each week. [2]

total marks for Question 1 [25]

2. (a) In standard form, we want to maximise a function. Thus, we want to maximise $z' = -6x_1 + x_2 + 3x_3$. [1]

As we have an equality constraint, we can use this to solve for $x_3$ giving $x_3 = 1 - x_1 - x_2$. Substituting for $x_3$ in $z'$ and the constraints gives

$$z' = 3 - 9x_1 - 2x_2$$

$$\begin{aligned} 2x_1 + 3x_2 &\leq 3 \\ 4x_1 - x_2 &\geq -1 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

[3]

Finally, introducing slack and surplus variables and multiplying the second constraint by $-1$ to ensure that the RHS is positive, we have the problem in standard form as follows:

Maximise

$$z' = 3 - 9x_1 - 2x_2$$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + x_4 &= 3 \\ -4x_1 + x_2 + x_5 &= 1 \end{aligned}$$

and

$$x_j \geq 0, \quad j = 1, 2, 4, 5, 6.$$

[3]

(b) i. In standard form, the problem is to maximize

$$z = 2x_1 + x_2 - x_3$$

subject to the constraints

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 21 \\ 6x_1 + x_2 + 2x_3 + x_5 &= 24 \end{aligned}$$

and

$$x_j \geq 0, \quad j = 1, \ldots, 5.$$

[2]

ii. An initial basic feasible solution is $x_1 = x_2 = x_3 = 0$, $x_4 = 21$, $x_5 = 24$. The basic variables are $x_4$ and $x_5$ and the non-basic variables are $x_1$, $x_2$ and $x_3$. [2]

iii. The initial tableau is as follows:

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Solution |
|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 1 | 3 | 1 | 1 | 0 | 21 |
| $x_5$ | 0 | 6 | 1 | 2 | 0 | 1 | 24 |
| $z$ | 1 | $-2$ | $-1$ | 1 | 0 | 0 | 0 |

[3]

iv. The largest negative coefficient in the $z$-row is $-2$ and so the entering variable is $x_1$. [1]

The quotients of the solution column divided by the $x_1$ column are 21 and 4 and so the departing variable is $x_5$. [1]

The new tableau is

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Solution |
|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 0 | $\frac{17}{6}$ | $\frac{2}{3}$ | 1 | $-\frac{1}{6}$ | 17 |
| $x_1$ | 0 | 1 | $\frac{1}{6}$ | $\frac{1}{3}$ | 0 | $\frac{1}{6}$ | 4 |
| $z$ | 1 | 0 | $-\frac{2}{3}$ | $\frac{5}{3}$ | 0 | $\frac{1}{3}$ | 8 |

[3]

The entering variable is now $x_2$ and the departing variable is $x_4$. The new tableau is then [3]

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Solution |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 0 | 1 | $\frac{4}{17}$ | $\frac{6}{17}$ | $-\frac{1}{17}$ | 6 |
| $x_1$ | 0 | 1 | 0 | $\frac{5}{17}$ | $-\frac{1}{17}$ | $\frac{3}{17}$ | 3 |
| $z$ | 1 | 0 | 0 | $\frac{31}{17}$ | $\frac{4}{17}$ | $\frac{5}{17}$ | 12 |

[3]

This is the optimal tableau as all the $z$-row coefficients are positive.

The optimal value is $z = 12$ and this occurs at $(x_1, x_2, x_3, x_4, x_5) = (3, 6, 0, 0, 0)$. [1]

total marks for Question 2 [25]

3. (a) i. The shadow price of a resource gives the rate of improvement of $z$ as that resource is altered.
If the shadow price is zero, then that resource is abundant and so an increase in that resource would not change the optimum solution.
If the shadow price is non-zero, then the resource is scarce and so an increase in that resource would be expected to make a difference to the optimum solution. [3]

ii. The shadow prices of land, finance and labour are 10, 0 and 5 respectively. [2]

iii. In order to increase the maximum profit by the largest amount, the largest shadow price is chosen. For the given example, this corresponds to the land, which should be increased. [2]

iv. Increasing the available land by 1 acre only increases the profit by £10, but costs £10,000, so the farmer should not buy more land. Increasing the available labour by one person-hour increases the profit by £5 and only costs £4.25, so the farmer should hire more labour and will make a net gain in his profit of £0.75 for every additional person-hour of labour he hires. [2]

(b) i. The objective function to be maximised is

$$z = 15x_1 + 35x_2 + 25x_3.$$

[2]

ii. From the optimal tableau, we see that $x_1 = 2000$, $x_2 = 0$ and $x_3 = 1000$, so the farmer should grow 2000 acres of barley, 1000 acres of wheat and no oats. The profit is then £55,000. [3]

iii. If the profit on barley falls by £t per acre, then the new objective function is

$$z' = (15 - t)x_1 + 35x_2 + 25x_3 = z - tx_1.$$

[2]

To determine whether the optimal solution for $z$ is also optimal for $z'$, we require $z'$ in terms of the non-basic variables from the optimal tableau of the original problem, which are $x_2$, $x_4$ and $x_6$.
From the optimal tableau, we have that

$$z = 55,000 - 10x_4 - 5x_6$$

and from the first row, we find that

$$x_1 = 2000 + x_2 - \frac{3}{2}x_4 + \frac{1}{2}x_6.$$

Substituting these into $z'$ gives

$$\begin{aligned} z' &= z + tx_2 \\ &= 55,000 - 10x_4 - 5x_6 - t\left(2000 + x_2 - \frac{3}{2}x_4 + \frac{1}{2}x_6\right) \\ &= 55,000 - 2000t - tx_2 - \left(10 - \frac{3}{2}\right)x_4 - \left(5 + \frac{1}{2}\right)x_6. \end{aligned}$$

[5]

If $z'$ is also optimal at this point, then all the coefficients must be negative or zero. Clearly, as $t$ increases, the coefficients for $x_2$ and $x_6$ become more negative. The $x_4$ coefficient will be non-positive provided that $t \leq 20/3$. [2]
Thus, the profit made on barley can fall by at most £$6\frac{2}{3}$ per acre if the existing optimal solution is to remain optimal. (Will accept £6.67 per acre, though to nearest penny the max fall that does NOT change optimum is £6.66 per acre.)

[2]

Total marks for Question 3 [25]

4. Solving the problem as stated, $x_2$ is the entering variable, the row quotients are $\frac{6}{5}$ and $\frac{4}{5}$ respectively so that $x_5$ is the leaving variable. The next tableau is then

| Basic | $z$ | $x_1$ | $x_2$ | $x_4$ | $x_5$ | Solution | |
|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | $\frac{14}{5}$ | 0 | 1 | $-\frac{3}{5}$ | $\frac{6}{5}$ | $r_1 := r_1 - 3r_2$ |
| $x_2$ | 0 | $\frac{2}{5}$ | 1 | 0 | $\frac{1}{5}$ | $\frac{4}{5}$ | $r_2 := \frac{1}{5}r_2$ |
| $z$ | 1 | $-\frac{1}{5}$ | 0 | 0 | $\frac{2}{5}$ | $-\frac{12}{5}$ | $r_3 := r_3 + 2r_2$ |

Now the entering variable is $x_1$ and the leaving variable $x_4$ (row quotients are $\frac{6}{14}$ and 2). the final tableau is

| Basic | $z$ | $x_1$ | $x_2$ | $x_4$ | $x_5$ | Solution | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 0 | 1 | 0 | $\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{6}{14}$ | $r_1 := \frac{5}{14}r_1$ |
| $x_2$ | 0 | 0 | 1 | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{44}{70}$ | $r_2 := r_2 - \frac{2}{5}r_1$ |
| $z$ | 1 | 0 | 0 | $\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{81}{35}$ | $r_3 + \frac{1}{5}r_1$ |

Hence we have $x_1 = \frac{6}{14}$, $x_2 = \frac{44}{70}$. But this gives $x_1 + x_2 = \frac{74}{70} > 1 \Rightarrow x_3 < 0$ which is not allowed.
We thus introduce the constraint

$$x_1 + x_2 + x_6 = 1$$

and rework the Simplex algorithm:

| Basic | $z$ | $x_1$ | $x_2$ | $x_4$ | $x_5$ | $x_6$ | Solution |
|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 4 | 3 | 1 | 0 | 0 | $\frac{18}{5}$ |
| $x_5$ | 0 | 2 | 5 | 0 | 1 | 0 | 4 |
| $x_6$ | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $z$ | 1 | -1 | -2 | 0 | 0 | 1 | -4 |

As before, $x_2$ enters and $x_5$ leaves.

| Basic | $z$ | $x_1$ | $x_2$ | $x_4$ | $x_5$ | $x_6$ | Solution | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | $\frac{14}{5}$ | 0 | 1 | $-\frac{3}{5}$ | 0 | $\frac{6}{5}$ | $r_1 := r_1 - 3r_2$ |
| $x_2$ | 0 | $\frac{2}{5}$ | 1 | 0 | $\frac{1}{5}$ | 0 | $\frac{4}{5}$ | $r_2 := \frac{2}{5}r_2$ |
| $x_6$ | 0 | $\frac{3}{5}$ | 0 | 0 | $-\frac{1}{5}$ | 1 | $\frac{1}{5}$ | $r_3 := r_3 - r_2$ |
| $z$ | 1 | $-\frac{1}{5}$ | 0 | 0 | $\frac{2}{5}$ | 0 | $-\frac{12}{5}$ | $r_4 := r_4 + 2r_2$ |

Now $x_1$ enters and $x_6$ leaves

| Basic | $z$ | $x_1$ | $x_2$ | $x_4$ | $x_5$ | $x_6$ | Solution | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{14}{3}$ | $\frac{4}{15}$ | $r_1 := r_1 - \frac{14}{5}r_3$ |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{8}{15}$ | $-\frac{2}{3}$ | $\frac{2}{3}$ | $r_2 := r_2 - \frac{2}{5}r_3$ |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $-\frac{5}{3}$ | $\frac{1}{3}$ | $r_3 := \frac{5}{3}r_3$ |
| $z$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{7}{3}$ | $r_4 := r_4 + \frac{1}{5}r_3$ |

The optimal solution is thus $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$ with $z_{min} = -\frac{7}{3}$
**Alternatively**
Replace the equality constraint with two inequalities and introduce two new slack variables,

$$x_1 + x_2 + x_3 \leq 1 \Rightarrow x_1 + x_2 + x_3 + x_6 = 1$$

$$x_1 + x -_2 + x_3 \geq 1 \Rightarrow x_1 + x_2 + x_3 - x_7 = 1$$

and then solve in the usual manner.

total marks for Question 4 [25]