Operations Research for Computing, Spring 2010
Coursework 2 Solutions

1. *A firm can manufacture four liquid speciality chemical products at its factory. Each product requires time on three different types of machine but production is limited by the time available on each of the machines. The firm wishes to maximise the profit made by manufacturing these products. The data is summarised below in hours and £Sterling profit per 100 litres..*

	Product				*Availability*
	1	2	3	4	
Machine 1 (hours)	1	4	2	3	*Up to 100 hours per week*
Machine 2 (hours)	3	1	5	3	*Up to 150 hours per week*
Machine 3 (hours)	2	3	1	3	*Up to 120 hours per week*
Profit	25	38	20	30	

(a) *Formulate this problem as a Linear Programming problem in standard form.*

maximise $z = 25x_1 + 38x_2 + 20x_3 + 30x_4$

subject to

$$x_1 + 4x_2 + 2x_3 + 3x_4 + x_5 = 100$$
$$3x_1 + 1x_2 + 5x_3 + 3x_4 + x_6 = 150$$
$$2x_1 + 43_2 + x_3 + 3x_4 + x_7 = 120$$

and

$$x_j \geq 0, j = 1 \ldots 7$$

(b) *Solve this problem using Excel Solver.*

See Excel spreadsheet

(c) *From your solution, state the amount of each type of product that should be made in order to maximise profits. What is the corresponding profit that is made?*
The optimal solution is to produce 3,786 litres of product 1, 1,321 litres of product 2 and 464 litres of product 3. The resulting profit is £154,143 per week.

(d) *Explain why the optimal solution will always involve making none of at least one product.*
The number of basic variables in a linear programming problem feasible solution must be less than or equal to the number of constraints. There are three constraints so only a maximum of three different products can be made at the optimal solution.

(e) *The firm can increase the available hours for each of the three machines at a cost of £1.50 per hour for machine 1, £4 per hour for machine 2 and £5 per hour for machine 3, but they can only increase the availability of time on one machine. On which machine should they increase the available time, by how much and how much extra profit would the firm make?*
The shadow prices for the three machines are $2, 1\frac{2}{7}$ and $9\frac{4}{7}$ for machine 1,2 and 3 respectively. Thus more time on machine 2 would lose $4 - 1\frac{2}{7} = 2\frac{5}{7}$ per hour.
Machine 1 could be utilised for another $75\frac{3}{7}$ hours per week - the extra profit would be $75\frac{3}{7} \times (2 - 1.50) = £3,786$ per week.
Machine 3 could be utilised for another $11\frac{9}{11}$ hours per week - the extra profit would be $11\frac{9}{11} \times (9\frac{4}{7} - 5) = £5,426$ per week.
The firm should therefore increase the availability of time on Machine 3.

(f) *The firm can further process product 4 so that it can be sold at a profit of £40 per 100 litres by acquiring machine 4 which is available for 105 hours per week at a cost of £12 per hour. Is this profitable for the firm and what would the new production schedule be?*

We are not told how much time at takes to process product 4 on machine 4. We thus examine the sensitivity of our solution to the processing time.

- If the profit on product 4 is £40 per 100 litres then with zero hours processing on machine 4 the optimal solution using Solver (see spreadsheet) gives net extra profit of £3,529 and requires that 2,467 litres of product 4 are made.
- If the time on machine 4 is t hours per 100 litres and $t \leq \frac{105}{24.67} = 4.26$ hours (the processing time at which the machine 4 constraint becomes binding) then the net increase in profit is
$$\pounds 3529 - (24.67 \times 12)t \approx 3529 - 295t$$
Thus for example if $t = 3$ hours the net extra profit is £2,636 and 2,467 litres of product 4 are made.
- If $t = 4.26$ the net extra profit is £2,264.
- If $t > 4.26$ the net extra profit decreases linearly until at $t = 11.9$ hours the extra profit from product 4 is equal to the cost of the additional hours processing - £$105 \times 12 = 1260$ so that the net extra profit is zero.

The company should acquire the additional hours of processing on machine 4 provided that the the time to process product 4 is less than 11.9 hours per 100 litres.=.

(g) *The firm wants to maintain the optimal production schedule determined in part (b). However, it wants to increase its profit as much as it can in order to fund a pay rise for the employees. To test the market, the firm has decided that it will only increase the profit on one of the four products. For which of the four products should the profit be increased and by how much, if the current solution is to remain optimal? How much extra profit would the firm make in this case.*
From the sensitivity report we see that the allowable increase in profit on products 1 to 3 is $4, 12$ and $24\frac{4}{11}$ respectively, The additional weekly profit at these levels, at the current optimal solution, is £$15,142$, $\pounds 15,857$ and £$11,311$ respectively so, provided it can sell all it produces of product 2, the firm should increase the profit of product 2. Note that the profit on Product 4 would have to rise by more than £$8\frac{4}{7}$ until it became profitable to produce it.

(h) *Half the work force has gone on strike and so production at the factory has been affected. Now a maximum of 4,500 litres of products per day can be made. What should the new production schedule be and what is the corresponding profit?*
We need a new constraint $x_1 + x_2 + x_3 + x_4 \leq 45$. The optimal solution is now $x_1 = 2,667$ litres and $x_2 = 1,833$ litres, resulting in a weekly profit of £136,333.

(i) *The company has the opportunity to manufacture product 5 which requires 3 hours on machine 1, 4 hours on machine 2 and 2 hours on machine 3. How much profit would need to be made per 100 litres of product 5 before it is profitable for the firm to produce it?*
We add a fifth column to the Solver data with a trial profit of £22 per litre. We run Solver and from the sensitivity report we see that the minimum increase in profit on product 5 has to be £8.29 per litre before it becomes profitable to manufacture it. Thus the profit on product 5 would need to be £30.29 for the firm to produce it.

2. *A firm assembles motherboards which contain the Euler chip. The firm uses 120,000 Euler chips each year at a constant rate and the factory works continuously. The chips must be stored in a refrigerator so their holding cost is high - £0.25 per chip per month. The chip supplier imposes a £500 delivery charge on each order.*

   (a) *Determine the optimal order quantity, the time between orders and the annual cost of inventory if no shortages are allowed.*
   We have $\beta = 10,000$ per month, $h = 0.25$ and $K = 500$. We take our unit of time as a month.
   The EOQ is

$$Q_0^\star = \sqrt{\frac{2K\beta}{h}} \Rightarrow Q_0^\star = \sqrt{4,000,000} = 6,325$$

   The cycle time is

$$t^\star = \frac{Q_0^\star}{\beta} = 2.74 \quad \text{weeks}$$

   The total cost per month is

$$T^\star = \sqrt{2K\beta h} = £1,581 \quad \text{per month}$$

   (b) *The present policy is to order stock every two months, how much would be the annual savings by ordering the optimal quantity.*
   At present every two months 20,000 chips are ordered. The total cost per month is then

$$\frac{K\beta}{Q_0} + \frac{hQ_0}{2} = \frac{500 \times 10,000}{20,000} + \frac{0.25 \times 20,000}{2} = £2,750$$

   The firm can thus save £12 $\times (2,750 - 1,581) = £14,028$ per annum on its inventory cost.

(c) *The purchase price of the Euler chip is usually £4 per chip but the supplier is offering a quantity discount of 10% on orders of 10,000 or more. Should the company accept this discount and if it does what is the new optimal strategy and annual inventory cost.*

First we see that since $q = 10,000$, we have $q > Q_m$ so we need to examine whether or not $Q_m < q < q1$ where q_1 is the root of

$$T_1(Q_m) = T_2(Q_1)$$

Thus

$$10,000 \times 3.6 + \frac{500 \times 10,000}{q_1} + \frac{q_1 \times 0.25}{2} = 10,000 \times 4 + 1,581$$

Solving this quadratic equation and taking the larger root we have $q_1 = 12,698$. Hence $Q_m < q < q_1$ so we are in zone II and the optimal order quantity is $10,000$. The cycle time is now 1 month and the monthly cost is £$37,750$ compared to £$41,581$.

(d) *If there is a shortage of Euler chips in the factory at any time the firm is obliged to pay an additional £3 per Euler chip to obtain them from another source (i.e. the shortage cost is £3 per chip). If the firm is willing to accept planned shortages, what would be the new optimal strategy and annual inventory cost. What would be the maximum shortage allowed?*

Using the formulae we have

$$S^\star = \sqrt{\frac{2 \times 500 \times 1000 \times 3}{0.25 \times 3.25}} = 6,076$$

and

$$Q^\star = \sqrt{\frac{2 \times 500 \times 1000 \times 3.25}{0.25 \times 3}} = 6,583.$$

The maximum shortage allowed is thus 507 chips.

The annual inventory cost is now $\sqrt{\frac{2 \times 500 \times 10{,}000 \times 0.25 \times 3}{3.25}} =$ £$1,519$ per month or £$18,228$ per annum.

(e) *How much of the time is there a shortage of chips?*

The monthly usage is 10,000. Assuming the factory works continuously this represents a shortage of $\frac{507}{10{,}000} \times 36512 \approx 1.5$ days.

3. *Consider the following transportation problem - five factories supply five warehouses, the costs are in £per ton and the supply and demand is in tons.*

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	6	4	5	6	10	35
F_2	11	7	9	8	7	10
F_3	3	10	8	13	11	20
F_2	9	4	11	2	3	10
F_3	8	6	7	0	6	20
Demand	10	20	30	12	28	

(a) *Use the* ***north-west corner method*** *to find an initial BFS (****No marks will be given for using the least-cost method****). How many basic variables are there. Why?*

We see that the supply is 95 tons but the demand is 100 tons so that we need to balance the problem. We introduce a dummy factory with supply of 5 tons at a transportation cost to each warehouse we will call A which is very large.

	6	4	5	10	11	Supply
0	10 6	20 4	5 5	-4 6	-1 10	35
4	1 11	-1 7	$10-\epsilon$ 9	-6 8	ϵ -8 7	10
3	-6 3	3 10	$15+\epsilon$ 8	$5-\epsilon$ 13	3 11	20
-8	11 9	8 4	14 11	$7+\epsilon$ 2	$3-\epsilon$ 3	10
-5	7 8	7 6	7 7	-5 0	20 6	20
$A-11$	5 A	7 A	6 A	1 A	5 A	5
Demand	10	20	30	12	28	

(b) *Check the initial BFS for optimality and continue iterating the transportation algorithm until you have found an optimal solution. Show all your working. Explain what is meant by shadow costs (s_{ij}). Clearly state your solution.*

The initial BFS, cost £583, is not optimal since there are multiple negative shadow costs.

The shadow costs (s_{ij}) represent the transportation cost saving per unit by a given non-basic variable entering the basis. The comprise the transportation cost for the cell (c_{ij}) minus the row dispatch cost (μ_I) and minus the column receiving cost (λ_j). Shadow costs for basic variables are zero.

We choose x_{25} as the entering variable and find $\epsilon = 3$. There are 10 variables in the basis, equal to the sum of the number of sources (6) and the number of destinations (5) minus 1. The -1 is due to the problem being balanced - once ten of the decision variables are known the eleventh is uniquely defined since the total supply = total demand with the introduction of the dummy factory.

	6	4	5	10	3	Supply
0	10 6	20 4	5 5	-4 6	7 10	35
4	1 11	-1 7	$7-\epsilon$ 9	-6 8	$3+\epsilon$ 7	10
3	-6 3	3 10	$18+\epsilon$ 8	$2-\epsilon$ 13	5 11	20
-8	11 9	8 4	14 11	10 2	8 3	10
3	-1 8	-1 6	-1 7	ϵ -13 0	$20-\epsilon$ 6	20
$A-3$	3 A	-1 A	-2 A	-7 A	5 A	5
Demand	10	20	30	12	28	

Now the cost is £538 but the solution is not optimal. We choose the entering variable is x_{54} and $\epsilon = 2$.

	6	4	5	-3	3	Supply
0	$10-\epsilon$ 6	20 4	$5+\epsilon$ 5	9 6	7 10	35
4	1 11	-1 7	5 9	7 8	5 7	10
3	ϵ -6 3	3 10	$20-\epsilon$ 8	13 13	1 11	20
-8	-2 9	-5 4	1 11	10 2	-5 3	10
3	-1 8	-1 6	-1 7	2 0	18 6	20
$A-3$	3 A	-1 A	-2 A	0 A	5 A	5
Demand	10	20	30	12	28	

The cost is now £498 but the solution is still not optimal. We choose x_{31} as the entering variable and $\epsilon = 10$.

	0	4	5	-3	3	Supply
0	6 6	$20-\epsilon$ 4	$15+\epsilon$ 5	9 6	7 10	35
4	7 11	-1 7	$5-\epsilon$ 9	7 8	$5+\epsilon$ 7	10
3	10 3	3 10	10 8	13 13	5 11	20
5	4 9	ϵ -5 4	1 11	$10-\epsilon$ 2	-5 3	10
3	5 8	-1 6	-1 7	$2+\epsilon$ 0	$18-\epsilon$ 6	20
$A-3$	3 A	-1 A	-2 A	6 A	5 A	5
Demand	10	20	30	12	28	

The cost is now £473, the solution is not optimal. The entering variable is x_{42} and $\epsilon \leq 5$. We choose x_{42} over x_{45} arbitrarily.

	0	4	5	2	8	Supply
0	6 6	15 4	20 5	4 6	2 10	35
-1	12 11	4 7	5 9	7 8	10 7	10
3	10 3	3 10	10 8	8 13	0 11	20
0	9 9	5 4	6 11	$5-\epsilon$ 2	ϵ -5 3	10
-2	10 8	4 6	4 7	$7+\epsilon$ 0	$13-\epsilon$ 6	20
$A-8$	8 A	4 A	3 A	6 A	5 A	5
Demand	10	20	30	12	28	

The cost is now £448 but the solution is not optimal. The entering variable is x_{45} and $\epsilon = 5$

	0	4	5	-3	3	Supply
0	6 6	15 4	20 5	49 6	7 10	35
4	12 11	4 7	5 9	7 8	10 7	10
3	10 3	3 10	10 8	13 13	5 11	20
0	9 9	$5-\epsilon$ 4	6 11	5 2	$5+\epsilon$ 3	10
3	5 8	ϵ -1 6	-1 7	12 0	$8-\epsilon$ 6	20
$A-3$	3 A	-1 A	-2 A	6 A	5 A	5
Demand	10	20	30	12	28	

The cost is £423. There are negative shadow costs so we choose x_{52} as the entering variable (ignoring the dummy row) and $\epsilon = 5$

	0	4	5	-2	4	Supply
0	6 6	$15+\epsilon$ 4	$20-\epsilon$ 5	8 6	7 10	35
3	8 11	0 7	1 9	7 8	10 7	10
3	10 3	3 10	10 8	12 13	4 11	20
-1	10 9	1 4	7 11	5 2	10 3	10
2	6 8	$5-\epsilon$ 6	0 7	12 0	$3+\epsilon$ 6	20
$A-4$	4 A	0 A	ϵ -1 A	2 A	$5-\epsilon$ A	5
Demand	10	20	30	12	28	

The cost is £418. There are negative shadow costs so we choose x_{63} as the entering variable (ignoring the dummy row) and $\epsilon = 5$

	0	4	5	-2	4	Supply
0	6 6	20 4	15 5	8 6	6 10	35
3	8 11	0 7	1 9	9 8	10 7	10
3	10 3	3 10	10 8	12 13	4 11	20
-1	10 9	1 4	7 11	5 2	10 3	10
2	6 8	0 6	0 7	12 0	8 6	20
$A-5$	5 A	1 A	5 A	3 A	1 A	5
Demand	10	20	30	12	28	

The cost is now £413. There are no negative shadow costs so we have the optimal tableau. Note that we have put $x_{52} = 0$ as a basic variable in order to ensure we have ten basic variables and to allow us to compute our shadow costs.
The solution is thus $x_{12} = 20, x_{13} = 15, x_{25} = 10, x_{31} = 10, x_{33} = 10, x_{45} = 10, x_{54} = 12, x_{55} = 8, x_{63} = 5$. The shortfall in supply has been allocated to Warehouse 3.

(c) *There has been a major fire at Factory 2 and the factory is closed. Find the new optimal solution* ***using the least-cost method.*** *How is the shortfall allocated?*

We now longer require the dummy factory as we include the shortfall in with the closed factory 2. We set the cost for any shipment from factory 2 to a very high penalty cost P. using the least-cost method we find the BFS to be:

	0	4	5	-1	5	Supply
0	6 6	20 4	15 5	7 6	5 10	35
$P-5$	5 P	1 P	5 P	6 P	10 P	[15]
3	10 3	3 10	10 8	11 13	3 11	20
-2	11 9	2 4	8 11	5 2	10 3	10
1	7 8	1 6	1 7	12 0	8 6	20
Demand	10	20	30	12	28	

The cost is £343, the shortage of 15 tons has been split between warehouse 3 and warehouse 5.

(d) *Due to a strike, Warehouse 4 is closed. The storage costs for excess production at factories 1 to 5 are £8,£8,£10,£12 and £10 respectively. Find the new optimal solution* ***using the least-cost method****. How is the excess production allocated.*

We set up a dummy warehouse in place of warehouse 4 with transportation costs equal to the storage costs of the appropriate warehouse and with capacity 7 to ensure that the problem is balanced (we eliminate the dummy factory). We find the BFS with the least cost method to be:

	-4	4	5	3	4	Supply
0	10 6	20 4	15 5	5 8	6 10	35
3	12 11	0 7	1 9	2 8	10 7	10
7	10 3	-1 10	ϵ -4 8	7 10	$3-\epsilon$ 11	20
-1	14 9	1 4	7 11	10 12	10 3	10
2	10 8	0 6	$15-\epsilon$ 7	5 10	$5+\epsilon$ 6	20
Demand	10	20	30	[7]	28	

The cost is £523, the solution is not optimal. x_{33} is the entering variable with $\epsilon = 3$

	0	4	5	7	4	Supply
0	10 6	20 4	15 5	1 8	6 10	35
3	8 11	0 7	1 9	ϵ -2 8	$10-\epsilon$ 7	10
3	10 3	3 10	$3+\epsilon$ -4 8	$7-\epsilon$ 10	4 11	20
-1	10 9	1 4	7 11	6 12	10 3	10
2	6 8	0 6	$12-\epsilon$ 7	1 10	$8+\epsilon$ 6	20
Demand	10	20	30	[7]	28	

The cost is £511, we choose x_{34} as the entering variable and $\epsilon = 7$

	-0	4	5	5	4	Supply
0	10 6	20 4	15 5	3 8	6 10	35
3	8 11	0 7	1 9	7 8	3 7	10
3	10 3	3 10	10 8	2 10	4 11	20
-1	10 9	4 4	7 11	8 12	10 3	10
2	6 8	0 6	5 7	3 10	15 6	20
Demand	10	20	30	[7]	28	

The cost is £497, 7 tons of factory 2's production is stocked. There are no negative shadow costs so we have the optimal solution.

(e) *Set up the problem in Excel solver and verify the overall transportation costs you have found in parts (b), (c) and (d). (You will may get a slightly different set of basic variables with the same overall transportation cost).*