MAT 1015 Techniques in Calculus I Autumn 2009

Coursework 1

Please hand in your solution to me no later than the 10:00 lecture on Friday November 13, clearly marked with **your name and that of your tutor**. You should show all your working as part of your answers. Questions with an asterisk (*) are a little more challenging.

1. By making the substitution $y = 3^x$ or otherwise, solve the equation
$$3^{2x} - 2 \times 3^{x+1} + 5 = 0.$$
Give your answer correct to 2 d.p.s.

2. Solve the equation
$$2\cos x(2\cos^2 x - 1) + 1 = 2\sin^2 x,$$
(i) for $x \in [0, 2\pi]$ and

   (ii) give the general solution for any x.

3. Find the set of real values of c for which the equation
$$\frac{x^2+1}{x^2-x+1} = c$$
is satisfied by at least one real value of x.

4. Given that $f : x \mapsto 1 + [x]$ and $g : x \mapsto \cos\left(\frac{\pi}{x}\right)$, find $f \circ g(4)$ and $g \circ f(\pi)$, giving your answer to 2 d.p.s.

5. For the following, sketch the graph and state with, justification whether the functions are odd, even, periodic (state the period), injective, surjective or bijective
   (a) $f : x \mapsto |\cos x|$
   (b) $g : x \mapsto \tan 3x + 1$
   (c) $h : x \mapsto 1 - \mathrm{sgn\,x\,e^{|x|}}$.

6. For all real x define
$$f : x \mapsto -3 + 5x, \quad g : x \mapsto x^2 + 5x + 3.$$
   (a) Find the range of each of f and g.
   (b) Define the inverse function f^{-1}.
   (c) Explain why g has no inverse. Restrict the domain and codomain so that $x \mapsto x^2 - 5x + 3$ is invertible and find the inverse function. How would you prove that your inverse function is correct?
   (d) Solve the equation $f \circ g = g \circ f$.

7. By substituting $x = z - \dfrac{p}{3z}$ into $x^3 + px + q = 0$ find a quadratic equation in z^3. Hence, by making a suitable substitution, transform the equation $x^3 - 30x - 36 = 0$ into a quadratic equation in z^3 and show that $z^3 = 18 \pm 26i$

   (a) Show that $(3 + i)^3 = 18 + 26i$

   (b) Deduce one root of $x^3 - 30x - 36 = 0$ and find the other two roots of the equation.

8. If $w = 2 - 3i$ and $z = -4 + 5i$, express the following complex numbers in the form $a + bi$ where a and b are real numbers. For each answer also find the modulus and the argument, in radians, between $-\pi$ and π, correct to 2 d.p.s.

   (a) $w - z$ (b) $w + z$ (c) wz (d) z^2 (e) $\dfrac{z}{w}$ (f) $\dfrac{1}{z^2}$

9. z_1 is the complex number with modulus 2 and argument $\dfrac{-3\pi}{4}$.

   z_2 is the complex number with modulus $2\sqrt{2}$ and argument $\dfrac{2\pi}{3}$

   (a) Find the modulus and argument (in radians between $-\pi$ and π) of

   (i) $z_1 z_2$ (ii) $\dfrac{z_1}{z_2}$ (iii) $\dfrac{1}{z_1}$

   (b) Express each of z_1 and z_2 in the form $a + bi$ where a and b are real.

10. Given that a and b are real numbers, prove that ai is a root of the equation

$$z^3 - bz^2 + a^2 z - a^2 b = 0$$

11. * Show that the area of a triangle with sides a, b, c is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter of the triangle, $s = \dfrac{1}{2}(a + b + c)$