CM 1021 Mathematical Methods for Computing I

Exercise Sheet 5 - Solutions

1. Work through as many of the examples and exercises in the Differentiation Workbook as you can. The ansewers are in the Workbook.

2. Find the derivative $\dfrac{dy}{dx}$ in each of the following cases, simplifying your answers:

   (a) $y = 3x^4, \quad y' = 12x^3$

   (b) $y = \frac{1}{2}x^{\frac{8}{3}}, \quad y' = \dfrac{4}{3}x^{\frac{5}{3}}$

   (c) $y = 5x^4 + x^3 + \dfrac{2}{x}, \quad y' = 20x^3 + 3x^2 - \dfrac{2}{x^2}$

   (d) $y = \cos 6x, \quad y' = -6\sin 6x$

   (e) $y = 2e^{-3x}, \quad y' = -6e^{-3x}$

   (f) $y = \sqrt{x} + \ln x^2, \quad y' = \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x}$

   (g) $y = x^4e^{2x}, \quad y' = 4x^3e^{2x} + 2x^4e^{2x} = 2x^3e^{2x}(2+x)$

   (h) $y = \dfrac{1-\cos x}{x}, \quad y' = \dfrac{x\sin x - 1 + \cos x}{x^2}$

   (i) $y = \dfrac{\cos x}{\sin x}, \quad y' = \sec^2 x$

   (j) $y = e^{-2x}(2x^2+2x-1), \quad y' = -2e^{-2x}(2x^2+2x-1) + e^{-2x}(4x+2) = 4e^{-2x}(1-x^2)$

   (k) $y = \ln(1+2x^2), \quad y' = \dfrac{4x}{1+2x^2}$

   (l) $y = e^{\sin x}, \quad y' = \cos x e^{\sin x}$.

3. For each of the following functions, find find the co-ordinates of all the local maxima and minima:
   (a) $-3x^2+4x+5$, (b) $y = 2x^3 - 5x^2 + 4x - 1$ (c) $y = x^4 - 2x^2$, (d) $y = x^2 + \dfrac{250}{x}$.

   (a) $y' = -6x + 4 \quad y' = 0 \quad \text{when} \quad x = \frac{3}{2}. \quad y\left(\frac{3}{2}\right) = \frac{23}{4}, \quad y''\left(\frac{3}{2}\right) = \frac{23}{4}$
   so the turning point at $\left(\frac{3}{2}, \frac{23}{4}\right)$ is a maximum.

   (b) $y' = 6x^2 - 10x + 4$, so $y' = 0$ when $3x^2 - 5x + 2 = 0$ or $(3x-2)(x-1) = 0$,
   $x = 1 \quad \text{or} \quad x = \frac{2}{3}. \; y'' = 6x - 5, \quad y''\left(\frac{2}{3}\right) = -1, \quad y''(1) = 1.$

Thus $x = \frac{2}{3}$ is a maximum and $x = 1$ is a minimum.
$y\left(\frac{2}{3}\right) = \frac{1}{27}, \quad y(1) = 0$, so the maximum is $\left(\frac{2}{3}, \frac{1}{27}\right)$ and the minimum $(1, 0)$.

(c) $y' = 4x^3 - 4x$ if $y' = 0$ then $x(x^2 - 1) = 0$ so $x = -1, \quad x = 0$ and $x = 1$.
$y'' = 12x^2 - 4$ so $y''(-1) = 8, \quad y''(0) = -4, \quad y''(1) = 8$.
$y(-1) = -1, \quad y(0) = 0 \quad y(1) = -1$ so we have two maxima at $(-1, -1)$ and $(1, -1)$ and a minimum at $(0, 0)$

(d) $y' = 2x - \dfrac{250}{x^2} \quad y' = 0$ when $x = 5, y = 75$. $y''(5) = 2 + \dfrac{250}{125} = 4$ so the function has a unique minimum at $(5, 75)$.

4. Find the coordinates of the points of inflection of the curve $y = x^4 - 6x^3 + 12x^2 + 5x - 3$.

$y' = 4x^3 - 18x^2 + 24x + 5, \quad y'' = 12x^2 - 36x + 24$.
We have $y'' = 0$ for $x^2 - 3x + 2 = 0, \quad (x - 1)(x - 2) = 0, \quad x = 1$ and $x = 2$
$y(1) = 9, \quad y(2) = 47$. The points of inflection are thus $(1, 9)$ and $(2, 47)$.

5. Find the equations of the tangent and the normal to the curve $y = e^{3x}(1 - x)$ at the point where $x = 0$.

We have $y(0) = 1$ and $y' = 2e^{3x} - 3xe^{3x}$ so $y'(0) = 2$
The gradient of the tangent is therefore 2 and that of the normal $-\frac{1}{2}$ at $(0, 1)$.
The tangent has equation $y - 1 = 2x, \quad y = 2x + 1$,
The normal has equation $y - 1 = -\dfrac{1}{2}x, \quad y = 1 - \dfrac{x}{2}$.

6. Find the second derivative $\dfrac{d^2y}{dx^2}$ of

(a) $y = \ln(1 + x^2), \quad y' = \dfrac{2x}{1 + x^2}, \quad y'' = \dfrac{2(1 - 2x^2)}{(1 + x^2)^2}$.

(b) $y = e^x \cos x \quad y' = e^x(\cos x - \sin x), \quad y'' = -2e^x \sin x$.

7. Find and characterise the turning point of $y = e^{-x^2}$. Sketch the graph.

$y' = -2xe^{-x^2}$. If $y' = 0$ then $x = 0$. $y(0) = 1$. $\quad y'' = -2x^2 + 4x^2e^{-x^2}$ so $y''(0) = -2$ and thus $(0, 1)$ is a maximum.
The points of inflection are found by solving $y'' = 0$, $-2 + 4x^2 = 0$ so $x = \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$.
The coordinates are $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

More Challenging Questions:

8. The binomial expansion of $(x + h)^n$ is

$$(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 \ldots$$

Using these first three terms show that when $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, from first principles.

We define

$$y' = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

so we can put

$$y' = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 \ldots - x^n}{h}$$

$$y' = nx^{n-1} + \lim_{h \to 0} \left(\frac{n(n-1)}{2}x^{n-2}h + O(h^2) \right) = nx^{n-1}$$

9. Find the first and second derivatives of $y = e^{-2x} \sin 5x$ and hence show that this function satisfies

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 29y = 0.$$

$$y' = -2e{-}2x \sin 5x + 5e^{-2x} \cos 5x$$

$$y'' = -21e^{-2x} \sin 5x - 20e^{-2x} \cos 5x$$

$$y''{+}4y'{+}29y = -21e^{-2x} \sin 5x{-}20e^{-2x} \cos 5x{-}8e{-}2x \sin 5x{+}20e^{-2x} \cos 5x{+}29e^{-2x} \sin 5x = 0$$

10. Find and characterise all the turning points of $y = x^3 e^{-x}$. Sketch the graph.

$y' = -x^3e^{-x} + 3x^2e^{-x}$ so $y' = 0$ when $-x^3 + 3x = 0$, $\quad x = 0 \quad$ and $\quad x = 3$.
$y(0) = 0, \quad y(3) = 27e^{-3}$.
$y'' = e^{-x}(x^3 - 6x^2 + 3x)$ and $y''(0) = 0$, so $(0, 0)$ is a point of inflection, while
$y''(3) = -18e^{-3}$ thus $(3, 27e^{-3})$ is a maximum.
There are two **more** points of inflection, where $x^2 - 6x + 3 = 0$, namely at $x = 3 \pm \sqrt{6}$.