Answers to Exercises in MS100 Notes (Part B)

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1. (a) $\mathrm{f}'(x) = \lim_{h\to 0} \dfrac{(x+h)^5 - x^5}{h}$

$= \lim_{h\to 0} \dfrac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h}$

$= \lim_{h\to 0}(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) \ = 5x^4, \quad$ for all $x \in \mathbf{R}$.

(b) $\mathrm{f}'(x) = \lim_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \dfrac{\left(\frac{x-(x+h)}{x(x+h)}\right)}{h}$

$= \lim_{h\to 0} \dfrac{-h}{hx(x+h)} = \lim_{h\to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}, \quad$ for all $x \in \mathbf{R} \setminus \{0\}$.

(c) $\mathrm{f}'(x) = \lim_{h\to 0} \dfrac{\sin 2(x+h) - \sin 2x}{h}$

$= \lim_{h\to 0} 2\dfrac{\sin 2x \cos 2h - \cos 2x \sin 2h - \sin 2x}{2h}$

$= \lim_{h\to 0} 2\left(\sin 2x \ \dfrac{\cos 2h - 1}{2h} - \cos 2x \ \dfrac{\sin 2h}{2h}\right)$

$= 2\sin 2x \lim_{h\to 0} \dfrac{\cos 2h - 1}{2h} - 2\cos 2x \lim_{h\to 0} \dfrac{\sin 2h}{2h}$

$= 2\sin 2x(0) - 2\cos 2x(1) = -2\cos 2x, \quad$ for all $x \in \mathbf{R}$.

2. (a) $\mathrm{f}'(x) = \frac{5}{2}(x^2+3)^{3/2} \ . \ 2x = 5x(x^2+3)^{3/2}$.

(b) $\mathrm{f}'(x) = \cosh(\cosh x) \ . \ \sinh x$.

(c) $\mathrm{f}'(x) = \dfrac{1}{\sqrt{1 - ((x+3)/2)^2}}.\dfrac{1}{2} = \dfrac{1}{\sqrt{4 - (x+3)^2}} = \dfrac{1}{\sqrt{-x^2 - 6x - 5}}$.

This is defined only when $x^2 + 6x + 5 < 0$, i.e. $-5 < x < -1$.

(d) $\mathrm{f}'(x) = \dfrac{1}{1 - (1-x^2)^2}(-2x) = \dfrac{-2x}{2x^2 - x^4} = \dfrac{2}{x^3 - 2x}$, for $x \neq 0$.

(e) $\mathrm{f}'(x) = -2 \ \coth\ (\mathrm{e}^x) \ . \ \mathrm{cosech}^2(\mathrm{e}^x) \ . \ (\mathrm{e}^x) = -2\mathrm{e}^x \ \coth\ (\mathrm{e}^x)\ \mathrm{cosech}^2(\mathrm{e}^x)$.

(f) Let $y = \mathrm{arsech}\ x$, so $x = \mathrm{sech}\ y$.

$\dfrac{\mathrm{d}x}{\mathrm{d}y} = -\ \mathrm{sech}\ y\ \tanh\ y = -x\sqrt{1-x^2}$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-1}{x\sqrt{1-x^2}}. \qquad (-1 < x < 1)$.

3. (a) Let $y = (\sin x)^x$, so $\ln y = x \ln \sin x$.

$\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \ln \sin x + x\dfrac{1}{\sin x}\cos x = \ln \sin x + x \cot x$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = (\sin x)^x(\ln \sin x + x \cot x)$.

(b) Let $y = 3^{\cosh x}$, so $\ln y = \cosh x \ln 3$.

$\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sinh x \ln 3$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3^{\cosh x} \sinh x \ln 3$.

(c) Let $y = \dfrac{x\tan^2 x}{x^3-1}$, so $\ln y = \ln x + \ln\tan^2 x - \ln(x^3-1)$

$= \ln x + 2\ln\tan x - \ln(x^3-1)$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} + 2\frac{1}{\tan x}\sec^2 x - \frac{1}{x^3-1}(3x^2)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\tan^2 x}{x^3-1}\left(\frac{1}{x} + 2\ \mathrm{cosec}\ x\sec x - \frac{3x^2}{x^3-1}\right).$$

4. The arcsin function has range $(-\pi/2, \pi/2)$. Since $-1 \le x^2-1 \le 0$ for x in the given domain, $\arcsin(x^2-1)$ has minimum value $\arcsin(-1) = -\pi/2$ and maximum value $\arcsin(0) = 0$.

   Note that differentiating does not help, as the maximum and minimum values do not occur at stationary points.

5. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t}\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{a(2+2\cos 2t)}{2a\sin 2t} = \dfrac{1+\cos 2t}{\sin 2t}$. When $t = \dfrac{\pi}{4}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{\sin 2t(-2\sin 2t) - (1+\cos 2t)(2\cos 2t)}{\sin^2 2t}.\frac{1}{2a\sin 2t} = \frac{-2(\sin^2 2t + \cos^2 2t) - 2\cos 2t}{2a\sin^3 2t}$$

$$= \frac{-2(1+\cos 2t)}{2a\sin^3 2t} = -\frac{1}{a} \text{ when } t = \frac{\pi}{4}.$$

6. When $x = 5$. $y^2 + 6y = 0$ so as $y < 0$, $y = -6$

   (a) $2x + 2yy' - 2 + 6y' = 0$ so $y' = (1-x)/(y+3) = 4/3$ at (5, –6).

   (b) $(x-1)^2 + (y+3)^2 == 25$, so centre is (1, –3), radius = 5.

   Radius from (1, –3) to (5, –6) has gradient –3/4, so gradient of tangent = 4/3.

7. $4yy' - 3xy' - 3y + 1 = 0$, so $y' = \dfrac{3y-1}{4y-3x}$.

   When $x = 1, 2y^2 - 3y - 5 = 0$, so $(2y-5)(y+1) = 0$. As $y > 0, y = \frac{5}{2}$.

   Then $y' = \frac{13}{14}$.

   Tangent is $y - \frac{5}{2} = \frac{13}{14}(x-1)$, i.e. $14y = 13x + 22$.

   Normal is $y - \frac{5}{2} = \frac{-14}{13}(x-1)$, i.e. $26y = -28x + 93$.

   Differentiating again, $4yy'' + 4(y')^2 - 3xy'' - 3y' - 3 = 0$, so $y'' = \dfrac{3 + 3y' - 4(y')^2)}{4y-3x}$.

8. $\mathrm{f}(x) = \dfrac{1}{3}\sinh x(3 + \sinh^2 x) = \sinh x + \dfrac{1}{3}\sinh^3 x$, so

   $\mathrm{f}'(x) = \cosh x + \dfrac{1}{3}\left(3\sinh^2 x\cosh x\right) = \cosh x + \cosh x(\cosh^2 x - 1) = \cosh^3 x.$

   $\mathrm{f}''(x) = 3\cosh^2 x\sinh x.$

   When $x = \ln 2, \sinh x = \dfrac{3}{4}, \cosh x = \dfrac{5}{4}$, so $\rho = \dfrac{\left(1 + (\frac{125}{64})^2\right)^{3/2}}{225/64} = \dfrac{19721^{3/2}}{921600} \approx 3.$

9. (a) The third derivative of $x^4\cos 2x$.

   is $24x(\cos 2x) + 3(12x^2)(-2\sin 2x) + 3(4x^3)(-4\cos 2x) + x^4(8\sin 2x)$

   $= 8(x^4 - 9x^2)\sin 2x + 24(x - 2x^3)\cos 2x.$

(b) The nth derivative of $x^3 \sinh x$.

Let $\mathrm{f}(x) = x^3, \quad \mathrm{g}(x) = \sinh x$.

$\mathrm{g}^{(n)}(x) = \sinh x$ if n is even, $\cosh x$ if n is odd.

$$\mathrm{h}^{(n)}(x) = x^3(\sinh x)^{(n)} + n(3x^2)(\sinh x)^{(n-1)}$$
$$+\frac{n(n-1)}{2}(6x)(\sinh x)^{(n-2)} + \frac{n(n-1)(n-2)}{6}(6)(\sinh x)^{(n-3)}.$$

If n is even, this equals $[x^3 + 3n(n-1)x]\sinh x + [3nx^2 + n(n-1)(n-2)]\cosh x$.

If n is odd, it equals $[x^3 + 3n(n-1)x]\cosh x + [3nx^2 + n(n-1)(n-2)]\sinh x$.

10. (a) $\displaystyle\lim_{x\to a} \frac{2x}{3x^2} = \frac{2}{3a}$.

(b) $\displaystyle\lim_{x\to 2} \frac{\pi\cos\pi x}{2x} = \frac{\pi\cos 2\pi}{4} = \frac{\pi}{4}$.

(c) $\displaystyle\lim_{x\to 0} \frac{1/(x^2+1)}{1/(x+1)} = 1$.

(d) $\displaystyle\lim_{x\to 1} \frac{1/x}{\mathrm{e}^x} = \frac{1}{\mathrm{e}}$.