Chapter 4 Exercises
Solutions

1. *A company which manufactures three products A, B and C, needs to solve the following LP problem in order to maximise their profit.*

$$\textbf{Maximise} z = \quad 3x_1 + x_2 + 5x_3$$

$$\textbf{subject to} \begin{cases} 6x_1 + 3x_2 + 5x_3 \leq 45 \\ 3x_1 + 4x_2 + 5x_3 \leq 30 \end{cases}$$

$$\textbf{and} \quad x_j \geq 0, \quad j = 1, 2, 3.$$

Thus x_1, x_2 and x_3 are the amounts of A, B and C to be produced. The first constraint is a labour constraint, and the second is a material constraint. The company solves the problem on a computer and obtains an optimal solution in which x_1 and x_3 are basic.

(a) *Express the problem in standard form*

$$\textbf{Maximise} \quad 3x_1 + x_2 + 5x_3$$

$$\textbf{subject to} \begin{cases} 6x_1 + 3x_2 + 5x_3 + x_4 = 45 \\ 3x_1 + 4x_2 + 5x_3 + x_5 = 30 \end{cases}$$

$$\textbf{and} \quad x_j \geq 0, \quad j = 1, 2, 3, 4, 5.$$

(b) *Find the company's optimal solution.*
We set up the initial tableau as follows with x_4 and x_5 as the initial basic variables.

Basic	z	x_1	x_2	x_3	x_4	x_5	Solution
x_4	0	6	3	5	1	0	45
x_5	0	3	4	5	0	1	30
z	1	-3	-1	-5	0	0	0

x_3 is the entering variable, x_5 the departing variable. With row operations we obtain the next tableau

Basic	z	x_1	x_2	x_3	x_4	x_5	Solution	
x_4	0	3	-1	0	1	-1	15	$r_1 := r_1 - 5r_2$
x_3	0	3/5	4/5	1	0	1/5	6	$r_2 := \frac{1}{5}r_2$
z	1	0	3	0	0	1	30	$r_3 := r_3 + 5r_2$

This is optimal : $x_3 = 6, x_1 = x_2 = 0$.

BUT this is not the solution that the company has. Notice that x_1 can enter the basis without changing the optimal value of z. We have **multiple optimal solutions**.

If x_1 enters we must have that x_4 leaves, iterating gives the company's solution:

Basic	z	x_1	x_2	x_3	x_4	x_5	Solution	
x_1	0	1	-1/3	0	1/3	-1/3	5	$r_1 := \frac{1}{3}r_1$
x_3	0	0	1	1	-1/5	2/5	3	$r_2 := r_2 - \frac{3}{5}r_1$
z	1	0	3	0	0	1	30	$r_3 := r_3$

This is also optimal, with $x_1 = 5, x_3 = 3, z_{max} = 30$.

(c) *How much can c_2, the unit profit for B, be increased without affecting the original optimal solution?*
Suppose c_2 increases by £t so now $c_2 = (1 + t)$. Then the new objective function is

$$z' = 3x_1 + (1 + t)x_2 + 5x_3 = z + tx_2 = 30 - (3 - t)x_2 - x_5.$$

The original solution is still optimal if $3 - t \geq 0$, i.e. $t \leq 3$. So c_2 can increase from 1 to 4 and the above solution is still optimal.

(d) *Find the range of c_1, the unit profit for A, for which the original solution remains optimal.*
Suppose c_1 increases by £t. Then the new objective function is

$$z' = z + tx_1$$

From the final tableau we can see that the equation for x_1 is

$$x_1 = 5 + \frac{1}{3}x_2 - \frac{1}{3}x_4 + \frac{1}{3}x_5,$$

so that substituting into the equation for z' we have

$$z' = 30 - 3x_2 - x_5 + t(5 + \tfrac{1}{3}x_2 - \tfrac{1}{3}x_4 + \tfrac{1}{3}x_5)$$

$$= (30 + 5t) - (3 - \tfrac{t}{3})x_2 - \tfrac{t}{3}x_4 - (1 - \tfrac{t}{3})x_5$$

This is optimal if all the coefficients are non-negative, which requires that $t \leq 9$ (from the x_2 coefficient), $t \geq 0$ (from the x_4 coefficient) and $t \leq 3$ (from the x_5 coefficient).

Hence $0 \leq t \leq 3$ and so c_1 can be between 3 and 6 for the solution still to be optimal.

(e) *Find the optimal solution when b_2, the amount of material available, is increased to 60 units.*

The new material constraint is $3x_1 + 4x_2 + 5x_3 \leq 30 + 30$, i.e. x_5 can be replaced by $x_5' - 30$ to get the same problem and solution as before.

Then the equations for the objective function and the basic variables form the final tableau in terms of the nonbasic variables are

$$\begin{aligned} z &= 60 - 3x_2 - x_5' \\ x_1 &= 5 + \tfrac{1}{3}x_2 - \tfrac{1}{3}x_4 + \tfrac{1}{3}(x_5' - 30) \\ x_3 &= 3 - x_2 + \tfrac{1}{5}x_4 - \tfrac{2}{5}(x_5' - 30) \end{aligned}$$

For x_1 to remain non-negative with $x_2 = x_4 = 0$ requires

$$5 + \frac{1}{3}(x_5\prime - 30) \geq 0,$$

which is $x_5' \geq 15$. For x_3 to remain non-negative requires

$$3 - \frac{2}{5}(x_5\prime - 3)) \geq 0,$$

or $x_5' \leq 75/2$.

Thus we have $15 \leq x_5\prime \leq \frac{75}{2}$. From the equation for z we see that we want $x_5\prime$ to be as small as possible (i.e. as little slack as possible in the use of material). Thus, we choose $x_5' = 15$. Then the new optimal solution is $z_{max} = 45$ at (0, 0, 9, 0, 15).

(f) *The constraint $3x_1 + 2x_2 + 3x_3 \leq 25$ is added to the original problem. How does this affect the original optimal solution?*

The new constraint in standard form is given by

$$3x_1 + 2x_2 + 3x_3 + x_6 = 25.$$

At the current optimum, $x_1 = 5$, $x_2 = 0$, $x_3 = 3$ and so $x_6 = 1$. So the current solution is still optimal. (Recall that provided that all the variables are non negative the solution will always be in the feasible region.)

(g) *A new product D is proposed. D has a unit profit of* 5, *and its labour and material requirements are 3 units and 4 units respectively. Is it profitable to produce D?*

Let x_6 be the amount of D. Then the new objective function is

$$z' = 3x_1 + x_2 + 5x_3 + 5x_6$$

The constraints become

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 + x_4' + 3x_6 &= 45 \\ 3x_1 + 4x_2 + 5x_3 + x_5' + 4x_6 &= 30 \end{aligned}$$

The new problem is then:

Maximise

$$z' = z + 5x_6$$

subject to

$$\begin{aligned} x_4' &= x_4 - 3x_6 \\ x_5' &= x_5 - 4x_6. \end{aligned}$$

The current solution has $z = 30 - 3x_2 - x_5$ and so

$$z' = 30 - 3x_2 - x_5' + x_6.$$

The optimal z' has $x_6 > 0$, i.e. putting $x_6 = 0$ does not give the optimum and so the company should produce product D.

(h) *An additional 15 units of material are available at a cost of £10. What should be done?*

The material constraint is now $3x_1 + 4x_2 + 5x_3 \leq 30 + 15$. Extra material costs £10.

As in (d), put $x_5 = x_5' - 15$ to give $z = 45 - 3x_2 - x_5'$.
Substituting for x_5 in the equations for x_1 and x_3 gives

$$x_1 = 0 + \frac{1}{3}x_2 - \frac{1}{3}x_4 + \frac{1}{3}x_5', \quad \text{so} \quad x_5' \geq 0.$$

Also

$$x_3 = 9 - x_2 + \frac{1}{5}x_4 - \frac{2}{5}x_5', \quad \text{so} \quad x_5' \leq \frac{45}{2}.$$

We want x_5' to be as small as possible (it has a negative coefficient in the equation for $z\prime$) so we choose it to be 0. Then $x_1 = 0, x_3 = 9$.

Thus the objective function increases by £15, so taking off the £10 cost still gives a profit of £5. So the company should buy the new material. In fact we could also have reached the same conclusion by looking at the shadow price of material, $\pounds 1$, which is the additional profit from one additional unit of material. Then $\pounds 15 \times 1 - \pounds 10 = \pounds 5$.

The new profit is £35 at the point (0, 0, 9, 0, 0).