Chapter 9 Exercises
Solutions

Exercises on p 115

1. (a) B 3, C 5, G 5, H 4

   (b) A-C-H-G-1 is an example of such a path

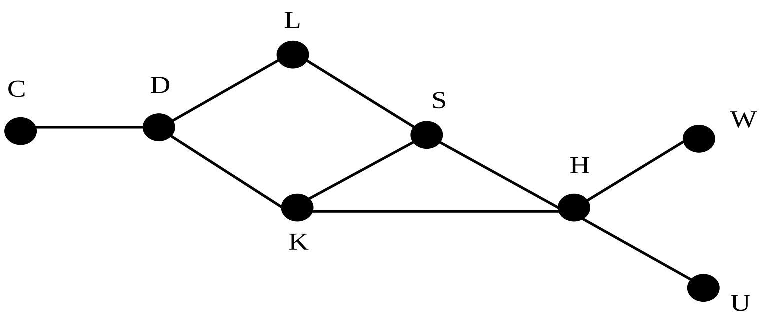
   (c) A-C-E-F-I

   (d) B-A-C-D-E-F-G-I-B

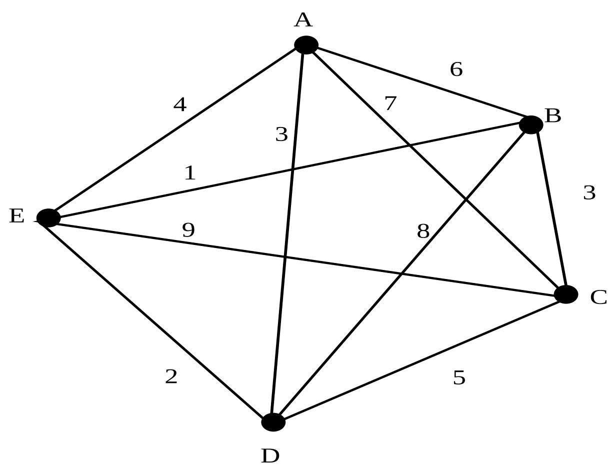
   (e)

	A	B	C	D	E	F	G	H	I
A	0	0	1	0	0	0	0	0	0
B	1	0	1	0	0	0	0	1	0
C	1	1	0	1	0	0	0	1	0
D	1	0	1	0	1	0	0	0	0
E	0	0	0	1	0	1	0	0	0
F	0	0	0	0	0	0	1	0	1
G	0	0	1	0	0	1	0	1	1
H	0	1	1	0	0	0	1	0	0
I	0	0	0	0	0	1	1	1	0

2. This is the network.



3. (a) This is the network:



(b) The possible paths from A to B include

AEB	5
AB	6
ADEB	6
ACB	10
ADB	11
ADCB	11
ACDB	11
AEDCB	14
AEDB	14
ACDEB	15
AECB	16
ADECB	17
ACEB	17
AECDB	26
ACEDB	26

(c) There are $4! = 24$ possible cycles

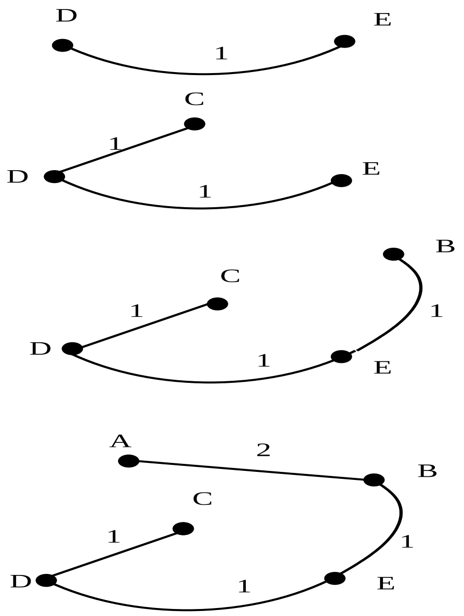
ABCDEA ABECDA ABDECA ABCEDA ABEDCA ABDCEA
ACBEDA ECABDA ACDEBA ACBDEA ACEDBA ACDBEA
ADBCEA ADCBEA ADEBCA ADBECA ADCEBA ADECBA
AEBCDA AECBDA AEDCBA AEBDCA AECDBA AEDBCA

4. (a) Prim's algorithm for the matrix

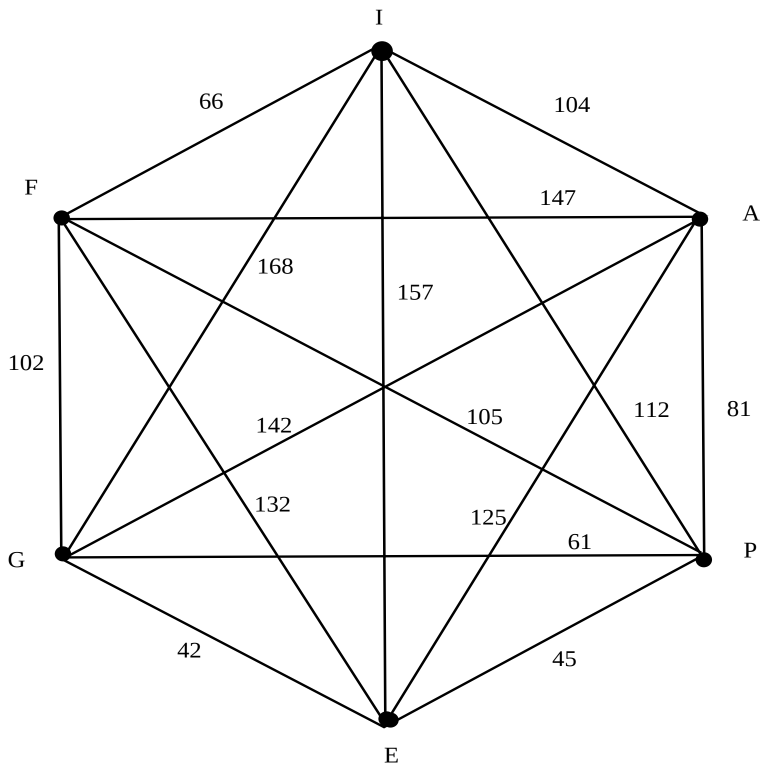
	[1] A	[2] B	C	[4] D	[3] E
A	-	1	4	2	-
B	[1]	-	3	-	1
C	4	3	-	[1]	3
D	2	-	1	-	[1]
E	-	[1]	3	1	-

The MST is **ABEDC**

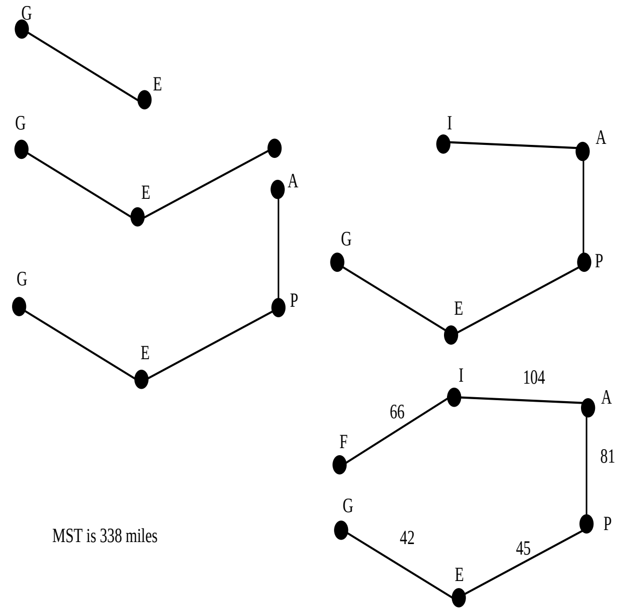
(b) Prim's algorithm applied to the graphical form



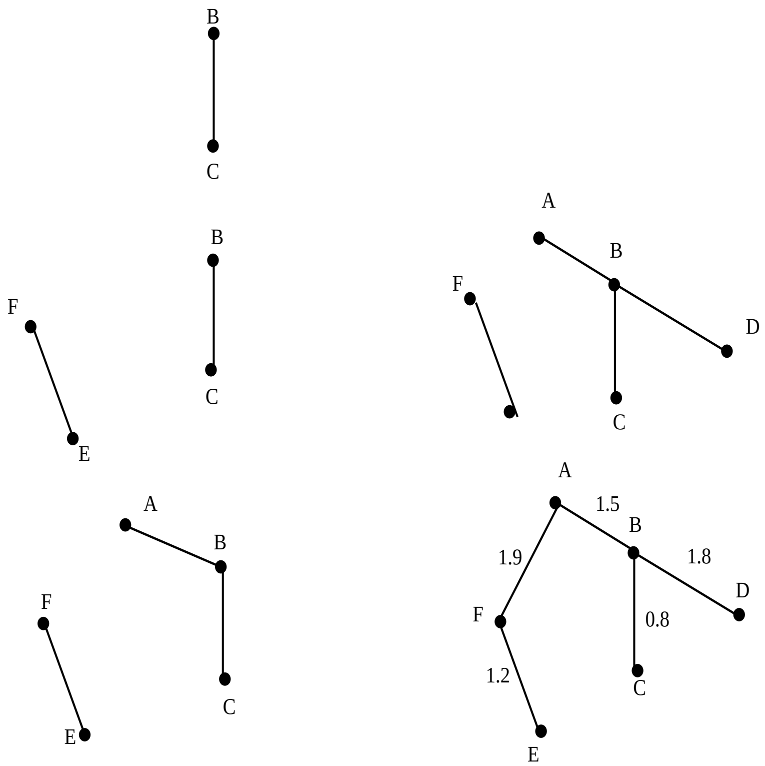
5. (a) This is the network



(b) This is the application of Prim's algorithm



6. (a) The application of Kruskals's algorithm, MST is £7.2m



(b) The matrix application of Prim's algorithm is

	[1] A	[2] B	[3] C	[5] D	E	[4] F
1-8 A	-	1.5	1.8	-	2.5	1.9
B	[1.5]	-	0.8	1.8	-	-
C	1.8	[0.8]	-	2.7	2.2	2.0
D	-	[1.8]	2.7	-	2.1	-
E	2.5	-	2.2	2.1	-	[1.2]
F	[1.9]	-	2.0	-	1.2	-

7. With 5 vertices in a connected network Prim's algorithm begins by choosing one node out of 5. Then 4 nodes are inspected to choose the second node. Now from 2 labeled nodes 3 nodes each are inspected, then from 3 labeled nodes 2 nodes each are inspected and finally from 4 labeled nodes the last node is inspected. This gives us $4 \times 1 + 3 \times 2 + 2 \times 3 + 1 \times 4 = 20$ inspections.

For 6 vertices the same argument leads to a total of 35 inspections.

With n nodes the total number of inspections is

$$S_n = (n-1) \times 1 + (n-2) \times 2 + (n-3) \times 3 \ldots (n-r) \times r \ldots 1 \times (n-1).$$

More succinctly

$$S_n = \sum_{r=1}^{n-1} (n-r)r = n \sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} r^2.$$

We have the formulae for the sum of the first m natural numbers and the squares of the first m natural numbers

$$\sum_{i=1}^{m} i = \frac{m}{2}(m(m+1)), \quad \sum_{i=1}^{m} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

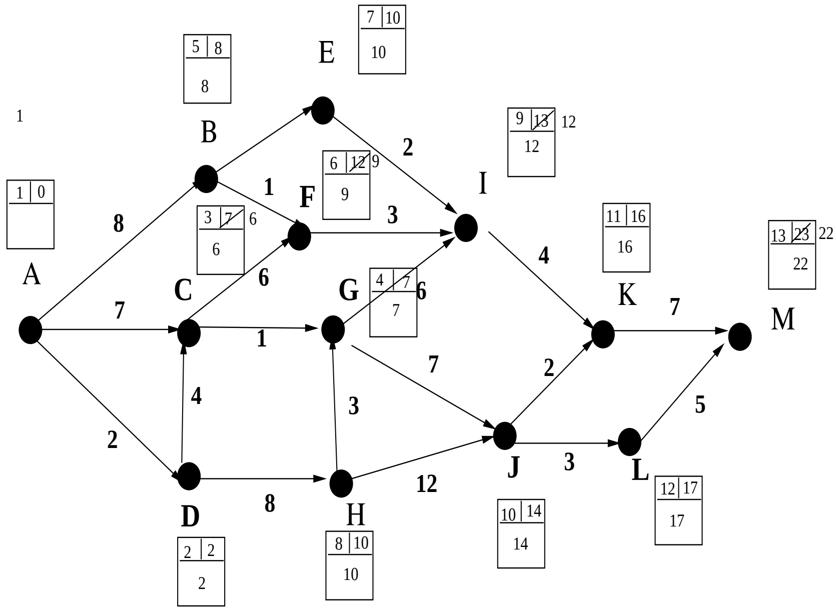
Making the substitutions we obtain

$$S_n = \frac{n(n^2-1)}{6}$$

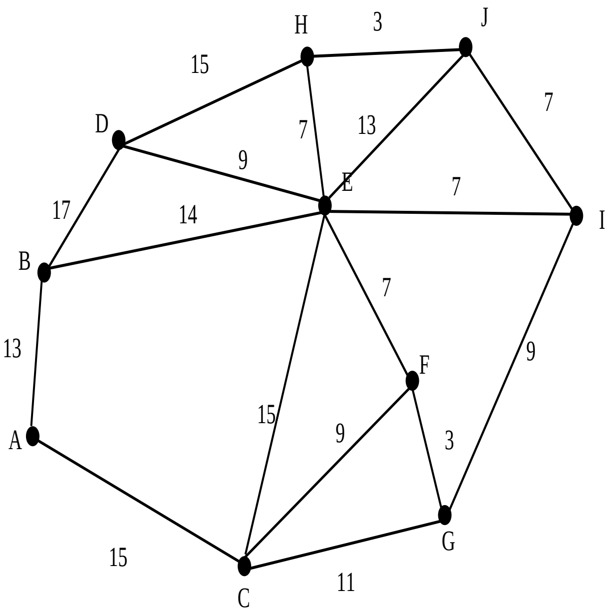
Thus the time spent in implementing Prim's algorithm is a increases as the cube of the number of vertices in the network.

Exercises on p 127

1. This is the network with the labelling. The shortest path is is **A-D-C-G-J-L-M** and the length of this path is 22.



2. The network is as shown

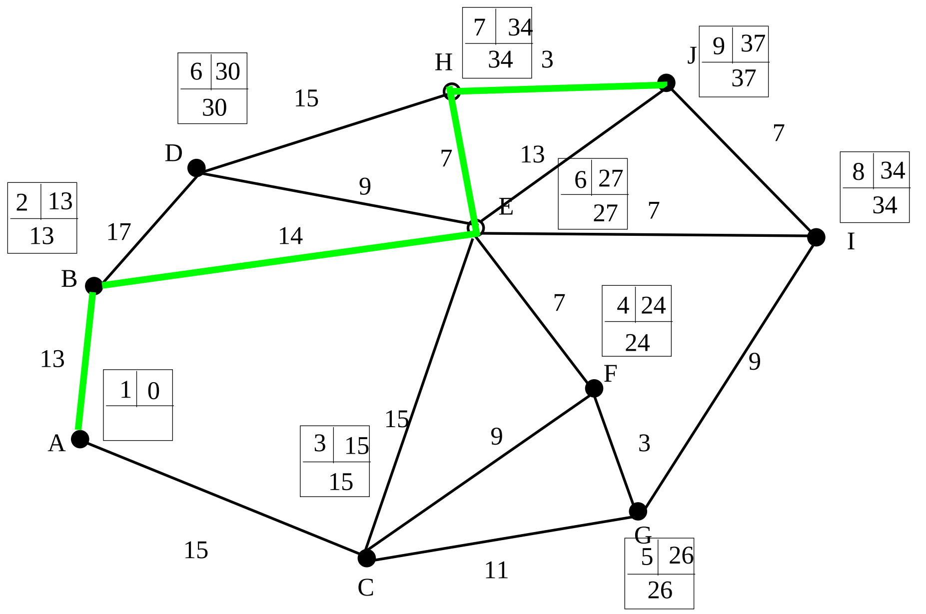


3. The algorithm in section 9.3.3 is applied as follows

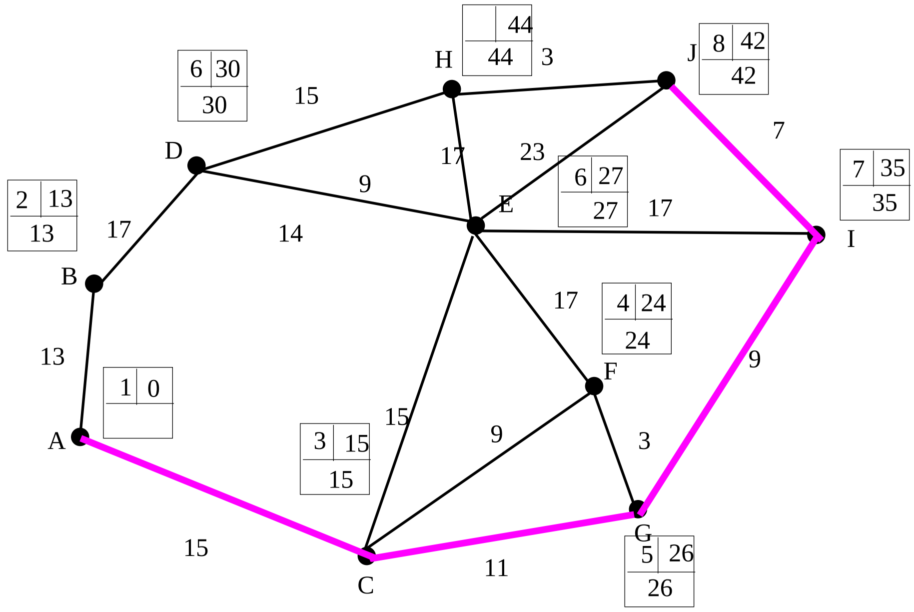
n	solved	closest unsolved	total distance	nth nearest	Min. distance	last arc
1	A	B	13	A	13	AB
2	B	E	13+14=27			
	A	C	15	C	15	AC
3	B	D	13+17=30			
	E	F	27+7=30			
		I	27+7=34			
		H	27+7=34			
	C	F	15+9=24	F	24	CF
4	F	G	24+3=27			
	H	J	34+3=37	J	37	HJ
	E	I	27+7=34			
	I	J	34+7=41			
5	G	I	27+9=36			
	E	J	27+13=40			

The shortest path is **ABEHJ** of length 37 miles

4. Using Dijkstra's algorithm we obtain the same result.



5. The driver will complete the journey in 37 minutes. A delay of 10 minutes at 8 is tantamount to increasing the distance of the arcs from E towards J by 10 miles. The reworked Dijkstra algorithm shows that the shortest route is now ACEIJ of length 42 miles, the journey now takes 42 minutes.



6. With 5 vertices in a connected network Dijkstra's algorithm begins by choosing one node out of 5. Then 4 nodes are inspected to choose the next nearest node. Now from the 2 labeled nodes 3 nodes each are inspected and a third labelled, then from 3 labeled nodes 2 nodes each are inspected and finally from 4 labeled nodes the last node is selected. This gives us $4 \times 1 + 3 \times 2 + 2 \times 3 + 1 \times 4 = 20$ inspections.

   For 6 vertices the same argument leads to a total of 35 inspections.

   With n nodes the total number of inspections is

$$S_n = (n-1) \times 1 + (n-2) \times 2 + (n-3) \times 3 \ldots (n-r) \times r \ldots 1 \times (n-1).$$

   More succinctly

$$S_n = \sum_{r=1}^{n-1} (n-r)r = n \sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} r^2.$$

   We have the formulae for the sum of the first m natural numbers and the squares of the first m natural numbers

$$\sum_{i=1}^{m} i = \frac{m}{2}(m(m+1)), \quad \sum_{i=1}^{m} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

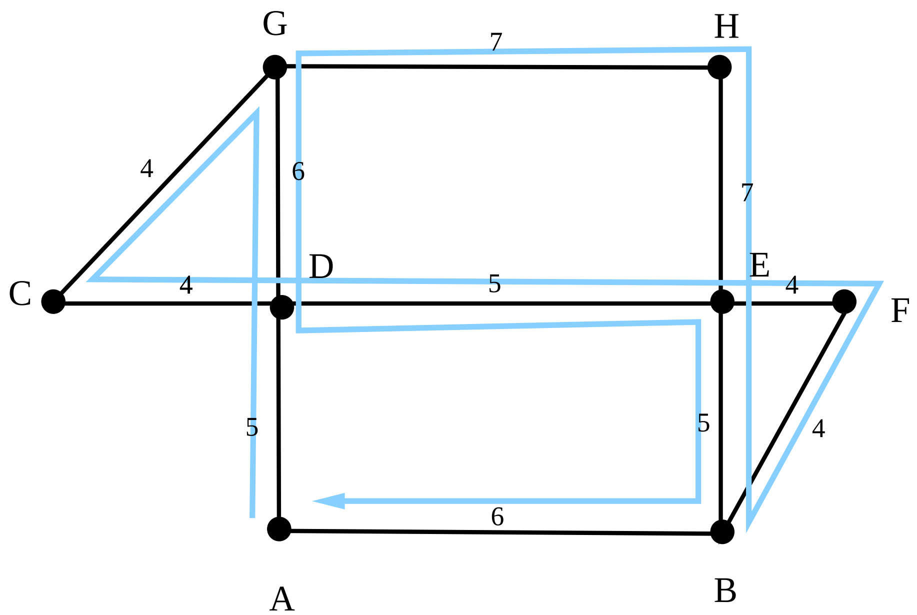
   Making the substitutions we obtain

$$S_n = \frac{n(n^2-1)}{6}$$

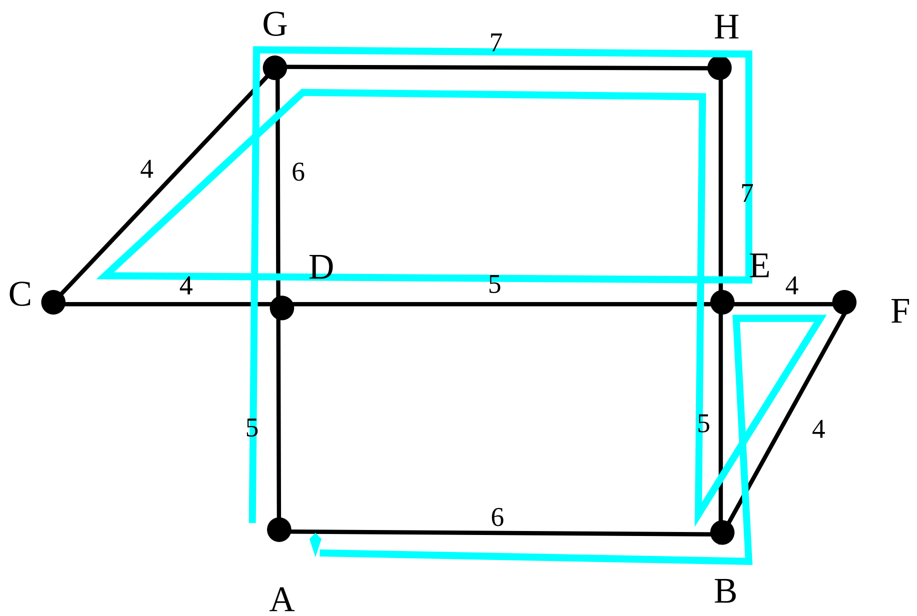
   Hence for 100 nodes we have $100 \times (10,000 - 1)/6 = 15,150$ computations. The number of computations increases as the cube of the number of nodes.

Solutions to exercises on p 138

1. (a) The sum of the times for each of the streets in the network is 57 minutes. There are two odd vertices - G and B. The minimum distance between B and G is the path GDEB of length 16 so the total times to traverse the network will be 73 minutes and streets GD, DE and EB will be traversed twice in each case.
The solution is shown:



(b) If the weight of GH is 3 then the shortest path from G to B is now GHEB of length 15 and the total length is now 53+15=58.Edges GH, HE and EB are traversed twice.

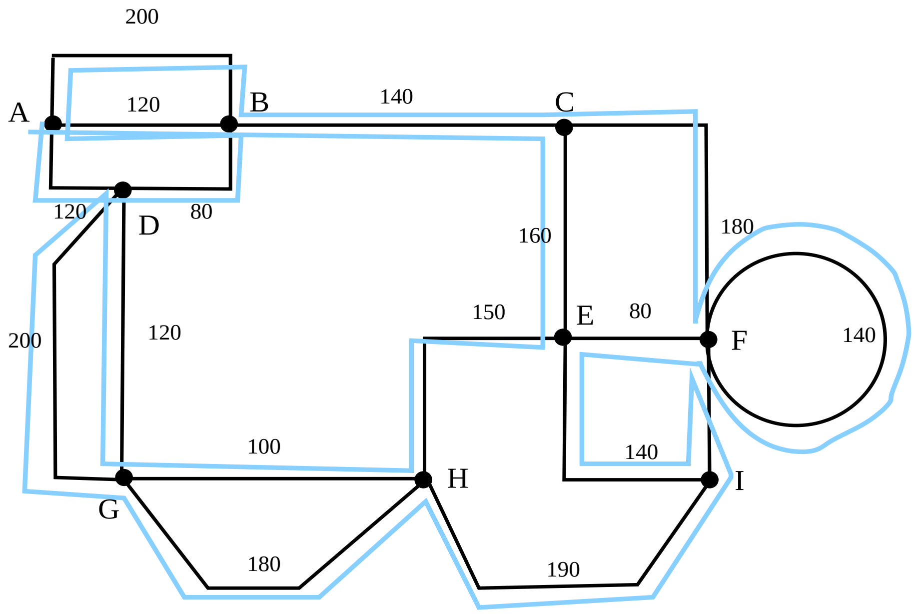


(c) Delivering to both sides of the street means that the weights of each street will be multiplied by 4. The minimum distance will still be that for the route in the first part of the problem. The time taken will be $73 \times 4 = 292$ minutes.

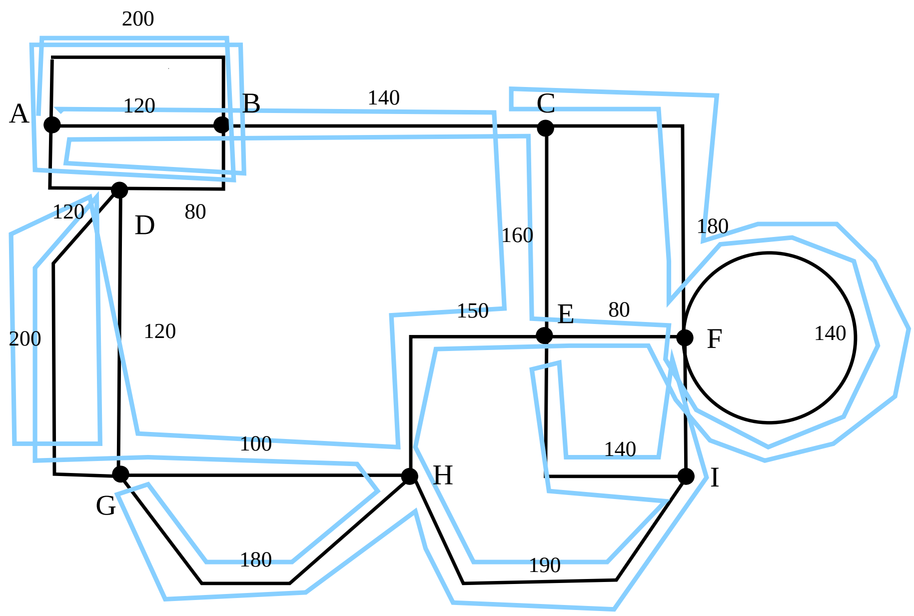
2. (a) The odd order nodes are A,C,F and I. The possible pairings are

- AF and CI minimum distances 440 and 240 respectively, total 680m
- AC and FI minimum distances 260 and 60 respectively, total 320m
- AI and CF minimum distances 500 and 180 respectively, total 680m

So we will need a minimum distance of 2300 (total length of all the roads)+320 =2620m, repeating the road connecting AC and FI. This is the route.



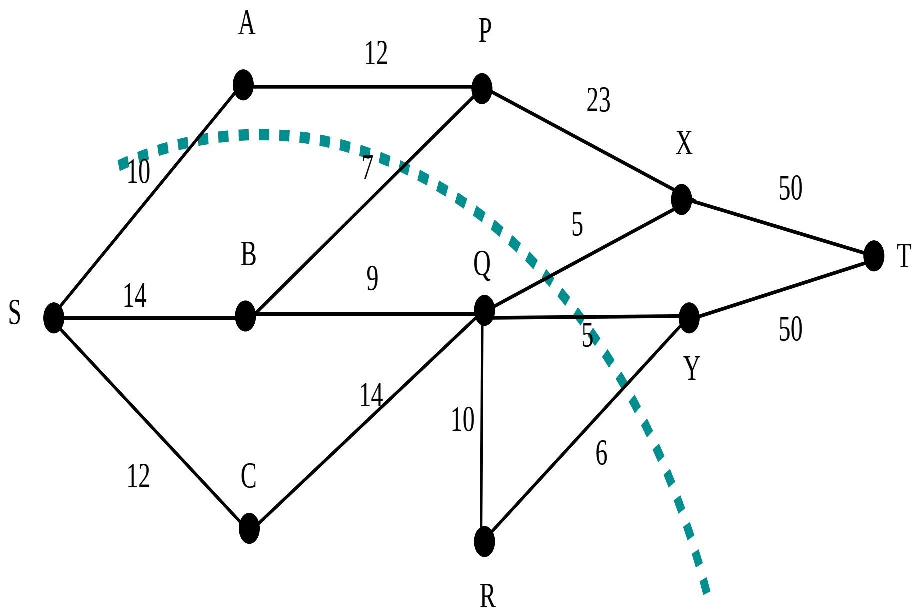
(b) The postman will have to cover each street twice, thus all of the nodes are now even, so the network is traversable and we need simply to find a route of total length 4600m



(c) The street cleaner has to follow directed edges so the in the network which models this process we will replace each of the arcs in the original network with a pair of directed edges in opposite directions. The total distance covered will be once more 4600m.

Solution to exercises on p 149

1. (a) This is the complete network, we assume that XT and YT are capacity 50.



We also show the minimum cut - $\{S, B, C, Q, R\}|\{A, P, X, Y, T\}$ with capacity 33.

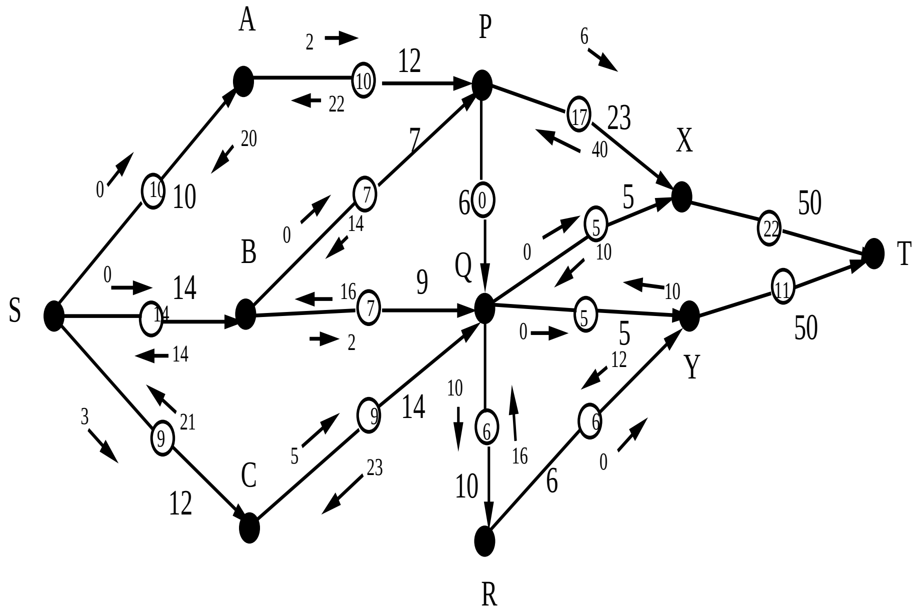
(b) This is a maximal flow, of 33.

Pipe link	SA	SB	SC	AP	BP	BQ	CQ	PX	QX	QY	QR	RY	XT	YT
Flow	10	14	9	10	7	7	9	17	5	5	6	6	22	11

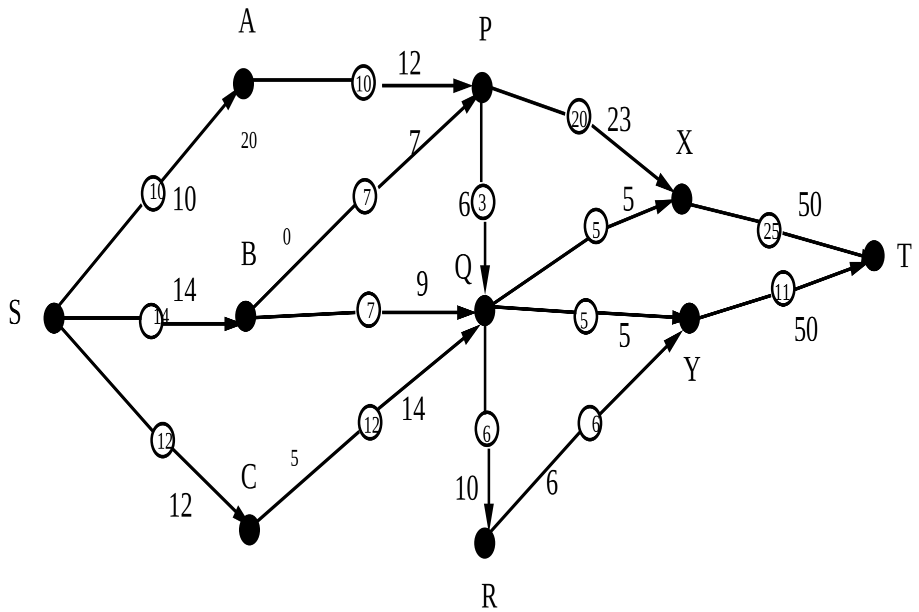
(c) Since the minimum cut = established flow we can be sure that we have found a maximum.

(d) A is supplying 10, B is supplying 14 and C is supplying 9, while X is receiving 22 and Y 11.

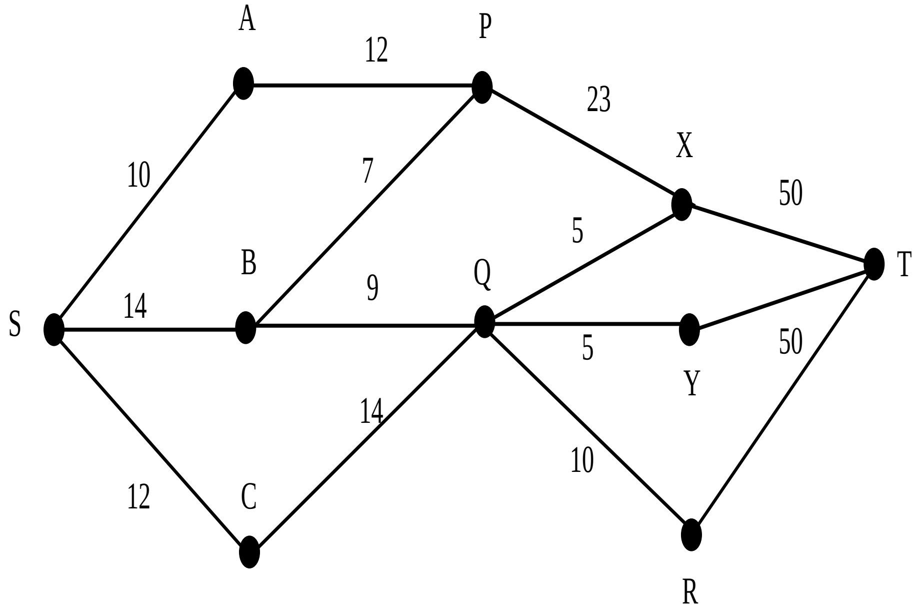
(e) This is the network with the forward and back capacities shown



The flow augmenting path is SCQPXT with a flow of 3 and the final pattern of flows is shown, giving a maximum flow of 36, which exhausts all the supplies and is thus the maximum flow, even though the minimum cut is larger than this figure



(f) We connect R and T and see that the resulting total delivery to X,Y and R is 43.

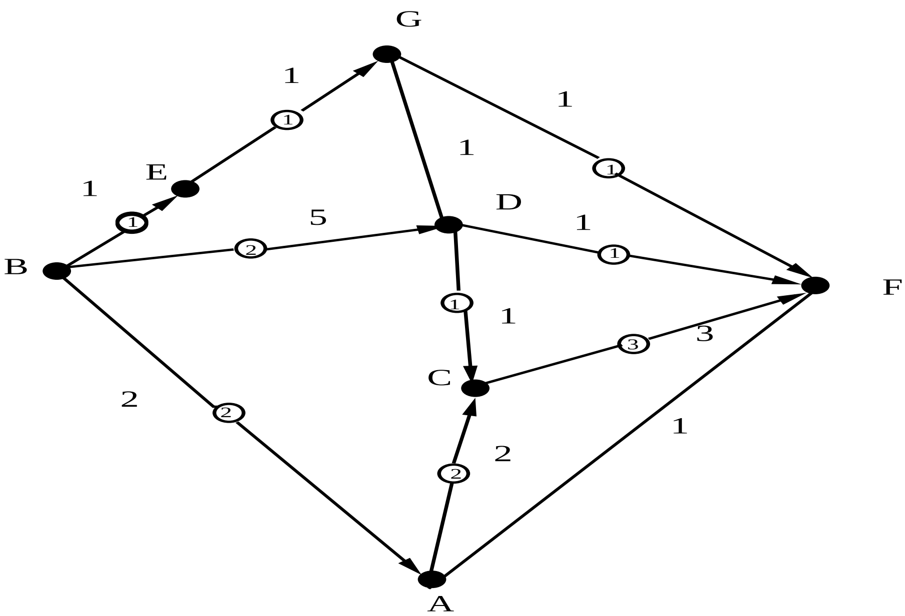


2. (a) There is one route that connects the source and the sink directly, there are $n-2$ routes which pass through 1 intermediate node, $(n-2)(n-3)$ routes which pass through two intermediate nodes, and so on.
   The total number of routes must therefore be

$$(n-2)+(n-2)(n-3)+(n-2)(n-3)(n-4)+\ldots(n-2)! = \sum_{r=0}^{n-3}\prod_{i=0}^{r}(n-2-i)$$

   (b) The algorithm will have to list all of the possible routes so will have factorial complexity. Thus, for n=10 this number is 10,960, for n=100 the number of possible paths is 2.56×10^{154}

3. The network is shown with a flow of 5000 cars from B to F.



The minimum cut is $\{B, D, E, G\}|\{A, C, F\}$ with capacity 5000 so we have found the maximum flow.