Calculus: Sheet 2

Please hand in your solutions to the starred questions at the START of our lecture on Thursday 7 November.

1. If $\cosh x = \frac{17}{15}$ and $x > 0$ find the values of $\sinh x$ and $\tanh x$.

2. Simplify the following expressions: (i) $2\cosh(\ln x)$, (ii) $\sinh(2\ln x)$.

3.* Differentiate the following functions: (i) $y = \sinh x^2$, (ii) $y = \ln(\cosh x)$.

4.* Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ and

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \qquad 0 < x \le 1.$$

5.* From the identity $\cosh^2 x - \sinh^2 x = 1$, show that $1 - \tanh^2 x = \operatorname{sech}^2 x$. Also find the derivative of $\operatorname{sech} x$. Hence show that if $y = \operatorname{sech} x$ then

$$\left(\frac{dy}{dx}\right)^2 + y^4 - y^2 = 0.$$

6.*

(i) Prove that $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$. [*You could do so either directly from the definition of sinh, or by proving a similar looking identity for* $\sin 3\theta$ *and then applying Osborn's rule to it.*]

(ii) We want to solve $y = x^3 + x$ for x in terms of y. Show that if $z = \frac{1}{2}\sqrt{3}\,x$ then the equation becomes $4z^3 + 3z = \frac{3}{2}\sqrt{3}\,y$. Then let $z = \sinh\theta$ and deduce using part (i) that

$$x = \frac{2}{\sqrt{3}}\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{3\sqrt{3}}{2}y\right)\right].$$

(iii) Following the idea in part (ii), but replacing the substitution $z = \frac{1}{2}\sqrt{3}\,x$ by something else, show that if $B > 0$ then the real root of the equation $x^3 + Bx = C$ is

$$x = 2\sqrt{\frac{B}{3}}\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{C}{2(B/3)^{3/2}}\right)\right].$$

7. Find

$$\text{(i)} \int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} \qquad \text{(ii)} \int_1^5 \frac{4}{\sqrt{x^2-2x+17}}\,dx \qquad \text{(iii)} \int \frac{\sqrt{1+x^2}}{x}\,dx$$

[Hint for (iii): substitute $u = \sqrt{1+x^2}$]

Answers for unstarred questions

1. $\sinh x = \frac{8}{15}$, $\tanh x = \frac{8}{17}$.
2. (i) $x + 1/x$, (ii) $\frac{1}{2}(x^2 - \frac{1}{x^2})$
7. (i) $\sinh^{-1}(\sqrt{3})$ (which equals $\ln(\sqrt{3}+2)$ by a result of exercise 4).
(ii) $4\sinh^{-1} 1$ or $4\ln(1+\sqrt{2})$.
(iii) $\sqrt{1+x^2} - \tanh^{-1}\sqrt{1+x^2} + c$. (There are other substitutions which produce a different looking but mathematically equivalent answer).