Calculus Coursework 2 (2013)

ADDITIONAL EXERCISES IF NEEDED

1. Solve the equation $3^{2x} - 2 \times 3^{x+1} + 5 = 0$. Give your answer correct to 2 d.p.s.

2. Solve the equation
$$2\cos x(2\cos^2 x - 1) + 1 = 2\sin^2 x,$$
   (i) for $x \in [0, 2\pi]$ and

   (ii) give the general solution for any x.

3. Find the set of real values of c for which the equation
$$\frac{x^2+1}{x^2-x+1} = c$$
is satisfied by at least one real value of x.

4. Given that $f : x \mapsto 1 + [x]$ and $g : x \mapsto \cos\left(\frac{\pi}{x}\right)$, find $f \circ g(4)$ and $g \circ f(\pi)$, giving your answer to 2 d.p.s.

5. For the following, sketch the graph and state with, justification whether the functions are odd, even, periodic (state the period), injective, surjective or bijective

   (a) $f : x \mapsto |\cos x|$

   (b) $g : x \mapsto \tan 3x + 1$

   (c) $h : x \mapsto 1 - \operatorname{sgn} x\, e^{|x|}$.

6. For all real x define
$$f : x \mapsto -3 + 5x, \quad g : x \mapsto x^2 + 5x + 3.$$
   (a) Find the range of each of f and g.

   (b) Define the inverse function f^{-1}.

   (c) Explain why g has no inverse. Restrict the domain and codomain so that $x \mapsto x^2 + 5x + 3$ is invertible and find the inverse function. How would you prove that your inverse function is correct?

   (d) Solve the equation $f \circ g = g \circ f$.

Calculus - **Coursework 1 Solutions** (2013)

1. By making the substitution $y = 3^x$ or otherwise, solve the equation

$$3^{2x} - 2 \times 3^{x+1} + 5 = 0.$$

Give your answer correct to 2 d.p.s.

Let $y = 3^x$. Then $y^2 - 6y + 5 = 0 \Rightarrow (y-5)(y-1) = 0 \Rightarrow y = 1, 5$. If $3^x = 5$ then $x = \frac{\ln 3}{\ln 5} = 1.46$ while if $y = 1$ then $x = 0$.

2. Solve the equation

$$2\cos x(2\cos^2 x - 1) + 1 = 2\sin^2 x,$$

(i) for $x \in [0, 2\pi]$ and

(ii) give the general solution for any x.

$$4\cos^3 x - 2\cos x + 1 = 2 - 2\cos^2 x$$

$$(2\cos x + 1)(2\cos^2 x - 1) = 0$$

Either $\cos x = -\dfrac{1}{2}, x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ i.e. $\pm\dfrac{2\pi}{3} + 2n\pi, n \in \mathbb{Z}$

Or $\cos x = \dfrac{1}{\sqrt{2}}, x = \dfrac{\pi}{4}, \dfrac{\pi}{4}$ i.e $\pm\dfrac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$

3. Find the set of real values of c for which the equation

$$\frac{x^2+1}{x^2-x+1} = c$$

is satisfied by at least one real value of x.

$$x^2(1-c) + cx + 1 - c = 0$$

The discriminant $\Delta = c^2 - 4(1-c)^2 = (3c-2)(2-c)$

We need $\Delta \geq 0$. The critical points are $c = \frac{2}{3}, c = 2$ so we need $\Delta \in \left[\frac{2}{3}, 2\right]$

4. Given that $f : x \mapsto 1 + [x]$ and $g : x \mapsto \cos\left(\frac{\pi}{x}\right)$, find $f \circ g(4)$ and $g \circ f(\pi)$, giving your answer to 2 d.p.s.

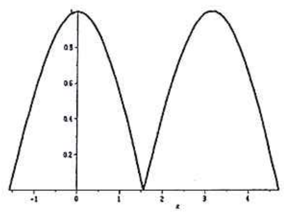
$f \circ g = 1 + [\cos\left(\frac{\pi}{x}\right) = 1$ for $x \neq 0$

$g \circ f = \cos\left(\frac{\pi}{1+[x]}\right), \quad \circ f(\pi) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

5. For the following, sketch the graph and state with, justification whether the functions are odd, even, periodic (state the period), injective, surjective or bijective

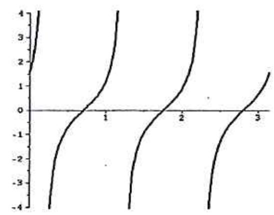
   (a) $f : x \mapsto |\cos x|$

   Not injective, surjective or bijective. Even. Periodic with period 2π.



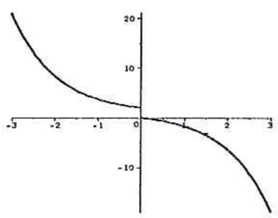
   (b) $g : x \mapsto \tan 3x + 1$

   Not odd or even, $\frac{\pi}{3}$ periodic, surjective, not injective



   (c) $h : x \mapsto 1 - \mathrm{sgn}\, \mathrm{x}\, \mathrm{e}^{|\mathrm{x}|}$.

   Not odd nor even. Not injective, surjective or bijective. Not periodic.



6. For all real x define

$$f : x \mapsto -3 + 5x, \quad g : x \mapsto x^2 + 5x + 3.$$

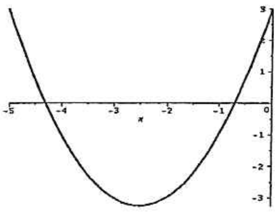
   (a) Find the range of each of f and g.
   (b) Define the inverse function f^{-1}.
   (c) Explain why g has no inverse. Restrict the domain and codomain so that $x \mapsto x^2 - 5x + 3$ is invertible and find the inverse function. How would you prove that your inverse function is correct?
   (d) Solve the equation $f \circ g = g \circ f$.

(a) Range of f is $\mathbb{R}$.

$g(x) = \left(x + \frac{5}{2}\right)^2 - \frac{13}{4}$ Range is $\left[-\frac{13}{4}, \infty\right)$

(b) $f^{-1}(x) = \dfrac{x+1}{3}$

(c) g is not bijective



We restrict the range to $\left[-\frac{5}{2}, \infty\right]$, codomain to $\left[-\frac{13}{4}, \infty\right)$

$$x = y^2 + 5y + 3 \Rightarrow y = \frac{-5 + \sqrt{25 + 4(3-x)}}{2} \Rightarrow f^{-1}(x) = \frac{1}{2}\left(-5 + \sqrt{13 - 4x}\right)$$

To show the solution is correct, show that $f(f^{-1}(x)) = 1$.

(d)

$$-3+5(x^2+5x+3) = (-3+5x)^2+5(-3+5x)+3 \Rightarrow 4x^2-6x-3 = 0 \Rightarrow x = \frac{3}{4} \pm \frac{\sqrt{21}}{5}$$