SOLUTIONS-FOR-MOCK-TEST

Question 1:

$$2u_x - 3u_y = 7u \qquad (*)$$

The characteristic equation is the ODE

$$\frac{dy}{dx} = -\frac{3}{2} \quad \therefore \quad 2y + 3x = \text{constant} = \eta(x,y)$$

For the other variable $\xi = \xi(x,y)$ let us choose $\xi = x$. With this choice we then have that the Jacobian J is given by

$$J = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2 \neq 0 \text{ everywhere.}$$

Thus $(*)$ becomes $2W_\xi = 7W$ and $\therefore 2W_\xi - 7W = 0$.

Integrating factor is $IF = e^{-\frac{7}{2}\xi}$.

Thus $\frac{d}{d\xi}\left(e^{-\frac{7}{2}\xi}W\right) = 0 \quad \therefore W = e^{\frac{7}{2}\xi}g(\eta)$.

Going back to the (x,y) variables we obtain

$$u = u(x,y) = e^{\frac{7}{2}x}g(2y+3x) \qquad (**)$$

$(**)$ is our general solution depending upon the arbitrary differential function $g = g(2y+3x)$.

For the particular solutions $u\left(x, y = \frac{2x}{3}\right) = e^{3x}$ one has:

$$u\left(x, \tfrac{2}{3}x\right) = e^{3x} = e^{\frac{7}{2}x}g\left(\frac{13x}{3}\right) \quad \therefore g\left(\tfrac{13}{3}x\right) = e^{-\frac{1}{2}x}.$$

Let $t = \frac{13}{3}x$ then $x = \frac{3t}{13}$ and $\therefore g(t) = e^{-\frac{3}{26}t}$. Hence one has

$$u(x,y) = e^{\frac{7}{2}x}e^{-\frac{3}{26}(2y+3x)} = e^{-\left(\frac{3}{13}y - \frac{41}{13}x\right)}.$$

Question-2-

$u_{xx} - u_{yy} = 4(x+y)^3 = G(x,y)$

$u_{xx} - u_{yy} - 4(x+y)^3 = 0.$

$A = 1;\ B = 0;\ C = -1;\ D = E = F = 0;\ G = +4(x+y)^3.$

$B^2 - AC = 0 - (1 \times -1) = 1 > 0$

Therefore the equation is hyperbolic every where on the plane (x,y).

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{0 \pm \sqrt{1}}{1} = \pm 1$$

For $\xi(x,y)$ take $\frac{dy}{dx} = +1$

$$\int dy = \int dx$$
$$y = x + c$$
$$y - x = c = \xi(x,y)$$

For $\eta(x,y)$ take $\frac{dy}{dx} = -1$

$$\int dy = \int -1\,dx$$
$$y = -x + \hat{c}$$
$$y + x = \hat{c} = \eta(x,y)$$

So characteristics are : $\begin{cases} \xi = y - x \\ \eta = y + x \end{cases}$. Jacobian $J = -1 \neq 0$.

$\xi_x = -1 \qquad \eta_x = 1 \qquad \xi_{xy} = 0$

$\xi_{xx} = 0 \qquad \eta_{xx} = 0 \qquad \eta_{xy} = 0$

$\xi_y = 1 \qquad \eta_y = 1$

$\xi_{yy} = 0 \qquad \eta_{yy} = 0$

By construction the $w_{\xi\xi}$ and $w_{\eta\eta}$ terms are zero.

Calculating the coefficients of the remaining terms :

- $2[A \xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + C \xi_y \eta_y] W_{\xi\eta}$
  $= 2[(1 \times -1 \times 1) + 0 + (-1 \times 1 \times 1)] W_{\xi\eta}$
  $= 2[-1 - 1] W_{\xi\eta}$
  $= -4 W_{\xi\eta}$

- $[A \xi_{xx} + 2B \xi_{xy} + C \xi_{yy}] W_\xi$ ✓
  $= [(1 \times 0) + 0 + (-1 \times 0)] W_\xi$
  $= 0$

- $[A \eta_{xx} + 2B \eta_{xy} + C \eta_{yy}] W_\eta$ ✓
  $= [(1 \times 0) + 0 + (-1 \times 0)] W_\eta$
  $= 0$

- As $D = E = F = 0$, terms involving these values are also zero.

- $G = +(x+y)^3$
  substitute $\eta = x + y$ ∴ One has :
  $G = +(\eta)^3$

✓

So, going back to transformed PDE gives,

$-4W_{\xi\eta} - 4\eta^3 = 0$ in canonical form of the hyperbolic equation.
$W_{\xi\eta} + \eta^3 = 0$
$W_{\xi\eta} = -\eta^3$.

Solving the equation,
$W_\xi = -\frac{1}{4}\eta^4 + f(\xi)$
$W = -\frac{1}{4}\eta^4 \xi + \hat{f}(\xi) + g(\eta)$ with $\hat{f}(\xi) = \int f(\xi) d\xi$.

✓

Hence, general solution is
$u(x,y) = -\frac{1}{4}(x+y)^4 (y-x) + \hat{f}(y-x) + g(x+y)$

-Question 3-

$u_t = u_{xx}$, $0 < x < 1$, $t \geq 0$,
$u(0,t) = u(1,t) = 0$, $t \geq 0$, $u(x,0) = \sin(7\pi x)$.
We use the method of separation of variables:

$$u(x,t) = X(x)T(t)$$

$$X_{xx}T = XT_t \Rightarrow \frac{X_{xx}}{X} = \frac{T_t}{T} = -\lambda$$

$$\Rightarrow X_{xx} + \lambda X = 0, \quad T_t + \lambda T = 0$$

$X(0)T(t) = X(1)T(t) = 0 \Rightarrow X(0) = X(1) = 0$ for non-trivial solutions.

1) $\lambda = 0$:

$X_{xx} = 0 \Rightarrow X = ax + b$.

$X(0) = 0 \Rightarrow b = 0$
$X(1) = 0 \Rightarrow a = 0$

$\Rightarrow X = 0$: trivial solution. So $\lambda = 0$ is not an eigenvalue.

2) $\lambda = -\nu^2$, $\nu > 0$:

$X_{xx} - \nu^2 X = 0 \Rightarrow X = a\cosh(\nu x) + b\sinh(\nu x)$

$X(0) = 0 \Rightarrow a\cosh(0) = 0 \Rightarrow a = 0.$
$X(1) = 0 \Rightarrow b\sinh(\nu) = 0 \Rightarrow b = 0.$

$\Rightarrow X = 0$: trivial solution. So there are no negative eigenvalues.

3) $\lambda = \mu^2$, $\mu > 0$.

$$X_{xx} + \mu^2 X = 0, \quad X(x) = a\cos(\mu x) + b\sin(\mu x).$$

$X(0) = 0$ gives $a = 0$

$X(1) = 0$ " $b\sin(\mu) = 0$ $\therefore \mu = n\pi$, $n = 1, 2, 3, \ldots$

$\therefore \lambda_n = n^2\pi^2$, $X_n(x) = b_n \sin(n\pi x)$.

Going back to the temporal component $T = T(t)$ one has $T_t + \lambda T = T_t + n^2\pi^2 T = 0$

$\therefore T(t) = e^{-n^2\pi^2 t}$ and $\therefore u_n(x,t) = b_n \sin(n\pi x) e^{-n^2\pi^2 t}$.

We now use linearit of our PDE to form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}.$$

We now use the initial condition

$$u(x,0) = \sin(7\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

with $b_n = 2\int_0^1 \sin(7\pi \xi)\sin(n\pi\xi)\, d\xi$.

Hence $b_n = 0$ for any $n \neq 7$. For $n = 7$ we have $b_2 = 2\int_0^1 \sin^2(7\pi\xi)\, d\xi = 1$.

$$\therefore u(x,t) = \sin(7\pi x) e^{-\pi^2 n^2 t}, \quad n = 7.$$

$$\therefore u(x,t) = \sin(7\pi x) e^{-49\pi^2 t}.$$