 Question - 1 - ASSESSED TEST - 2014

$u_x - 2u_y = u$

$\frac{dy}{dx} = -2 \quad y = -2x + c \quad c = y + 2x = \eta(x,y).$

5 $\left.\begin{matrix} \eta = y + 2x \\ \xi = x \end{matrix}\right\}$ $\quad J = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0 \quad \forall (x,y)$

OR $J = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \neq 0.$

13 $u(x,y) = w(\eta, \xi)$

$w_\xi = w$

$\ln w = \xi + \hat{g}(\eta)$

$w = e^{\xi} g(\eta)$ with $g(\eta) = \int \hat{g}(\eta)\, d\eta.$

$\therefore u(x,y) = e^{x} g(y+2x)$

$u_x = e^{x} g(y+2x) + 2e^{x} g'$

$u_y = e^{x} g'$

$u_x - 2u_y = u$ ✓

For the solution $u(0,y) = y$ one has:

$u(0,y) = y = g(y)$

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$\therefore u(x,y) = e^{x}(y + 2x).$

$u(0,y) = y.$ ✓

Question-2-

$u_{xx} - u_{yy} = (x+y)^2$

$u_{xx} - u_{yy} - (x+y)^2 = 0$

$A = 1$; $B = 0$; $C = -1$; $D = E = F = 0$; $G = +(x+y)^2$

4 $B^2 - AC = 0 - (1 \times -1) = 1 > 0$

Therefore the equation is hyperbolic.

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{0 \pm \sqrt{1}}{1} = \pm 1$$

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For $\xi(x,y)$ take $\frac{dy}{dx} = +1$

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$$\int dy = \int dx$$
$$y = x + c$$
$$y - x = c = \xi(x,y)$$

For $\eta(x,y)$ take $\frac{dy}{dx} = -1$

$$\int dy = \int -1\, dx$$
$$y = -x + \hat{c}$$
$$y + x = \hat{c} = \eta(x,y)$$

4 So characteristics are : $\begin{cases} \xi = y - x \\ \eta = y + x \end{cases}$

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$\xi_x = -1$ $\quad \eta_x = 1$ $\quad \xi_{xy} = 0$

$\xi_{xx} = 0$ $\quad \eta_{xx} = 0$ $\quad \eta_{xy} = 0$

$\xi_y = 1$ $\quad \eta_y = 1$

$\xi_{yy} = 0$ $\quad \eta_{yy} = 0$

By construction the $w_{\xi\xi}$ and $w_{\eta\eta}$ terms are zero.

Calculating the coefficients of the remaining terms :

- $2[A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y]W_{\xi\eta}$
$= 2[(1\times -1\times 1) + 0 + (-1\times 1\times 1)]W_{\xi\eta}$
$= 2[-1-1]W_{\xi\eta}$
$= -4W_{\xi\eta}$

- $[A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy}]W_\xi$
$= [(1\times 0) + 0 + (-1\times 0)]W_\xi$
$= 0$

- $[A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy}]W_\eta$
$= [(1\times 0) + 0 + (-1\times 0)]W_\eta$
$= 0$

- As $D = E = F = 0$, terms involving these values are also zero.

- $G = +(x+y)^2$
substitute $\eta = x+y$:
$G = +(\eta)^2$

So, going back to transformed PDE gives,

8 $-4W_{\xi\eta} - \eta^2 = 0$ in canonical form of the hyperbolic equation.
$W_{\xi\eta} + \frac{1}{4}\eta^2 = 0$
$W_{\xi\eta} = -\frac{1}{4}\eta^2$

Solving the equation,
$W_\xi = -\frac{1}{4}\eta^3 \times \frac{1}{3} + f(\xi)$
$W = -\frac{1}{12}\eta^3\xi + \hat{f}(\xi) + g(\eta)$

Hence, general solution is
$u(x,y) = -\frac{1}{12}(x+y)^3(y-x) + \hat{f}(y-x) + g(x+y)$

QUESTION-3-

1. Using separation of variables:

$$u(x,t) = X(x)\,T(t)$$

$$X_{xx} T = X T_t \Rightarrow \frac{X_{xx}}{X} = \frac{T_t}{T} = -\lambda$$

5 $\Rightarrow X_{xx} + \lambda X = 0, \quad T_t + \lambda T = 0$

$X(0)\,T(t) = X(1)\,T(t) = 0 \Rightarrow X(0) = X(1) = 0$ for non-trivial solutions.

1) $\lambda = 0$:

$X_{xx} = 0 \Rightarrow X = ax + b.$

3 $X(0) = 0 \Rightarrow b = 0$
$X(1) = 0 \Rightarrow a = 0$

$\Rightarrow X = 0$: trivial solution. So $\lambda = 0$ is not an eigenvalue.

2) $\lambda = -\nu^2, \ \nu > 0$:

5 $X_{xx} - \nu^2 X = 0 \Rightarrow X = a\cosh(\nu x) + b\sinh(\nu x)$

$X(0) = 0 \Rightarrow a\cosh(0) = 0 \Rightarrow a = 0.$
$X(1) = 0 \Rightarrow b\sinh(\nu) = 0 \Rightarrow b = 0$, as $\nu > 0$ and $\sinh(\nu) \neq 0$.

$\Rightarrow X = 0$: trivial solution. So there are no negative eigenvalues.

3) $\lambda = \mu^2$, $\mu > 0$.

10 $X_{xx} + \mu^2 X = 0$, $X(x) = a\cos(\mu x) + b\sin(\mu x)$.

$X(0) = 0$ gives $a = 0$

$X(1) = 0$ " $b\sin(\mu) = 0$ $\therefore \mu = n\pi$, $n = 1, 2, 3, \dots$

$\therefore \lambda_n = n^2\pi^2$, $X_n(x) = b_n \sin(n\pi x)$.

4 Going back to the temporal component $T = T(t)$
one has $T_t + \lambda T = T_t + n^2\pi^2 T = 0$

$\therefore T(t) = e^{-n^2\pi^2 t}$ and $\therefore u_n(x,t) = b_n \sin(n\pi x) e^{-n^2\pi^2 t}$.

3 We now use linearit of our PDE to form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}.$$

We now use the initial condition

2 $u(x,0) = \sin(2\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

2 with $b_n = 2\int_0^1 \sin(2\pi\xi)\sin(n\pi\xi)\,d\xi$.

2 Now for $n \neq 2$, $b_n = \frac{2}{2}\int_0^1 \cos[(2\pi - n\pi)\xi] - \cos[(2\pi + n\pi)\xi]\,d\xi =$

$$= \frac{1}{(2-n)\pi}\left[\sin((2\pi - n\pi)\xi)\right]_0^1 - \frac{1}{(2+n)\pi}\left[\sin((2\pi + n\pi)\xi)\right]_0^1 = 0$$

2 For $n = 2$, $b_2 = 2\int_0^1 \sin^2(2\pi\xi)\,d\xi = 1$

2 $\therefore u(x,t) = \sin(2\pi x)\, e^{-4n^2 t}$.