PDEs: UN-ASSESSED HOMEWORK III
Please solve all the questions below
Please hand (the solutions of) it in at the end of our second lecture on Thursday 3/4/2014

1) Solve the heat equation on bounded intervals in each of the problems below, with the given initial and boundary conditions:

✓ (a) $u_{xx} = u_t, \quad u(0,t) = u(1,t) = 0, \;\; t \geq 0; \quad u(x,0) = \sin(\pi x), \;\; 0 \leq x \leq 1.$

✓ (b) $ku_{xx} = u_t, \quad u_x(0,t) = u_x(4,t) = 0, \;\; t \geq 0; \quad u(x,0) = x^2, \;\; 0 \leq x \leq 4;$

2) Solve the heat equation on unbounded intervals with the given initial and boundary conditions:

✓ (a) $-ku_{xx} + u_t = 0, \;\; x \geq 0, \quad u(0,t) = 0, \;\; t \geq 0,$

$$u(x,0) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & x > 1. \end{cases}$$

✓ (b) $-ku_{xx} + u_t = 0, \;\; x \geq 0, \quad u(0,t) = 0, \;\; t \geq 0, \;\; u(x,0) = e^{-\alpha x}, \;\; \alpha > 0.$

✓ 3) Suppose u is a solution of the heat equation

$$-ku_{xx} + u_t = 0$$

for $0 < x < L$, $t > 0$. Let $0 < a < b < L$. Prove that, for any positive t,

$$\frac{d}{dt}\int_a^b \frac{1}{2}u(x,t)^2 dx = k[uu_x]_a^b - \int_a^b ku_x(x,t)^2 dx.$$

a) Question -1- COURSEWORK III -

$$\begin{cases} u_t = u_{xx} \ ; & t > 0 \ ; \ 0 < x < 1 \ ; \\ u(0,t) = u(1,t) = 0 \ ; & t \geq 0 \\ u(x,0) = \sin(\pi x) \end{cases}$$

Use the separation of variables method:

$u = X(x)\,T(t)$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$\therefore \ X'' + \lambda X = 0 \ ; \ X(0) = X(1) = 0$

$T' + \lambda T = 0$

$\lambda \in (-\infty, +\infty)$

$\lambda \leq 0$ No eigenvalues

$\lambda = \mu^2 > 0 \ ; \ \mu > 0$

$X = a\cos(\mu x) + b\sin(\mu x)$

using $X(0) = X(1) = 0$ we have

$X_n = b_n \sin(n\pi x) \ ; \ n = 1, 2, 3$

$\lambda_n = n^2\pi^2$

$T = c\, e^{-n^2\pi^2 t}$

$$u_n(x,t) = X_n(x)\,T_n(t) =$$

$$= b_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

We then try an infinite superposition

$$u(x,t) = \sum_1^\infty u_n(x,t) = \sum_1^\infty b_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$$u(x,0) = \sin(\pi x) = \sum_1^\infty b_n \sin(n\pi x)$$

$$\therefore\ b_n = \frac{2}{1}\int_0^1 \sin(\pi \xi)\sin(n\pi \xi)\, d\xi$$

$$= \begin{cases} 0 & n > 1 \\ 1 & n = 1 \end{cases}$$

$$\therefore\ u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$$

b) Question 1 ③

$$\begin{cases} u_t = k\,u_{xx} \ ; & t>0 \ ; \ 0<x<4 \ ; \\ u_x(0,t) = u_x(4,t) = 0 \ ; & t \geq 0 \ ; \\ u(x,0) = x^2 & \end{cases}$$

$u = X\,T$

$X'' + \lambda X = 0 \ ; \quad X_x(0) = X_x(4) = 0$

$T' + \lambda k T = 0$

$\lambda < 0$ NO EIGENVALUES

$\lambda = 0$ yes $\quad X = b \neq 0$ Eigenfunction

$\lambda = \mu^2 > 0 \ ; \ \mu > 0$

$X = a\cos(\mu x) + b\sin(\mu x)$

$X'(0) = 0 = b$

$X'(4) = 0 = -a\mu\sin(\mu 4) = 0$

$$\mu_n = \frac{n\pi}{4} \ ; \quad n = 1, 2, 3, \ldots \quad \lambda_n = \frac{n^2\pi^2}{16}$$

$$u_n(x,t) = a_n\cos\left(\frac{n\pi}{4}x\right)e^{-\frac{n^2\pi^2}{16}kt}$$

④

$$u(x,t) = b + \sum_1^\infty a_n \cos\left(\frac{n\pi}{4}x\right) e^{-\frac{n^2\pi^2}{16}kt}$$

$$u(x,0) = x^2 = \frac{a_0}{2} + \sum_1^\infty a_n \cos\left(\frac{n\pi}{4}x\right)$$

$$\therefore\ a_n = \frac{1}{2}\int_0^4 \xi^2 \cos\left(\frac{n\pi}{4}\xi\right) d\xi \quad , \quad n = 0, 1, 2$$

$$u(x,t) = \frac{16}{3} + \sum_1^\infty \frac{(-1)^n 64}{n^2\pi^2} \cos\left(\frac{n\pi}{4}x\right) e^{-\frac{n^2\pi^2}{16}kt}$$

Question - 2 — We first solve b) as 2-a) is easier.

b) $u_t = k u_{xx}$, $x \geq 0$, $t > 0$;

$u(x,0) = e^{-\alpha x}$, $\alpha > 0$;

$u(0,t) = 0$; $t \geq 0$.

Use separation of variables

$X'' + \lambda X = 0$

$T' + \lambda k T = 0$

$u(0,t) = 0 = X(0)\,T(t)$ $\therefore X(0) = 0$

We look for BOUNDED solutions

$X(0) = 0$ gives $X \equiv 0$ for $\lambda = 0$

$\lambda = -\omega^2$ then $X = a e^{-\omega x} + b e^{\omega x}$

$X(0) = 0 = a + b = 0$

But X bounded at $x = \pm \infty$ gives $a = b = 0$

Therefore no eigenvalues for $\lambda \leq 0$.

for $\lambda = \omega^2 > 0$; $\omega > 0$

$X = a\cos(\omega x) + b\sin(\omega x)$

$X(0) = 0 = a$

$X_\omega = b_\omega \sin(\omega x)$ for every $\omega > 0$

any real number

$T = e^{-\omega^2 kt}$

$u = b_\omega \sin(\omega x)\, e^{-\omega^2 kt}$

Attempt a continuous superposition

$$u = \int_0^\infty b_\omega \sin(\omega x)\, e^{-\omega^2 kt}\, d\omega$$

$$u(x,0) = f(x) = e^{-\alpha x} = \int_0^\infty b_\omega \sin(\omega x)\, d\omega$$

$\therefore$ b_ω is the coefficient in the Fourier sine integral expansion of $f(x) = e^{-\alpha x}$.

$$b_\omega = \frac{2}{\pi}\int_{-\infty}^{+\infty} e^{-\alpha \rho} \sin(\omega \rho)\, d\rho = \frac{2}{\pi}\,\frac{\omega}{\alpha^2+\omega^2}$$

$$\therefore\; u = \frac{2}{\pi}\int_0^\infty \frac{\omega}{\alpha^2+\omega^2}\sin(\omega x)\, e^{-\omega^2 kt}\, d\omega$$

Question-2-

a) $U_t = kU_{xx}; \quad t>0 \quad x \geq 0$

$U(0,t) = 0; \quad t \geq 0;$

$$U(x,0) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x > 1 \end{cases}$$

This question is very similar to question 2-b) and the final result reads

$$U(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\omega)}{\omega} \sin(\omega x)\, e^{-\omega^2 kt}\, d\omega.$$

Question - 3 -

$$u_t = k u_{xx}$$

$$u u_t = k u u_{xx}$$

Integrete in space

$$\int_a^b u u_t \, dx = k \int_a^b u u_{xx} \, dx$$

$$\therefore \frac{1}{2} \frac{d}{dt} \int_a^b (u)^2 \, dx = \left[k u u_x \right|_a^b - k \int_a^b (u_x)^2 \, dx,$$

where we have first differentiated and then we have integrated by parts.