UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT2011; 15 Credits

Linear PDEs

FHEQ Level 5 (Year 2) Examination

Time allowed: **2 hours** Semester 2 2012/13

Answer **THREE** questions only.

If a candidate attempts more than THREE questions,
then only the best THREE solutions will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual
part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Explain how to reduce the partial differential equation

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \quad \text{with} \quad a(x,y) \neq 0$$

to the form

$$aw_\xi + cw = d,$$

where $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. [10]

(b) Given the first order linear non-homogeneous partial differential equation

$$x^2u_x - xyu_y - xu = -y^2.$$

(i) Find its characteristic curves. [3]

(ii) Show that its general solution is given by

$$u(x,y) = \frac{y^2}{4x} + xg(xy),$$

where g is an arbitrary differentiable function of its argument. [7]

(iii) Show that there are infinitely many solutions if we require that for $x > 0$

$$u(x, y = \frac{1}{x}) = \frac{1}{4x^3} + x.$$

[5]

Question 2

The function $u(x, y)$ satisfies the partial differential equation

$$u_{xx} - 4u_{xy} + 3u_{yy} - 4xu_x - 4yu_y - 4(3x^2 + 4xy + y^2 + 1)u = 0.$$

(i) Show that it is everywhere hyperbolic and $\xi = 3x + y, \eta = x + y$ are suitable characteristic coordinates. [3]

(ii) Transform the equation to the canonical form. [7]

(iii) Verify that this canonical form can be written as

$$v_\xi + \eta v = 0, \quad \text{where} \quad v = w_\eta + \xi w.$$

[2]

(iv) Hence obtain its general solution in the form

$$u(x, y) = e^{-(3x+y)(x+y)}[f(3x + y) + g(x + y)],$$

where f and g are arbitrary twice differentiable functions of their argument. [7]

(v) Find the particular solution which satisfies the Cauchy data

$$u(x, 0) = 6, \quad u_y(x, 0) = 0, \quad -\infty < x < \infty.$$

[6]

Question 3

(a) Consider the Dirichlet problem for a disk of radius ρ about the origin in the plane. By using polar coordinates the problem reduces to

$$\begin{aligned} \Delta u(r,\theta) &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \\ & \quad 0 \le r < \rho, \quad -\pi \le \theta \le \pi, \\ u(\rho,\theta) &= f(\theta), \quad -\pi \le \theta \le \pi. \end{aligned}$$

(i) By using the method of separation of variables, show that a solution of the above problem is given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k r^k \cos(k\theta) + b_k r^k \sin(k\theta)],$$

where the coefficients a_k and b_k are given by

$$a_k = \frac{1}{\pi\rho^k}\int_{-\pi}^{\pi} f(\xi)\cos(k\xi)d\xi, \qquad k = 0, 1, 2, ...,$$

$$b_k = \frac{1}{\pi\rho^k}\int_{-\pi}^{\pi} f(\xi)\sin(k\xi)d\xi, \qquad k = 1, 2, 3...$$

[12]

(ii) Find the solution for the particular function $f(\theta) = \cos^2(\theta)$. [4]

(b) The function $u(x,t)$ satisfies the equation

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < L, \quad t > 0, \quad c \ne 0,$$

the boundary conditions $u(0,t) = A(t)$, and $u(L,t) = B(t)$ for all $t \ge 0$, and the initial data $u(x,0) = \phi(x)$, and $u_t(x,0) = \psi(x)$. The functions f, A, B, ϕ, ψ are continuous and the boundary and initial conditions are compatible at the end points.

(i) Let $w = u - v$, with u and v two solutions of the equation satisfying the same data above, and let

$$E(t) = \frac{1}{2}\int_0^L (w_t^2 + c^2 w_x^2)dx,$$

show that

$$\frac{dE}{dt} = 0.$$

[6]

(ii) Deduce that $E \equiv 0$ for all $t \ge 0$, and hence derive a uniqueness theorem for the above problem. [3]

Question 4

(a) Consider the linear parabolic partial differential equation

$$u_{xx} - u_t = 0, \quad t > 0, \quad x > 0,$$

satisfying the conditions

$$u(0,t) = 0, \quad t \geq 0, \quad u(x,0) = xe^{-\frac{x^2}{4}}, \quad 0 \leq x < \infty,$$

and $u(x,t)$ bounded for all $t \geq 0$ and for all $x \geq 0$.
By using the separation of variables method, find its general solution. It may be assumed without proof that

$$\int_0^\infty \phi e^{-\frac{a\phi^2}{4}} \sin(\eta\phi) d\phi = 2\pi^{1/2} a^{-3/2} \eta e^{-\eta^2/a}, \quad a > 0.$$

[18]

(b) Let Ω be a regular bounded domain in $\mathbf{R}^n$ and let $f(x)$ with $x \in \mathbf{R}^n$ be continuous on the domain boundary $\partial\Omega$. Prove that the Dirichlet problem for the Laplace equation in $\mathbf{R}^n$

$$\Delta u = 0 \text{ in } \Omega,$$
$$u = f(x) \text{ on } \partial\Omega,$$

has at most one solution which is continuous in the closure of the domain $\overline{\Omega} = \Omega \cup \partial\Omega$.
[7]

END OF PAPER

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