UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematics

Module MAT2011; 15 Credits

Linear PDEs

FHEQ Level 5 (Year 2) Examination

Time allowed: **2 hours** Summer Resit 2012/13

Answer **THREE** questions only.

If a candidate attempts more than THREE questions,
then only the best THREE solutions will be taken into account.

Each question carries 25 marks.

Where appropriate the mark carried by an individual
part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Consider the linear first order non-homogeneous partial differential equation

$$u_x + yu_y = u + ye^{-x}.$$

(i) Show that its general solution is given by

$$u(x, y) = -ye^{-x} + e^x g(ye^{-x}),$$

in which g is any differentiable function of its argument. [7]

(ii) Find the solution such that

$$u = 1 \qquad \text{on} \qquad x = 0.$$

[3]

(b) Given the first order linear homogeneous partial differential equation

$$(x + y)u_x - (x + y)u_y - u = 0.$$

(i) Show that its general solution is given by

$$u(x, y) = e^{\frac{x}{x+y}} g(x + y),$$

where g is an arbitrary differentiable function of its argument. [7]

(ii) Show that there are infinitely many solutions if we require that

$$u(x, y = 1 - x) = e^x.$$

[3]

(c) Explain why in case (a) we obtain one and only one solution, while in case (b) we obtain an infinite number of solutions. [5]

Question 2

(a) The function $u(x,t)$ satisfies the equation

$$u_{xx} - u_t = 0, \quad 0 < x < 1, \quad t > 0.$$

Assume that the functions $u_1(x,t)$ and $u_2(x,t)$ are two solutions of the above equation, both satisfying the data

$$u(x,0) = f(x), \quad 0 \leq x \leq 1, \qquad u(0,t) = g(t), \qquad u(1,t) = h(t), \quad t > 0,$$

where the functions f, g, h are continuous and compatible at the end points. Derive a uniqueness theorem for the above problem.

Hint: Let $w(x,t) = u_1(x,t) - u_2(x,t)$ and $I(t) = \dfrac{1}{2}\displaystyle\int_0^1 w(x,t)^2 dx.$ [11]

(b) The function $u(x,y)$ satisfies the partial differential equation

$$2u_{xx} - u_{xy} - 3u_{yy} + u_x + u_y = 5e^{x-y}.$$

(i) Show that it is hyperbolic and that $\xi = x - y, \eta = 3x + 2y$ are suitable characteristic coordinates. [4]

(ii) Transform the equation to canonical form and hence obtain its general solution. [10]

Question 3

(a) The steady state temperature u, in a thin flat plate is modelled by the Dirichlet problem for Laplace's equation on a rectangle:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 < x < a, \quad 0 < y < b, \\ u(x,0) &= 0, \quad 0 \le x \le a, \\ u(0,y) &= u(a,y) = 0, \quad 0 \le y \le b, \\ u(x,b) &= f(x), \quad 0 < x < a, \end{aligned}$$

where a and b are positive constants. The temperature on the lower and vertical sides of the rectangle is kept at zero value, and it is kept at the value $f(x)$ along the top side.

(i) By using the method of separation of variables, show that a solution of the above problem is given by

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

where the coefficients b_n are given by

$$b_n = \frac{2}{a\sinh(\frac{n\pi b}{a})} \int_0^a f(\xi) \sin\left(\frac{n\pi \xi}{a}\right) d\xi,$$

[12]

(ii) Find the solution for the particular function $f(x) = x$. [6]

(b) Consider the function $u(x,y)$ which is twice differentiable in both variables (x,y) with continuous second partial derivatives, and such that it satisfies the equation

$$u_{xx} + u_{yy} + 2u_x + u_y = (x^2+1)(y^2+1)$$

in a bounded region R of the (x,y) plane. The region R consists of an interior domain D with closed and distinct boundary curve C; namely $R = D \cup C$. Show that u cannot have a maximum in D. [7]

Question 4

(a) The temperature distribution in a conducting bar of length L, in which the ends are kept at different temperatures is modelled by the following initial-boundary value problem for the parabolic linear partial differential equation of the form

$$u_t = k u_{xx}, \quad t > 0, \quad 0 < x < L,$$

satisfying the conditions

$$u(0,t) = T_1, \quad u(L,t) = T_2, \quad t \geq 0,$$

and

$$u(x,0) = f(x), \quad 0 \leq x \leq L.$$

Here k is any positive constant and T_1 and T_2 are distinct positive numbers. Find the general solution of the above initial-boundary value problem. [15]

(b) Find the particular solution in the case where $k = 1$, $L = 1$, $T_1 = 1$, $T_2 = 2$ and

$$f(x) = \begin{cases} 2x+1, & 0 \leq x \leq \frac{1}{2}, \\ 2, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

[10]

END OF PAPER

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