**MS315: LAGRANGIAN AND HAMILTONIAN DYNAMICS: ASSESSED COURSEWORK I**
**Please hand (the solutions of) it in at the end of our lecture on Friday 24/10/2008**

Thought of the coursework. To some-One who could view the universe from a unified standpoint, the entire creation would appear as a unique truth and necessity.
D'Alembert, L'Encyclopédie (1751)

1) A particle of mass $m$ is constrained to move in a vertical plane under the influence of gravity along a smooth given curve with parametric equations $z = h(\theta)$, $x = f(\theta)$, where the $z-$axis is vertically upward. Taking $\theta$ as the generalised coordinate show that the Lagrangian is

$$L = \frac{m}{2}\dot{\theta}^2 \left[ \left( \frac{dh}{d\theta} \right)^2 + \left( \frac{df}{d\theta} \right)^2 \right] - mgh(\theta).$$

Study the particular case when $z = l\theta$, $\quad x = l\cos\theta$, where $l$ is constant.

2) a) Two Lagrangians $L$ and $\overline{L}$ are said to be equivalent if they are related by

$$\overline{L}(\dot{q}_i, q_i, t) = L(\dot{q}_i, q_i, t) + \frac{dF}{dt}(q_i, t),$$

where $F(q_i, t)$ is an arbitrary differentiable function of the generalised coordinates $q_i$ and time $t$. What is the relationship between the equations of motion derived from $L$ and $\overline{L}$?

b) A simple pendulum comprises of a particle of mass $m$ attached to the end of a light stiff rod of length $l$. The simple pendulum is in a constant gravitational field, with gravity acting downwards. The massless pivot $P$ executes an anticlockwise circular motion on a circumference with centre $O$ and radius $a$ with constant angular velocity $\omega$ in a vertical plane, so that at time $t$, $OP$ makes an angle $\omega t$ with the downward drawn vertical. Hence the displacement of the pivot $P$ is

$$x_p = a\sin\omega t, \quad y_p = a\cos\omega t,$$

where $x_p$ and $y_p$ are its cartesian coordinates, relative to axes $(Ox, Oy)$ with the $Oy$ direction vertically downwards.

i) Show that the Lagrangian for the motion of the simple pendulum is

$$L = \frac{ma^2}{2}\omega^2 + \frac{ml^2}{2}\dot{\theta}^2 + mga\cos(\omega t) + mgl\cos\theta + mla\omega\dot{\theta}\cos(\theta - \omega t),$$

where $\theta$ is the angle made by the stiff rod with the downward vertical, $g$ is the constant acceleration due to gravity, and the dots denote differentiation with respect to time.

ii) Show that an equivalent Lagrangian for the system is given by

$$\overline{L} = \frac{ml^2}{2}\dot{\theta}^2 + mla\omega^2\cos(\theta - \omega t) + mgl\cos\theta.$$

iii) From $\overline{L}$, find the Lagrangian equation of motion.

iv) Assume now that $\omega$ and $\theta$ remain small throughout the motion of the pendulum, so that, in the equation of motion, we can neglect all the terms of order $\omega^2\theta$ or smaller. Linearize the system under this assumption and find its general solution under the hypothesis $\frac{g}{l} \neq \omega^2$.

3) A massless non-extensible string passes over a small pulley and carries at one end a mass $2m$ and beneath it, supported by a spring with spring constant $k > 0$, a second mass $m$. On the other end there is a mass $m$, and beneath it, supported by a spring with spring constant $k > 0$, a second mass $m$.

i) Find the Lagrangian of this system.

ii) Find the corresponding Hamiltonian function and Hamilton's equations of motion.

iii) Assume now that the system is released from rest with the springs unextended. Find the positions of the point mass particles at any later time.