UNIVERSITY OF SURREY©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT3008 — 15 Credits

LAGRANGIAN AND HAMILTONIAN DYNAMICS

Level HE3 Examination

Time allowed: Two hours

Semester 1, 2011/12

Answer **THREE** questions only
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Each question carries 25 marks.

Approved calculators are allowed.

Additional material:
1 handout

Question 1

A simple pendulum comprises of a particle of mass m attached to the end of a light stiff rod of length l. The simple pendulum is in a constant gravitational field, with gravity acting downwards. The massless pivot P executes an anticlockwise circular motion on a circumference with centre O and radius a with constant angular velocity ω in a vertical plane, so that at time t, OP makes an angle ωt with the downward drawn vertical. Hence the displacement of the pivot P is

$$x_p = a \sin \omega t, \quad y_p = a \cos \omega t,$$

where x_p and y_p are its cartesian coordinates, relative to axes (Ox, Oy) with the Oy direction vertically downwards.

(a) Show that the Lagrangian for the motion of the simple pendulum is

$$L = \frac{ma^2}{2}\omega^2 + \frac{ml^2}{2}\dot{\theta}^2 + mga\cos(\omega t) + mgl\cos\theta + mla\omega\dot{\theta}\cos(\theta - \omega t),$$

where θ is the angle made by the stiff rod with the downward vertical, g is the constant acceleration due to gravity, and the dots denote differentiation with respect to time. [8]

(b) Show that an equivalent Lagrangian for the system is given by

$$\overline{L} = \frac{ml^2}{2}\dot{\theta}^2 + mla\omega^2\cos(\theta - \omega t) + mgl\cos\theta.$$

[5]

(c) From $\overline{L}$, find the Lagrangian equation of motion. [4]

(d) Assume now that ω and θ remain small throughout the motion of the pendulum, so that, in the equation of motion, we can neglect all the terms of order $\omega^2\theta$ or smaller. Linearize the system under this assumption and find its general solution under the hypothesis $\frac{g}{l} \neq \omega^2$. [8]

Question 2

The vector angular momentum about the origin O of a particle of unit mass at position $\mathbf{r}$, and moving with linear momentum $\mathbf{p} = \dot{\mathbf{r}}$ in a central force field having the potential $V = V(r)$, is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \dot{\mathbf{r}},$$

where r is the distance of the particle from the fixed centre of force O, and the dots denote differentiation with respect to time.

(a) Show that the time derivative $\dot{\mathbf{J}}$ is zero. [2]

(b) Thus show that the particle moves on the plane generated by the vectors $\mathbf{r}$ and $\mathbf{p} = \dot{\mathbf{r}}$. [6]

(c) Introducing polar coordinates (r, θ) on this plane, show that the total energy of the particle is given by

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r),$$

and the magnitude of its angular momentum is given by

$$J = r^2\dot{\theta}.$$

[4]

(d) By using $J = r^2\dot{\theta}$ show that the energy takes the form

$$E = \frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + V(r) = \frac{1}{2}\dot{r}^2 + U(r),$$

where $U(r)$ is the effective potential. [2]

(e) Take the particular case in which the above particle moves in a potential having the form $V = \dfrac{k}{2}r^2$, where k is a positive constant. Show that the effective potential has a minimum at $r_e = \left(\dfrac{J^2}{k}\right)^{1/4}$, where r_e is such that $E = U(r_e)$ and therefore $\dot{r} = 0$. [6]

(f) Thus show that the particle moves on a circle about the origin and find the period of its motion. [5]

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Question 3

(a) Given two sufficiently regular functions $f(\mathbf{q}, \mathbf{p}, t)$ and $g(\mathbf{q}, \mathbf{p}, t)$, where $\mathbf{q} \equiv (q_1, q_2....q_n)$ and $\mathbf{p} \equiv (p_1, p_2....p_n)$, define their Poisson bracket $[f, g]$. [1]

(b) Consider a sufficiently regular function $f(\mathbf{q}, \mathbf{p}, t)$, with $(\mathbf{q}, \mathbf{p})$ as above. Give the definition for $f(\mathbf{q}, \mathbf{p}, t)$ to be a first integral for the motion of the system with Hamiltonian $H = H(\mathbf{q}, \mathbf{p}, t)$. [1]

(c) Show that if a time independent function $f(\mathbf{q}, \mathbf{p})$ is a first integral for a Hamiltonian system with Hamiltonian $H = H(\mathbf{q}, \mathbf{p}, t)$, then its Poisson bracket with the Hamiltonian vanishes. [2]

(d) Given two sufficiently regular functions $f(\mathbf{q}, \mathbf{p}, t)$ and $g(\mathbf{q}, \mathbf{p}, t)$, by using the Jacobi identity, show that

$$\frac{d}{dt}[f, g] = [\frac{df}{dt}, g] + [f, \frac{dg}{dt}].$$

[10]

(e) Consider the type-one generating function $F_1(Q, q) = \lambda q^2 \cot Q$, where λ is a constant.

i) Find the canonical transformation generated by the above generating function from the (Q, P) representation to the (q, p) representation, and verify directly that it is canonical. [4]

ii) Then consider the Hamiltonian in the (q, p) representation given by $H(q, p) = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2m}$, where ω and m are positive constants. Find the new Hamiltonian $K(Q, P)$ in the (Q, P) representation and then choose λ so as to make this new Hamiltonian independent of Q. [4]

iii) Hence find the motion in both the representation (Q, P) and (q, p). [3]

Question 4

Consider the Lagrangian

$$L = \frac{\dot{q}^2}{2(1+q)^2} - \ln(1+q), \qquad q \geq 0.$$

(i) Show that the corresponding Hamiltonian function is given by

$$H(q,p) = (1+q)^2\, \frac{p^2}{2} + \ln(1+q)$$

and find Hamilton's equations of motion. [5]

(ii) Find the canonical transformation generated by a generating function of the type $F_3(p,Q) = -p(e^Q - 1)$ and find the transformed Hamiltonian function $K(P,Q)$. [5]

(iii) By using the Hamiltonian $K(P,Q)$ found in (ii), solve the equations of motion with initial conditions $q(0) = q_0 \geq 0,\ p(0) = p_0$. [6]

(iv) Solve the equations of motion found in (i) by using the method of Hamilton-Jacobi. [9]

END OF PAPER

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