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MAT3008 and MAT3031: LAGRANGIAN AND HAMILTONIAN DYNAMICS. ASSESSED TEST, 2 DECEMBER 2011

Write your answers in the answer book provided. Write (in capital letters and clearly please) your name on the outside cover. Do any rough working you need in the answer book, but indicate clearly that it is rough working. You should show sufficient working to demonstrate the method you have used in arriving at your solution.

You have 60 minutes.

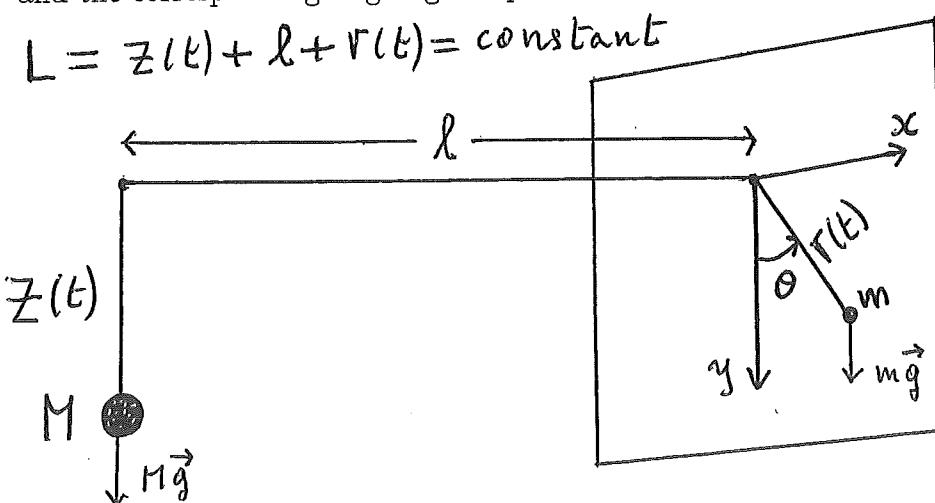
- 1) A particle of mass $m > 0$ moves on a vertical plane under the action of gravity of intensity g , along a smooth curve of the form $f(x) = e^{x^2}$.

- i) Compute its Lagrangian.
 - ii) Then compute the corresponding Hamiltonian.
- 2) Two Lagrangians L and \bar{L} are said to be equivalent if they are related by

$$\bar{L}(\dot{q}_i, q_i, t) = L(\dot{q}_i, q_i, t) + \frac{dF}{dt}(q_i, t), \quad i = 1, 2, \dots, n;$$

$F(q_i, t)$ is an arbitrary differentiable function of the generalised coordinates q_i and time t . Prove that L and \bar{L} have the same Lagrangian equations of motion and hence they describe the same motion.

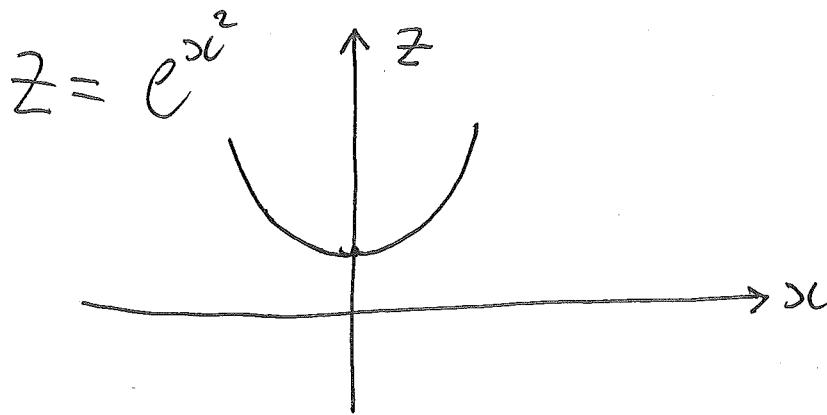
- 3) Consider the so-called Swinging Atwood's Machine (SAM): the smaller mass, labelled m , is allowed to swing freely in a vertical plane under the action of gravity and the action of the larger mass M , whereas the larger mass, M , can only move up and down vertically, also under the action of gravity and the smaller mass m (see figure below). The two masses are connected by an inextensible string of length L . Assume the pivots to be massless points. Compute the Lagrangian of the SAM and the corresponding Lagrange's equations of motion.



- 4) The potential energy of a particle of unit mass is $V(r)$. Show that the motion is planar. By using polar coordinates (r, θ) on the plane of motion, find the total energy E of the particle. Take the particular case in which the above particle moves in a potential having the form $V(r) = -\frac{1}{r} + \frac{1}{r^2}$. By using the effective potential energy theory, discuss the possible type of motion for the particle in the above potential. Find also the minimum value $U(r_0)$ of the effective potential, where r_0 is such that $E_0 = U(r_0)$ and therefore $\dot{r} = 0$ for $E = E_0$. Thus show that for $E = E_0 = U(r_0)$ the particle moves of uniform motion on a circle about the origin and find the period of its motion.

SOLUTIONS - TEST

Question - 1 -



$$T = \frac{m}{2} (\dot{x}^2 + \dot{z}^2) = \frac{m}{2} \dot{x}^2 \left(1 + \left(\frac{dz}{dx} \right)^2 \right) = \frac{m}{2} \dot{x}^2 \left(1 + 4x^2 e^{2x^2} \right).$$

$$V(z) = mgz = mg e^{x^2}$$

$$L = T - V = \frac{m}{2} \dot{x}^2 \left(1 + 4x^2 e^{2x^2} \right) - mg e^{x^2}$$

$$H = \dot{x} P_x - L(x, \dot{x}(x, P_x))$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \left(1 + 4x^2 e^{2x^2} \right)$$

$$\therefore \dot{x} = \frac{P_x}{m(1 + 4x^2 e^{2x^2})}$$

$$\therefore H = \frac{P_x^2}{2m(1 + 4x^2 e^{2x^2})} + mg e^{x^2}$$

Question - 2 -

Two equivalent Lagrangian describe the same motion in the sense that they have the same equations of motion. In fact given $\bar{L} = L + \frac{dF}{dt}(\underline{q}, t)$ with $\underline{q} \equiv (q_1, q_2, \dots, q_n)$ and $L = L(\underline{q}, \dot{\underline{q}}, t)$. We have $\frac{\partial \bar{L}}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i} \left(\frac{dF}{dt} \right) =$

$$= \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i} \left(\sum K \frac{\partial F}{\partial q_K} \dot{q}_K + \frac{\partial F}{\partial t} \right) = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i} + 0$$

$$\therefore \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d}{dt} \left(\frac{\partial F}{\partial q_i} \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial}{\partial q_i} \left(\frac{dF}{dt} \right)$$

$$\therefore \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{q}_i} \right) - \frac{\partial \bar{L}}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \blacksquare$$

Question - 3 -

The SAM (Swinging Atwood's Machine)



$$l + Z(t) + r(t) = L = \text{constant} > 0.$$

$$T_m = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2); V_m = -mg r(t) \cos \theta$$

$$T_M = \frac{M}{2} (\dot{z}^2) = \frac{M}{2} (\dot{r}^2); V_M = -Mg z = -Mg(L - l - r(t))$$

$$L = (T_m + T_M) - (V_m + V_M) =$$

$$= \left(\frac{M+m}{2} \right) \dot{r}^2 + \frac{m}{2} r^2 \dot{\theta}^2 + mg r \cos \theta + Mg L - Mg l - Mg r$$

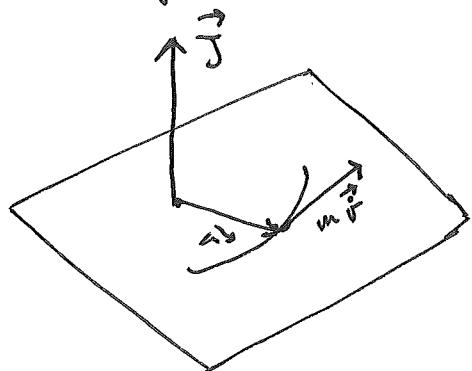
The Lagrange's equations are Neglect

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 = m r^2 \ddot{\theta} + 2mr \dot{r} \dot{\theta} + m g r \sin \theta = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = (M+m) \ddot{r} - m r \dot{\theta}^2 - mg \cos \theta + Mg = 0$$

Question - 4-

Define the angular momentum vector $\vec{J} = \vec{r} \times m\vec{v}$



The force associated to the potential $V = V(r)$ is $\vec{F} = -\frac{\partial V}{\partial r} \vec{U}_r$ where \vec{U}_r is the unit vector along \vec{r} .

Thus if $\frac{d\vec{J}}{dt} = \vec{0}$ then $\vec{J} = \overrightarrow{\text{constant}}$ and therefore the vectors \vec{r} and $m\vec{v}$ vary in time so as to keep $\vec{J} = \overrightarrow{\text{constant}}$ and \therefore by one of the vector product properties it follows that they must vary on a geometrical plane, the plane of motion. Thus we have to show that $\frac{d\vec{J}}{dt} = \vec{0}$.

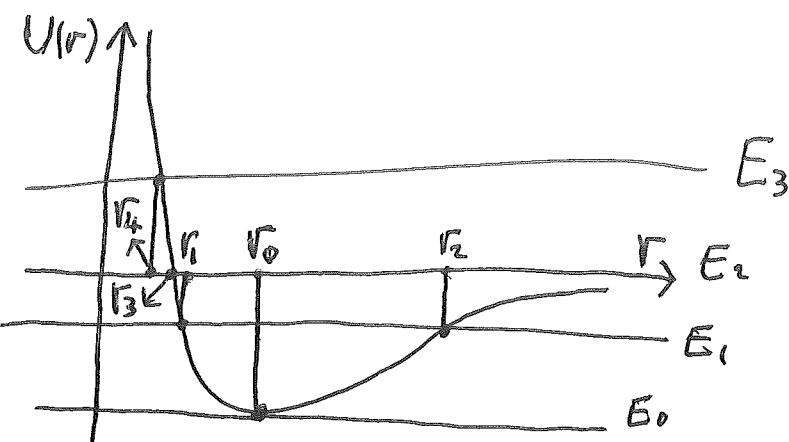
$$\frac{d\vec{J}}{dt} = \vec{r} \times m\vec{v} + \vec{r} \times m\ddot{\vec{v}} = \vec{0} + \vec{r} \times \left(-\frac{\partial V}{\partial r} \vec{U}_r \right) = \vec{0} + \vec{0} = \vec{0}.$$

The total energy of the particle is

$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r).$$

Consider $V(r) = -\frac{1}{r} + \frac{1}{r^2}$. The effective potential energy

$$U(r) = \frac{J^2}{2mr^2} - \frac{1}{r} + \frac{1}{r^2}$$



For $E = E_0$ we have $E_0 = V(r_0)$ and $\dot{r} = 0$ and $\therefore r_0 = \text{constant}$. For non-trivial motion we take $r_0 > 0$. Then from $J = mr^2\dot{\theta} = \text{constant}$ we have that $\dot{\theta} = \frac{J}{mr_0^2}$ for $E = E_0$.

Hence from $r_0 = \text{constant} > 0$ we have that the particle moves on a circle centred on the centre of the field with uniform circular motion with angular velocity $\dot{\theta} = \omega = \frac{J}{mr_0^2} = \text{constant} > 0$ by assuming that $J \neq 0$. The period of motion

- $T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\omega} = \left(\frac{2\pi}{J}\right)mr_0^2$. The value of r_0 is obtained from $\frac{dV}{dr}|_{r=r_0} = 0$. We obtain ($m=1$):

$$\frac{dV}{dr} = -\frac{J^2}{r^3} + \frac{1}{r^2} - \frac{2}{r^3} = 0 \quad \therefore r_0 = J^2 + 2$$

For $E_0 < E < E_2$ we have two turning points r_1 and r_2 where $\dot{r} = 0$ and $E(r_1) = E(r_2) = V(r_1) = V(r_2)$. Because the force at $r=r_1$ and $r=r_2$ is not-zero then the particle is forced to move back and forth between $r_1 \leq r \leq r_2$ while at the same time "rotates" with angular velocity $\dot{\theta} = \frac{J}{mr^2}$. So for $E_0 < E < E_2$ the motion is bounded. However the bounded path is generically a not closed

curve lying within the annulus bounded by the values $r=r_1$ and $r=r_2$. For the curve to be a closed curve, the angle $\Delta\theta$, which expresses the "turning" of the radius vector \vec{r} during the time in which it varies from r_2 to r_1 and back, must be a rational fraction of 2π , namely

$$\Delta\theta = \frac{p}{q} 2\pi \text{ with } p \text{ and } q \text{ relatively prime integers.}$$

For $E = E_2$ the motion is unbounded but "when" the particle reaches infinity we have

$\lim_{r \rightarrow +\infty} E(r) = U(r)$ and so $\lim_{r \rightarrow +\infty} \dot{r} = 0$, and
so the particle stops moving asymptotically.

For $E > E_2$ the motion is similar to the case $E = E_2$ but here the particle "reaches" infinity with a positive speed.