MAT3008: LAGRANGIAN AND HAMILTONIAN DYNAMICS
UN-ASSESSED COURSEWORK II
We politely but strongly advise you to solve ALL the questions below.
However, please hand in only the solutions of the starred questions at the end of our lectures on Tuesday 12/03/2013.
These questions will be marked and returned to you with appropriate feedback.

Thoughts of the coursework:
To some-One who could view the universe from a unified standpoint, the entire creation would appear as a unique truth and necessity.
D'Alembert, L'Encyclopédie (1751)

Who by a vigour of mind almost divine, the motions and figures of planets, the paths of comets, and the tides of the sea, first demonstrated.
Newton's Epitaph.

1*) Integrate the equation of motion for a spherical pendulum in the gravity field. Identify one conserved quantity other than the total energy and relate it to a symmetry in the system. Finally, by using the energy equation or otherwise, show that a particular solution is possible where the particle executes uniform circular motion at a constant height.

2*) Consider a particle of unit mass moving in an attractive potential having the form $V = -\dfrac{\mu}{4r^4}$, where μ is a positive constants. The particle is projected from a point P, at a distance a from the origin O othogonally to the polar axis OP and its initial speed is given by

$$u = \sqrt{\frac{\mu}{2a^4}}$$

The square of the modulus of the angular momentum with respect to the centre of the field is conserved (the vector angular momentum $\mathbf{J}$ is along the vertical) and its value is $J^2 = a^2u^2$.
a) Use polar coordinates with centre at O and polar axis along OP and find the energy equation.
b) Find the equation of the path.
c) Integrate the equation of the path to show that the orbit is given by a semicircle with diameter a.
d) Inspecting the formula for the orbit, one can see that the particle reaches the centre of force O in finite time. Prove that the time T it takes to reach O is $T = \dfrac{\pi a^3}{\sqrt{8\mu}}$.

3*) The potential energy of a particle of unit mass is $V(r)$. Show that the motion is planar. By using polar coordinates (r, θ) on the plane of motion, find the total energy of the particle. Take the particular case in which the above particle moves in a potential having the form $V(r) = -\dfrac{1}{r} + \dfrac{1}{r^2}$. By using the effective potential energy theory, discuss the possible type of motion for the particle in the above potential. Hence in particular show that there exists a value of the total energy $E = E_0$ such that the particle moves on a circle about the centre of the field with uniform circular motion and find the period of this circular motion.

4) The vector angular momentum about the origin O of a particle of unit mass at position $\mathbf{r}$, and moving with linear momentum $\mathbf{p} = \dot{\mathbf{r}}$ in a central force field having the potential $V = V(r)$, is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \dot{\mathbf{r}},$$

where r is the distance of the particle from the fixed centre of force O, and the dots denote differentiation with respect to time.

a) Show that the time derivative $\dot{\mathbf{J}}$ is zero.

b) Thus show that the particle moves on the plane generated by the vectors $\mathbf{r}$ and $\mathbf{p} = \dot{\mathbf{r}}$.

c) Introducing polar coordinates (r, θ) on this plane, show that the total energy of the particle is given by

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r),$$

and the magnitude of its angular momentum is given by

$$J = r^2\dot{\theta}.$$

d) By using $J = r^2\dot{\theta}$ show that the energy takes the form

$$E = \frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} + V(r) = \frac{1}{2}\dot{r}^2 + U(r),$$

where $U(r)$ is the effective potential.

e) Take the particular case in which the above particle moves in a potential having the form $V = \frac{k}{2}r^2$, where k is a positive constant. Show that the effective potential has a minimum at $r_e = \left(\frac{J^2}{k}\right)^{1/4}$, where r_e is such that $E = U(r_e)$ and therefore $\dot{r} = 0$.

f) Thus show that the particle moves on a circle about the origin and find the period of its motion as a function of k.

5) A particle of mass m moves in space so that its coordinates at time t are $[x(t), y(t), z(t)]$. The Lagrangian of the particle is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V + \dot{x}A + \dot{y}B + \dot{z}C,$$

where V, A, B, C are given smooth functions of x, y only. Find Lagrange's equations of motion for the coordinates $[x(t), y(t), z(t)]$.

6*) A particle of unit mass is moving in a plane under the action of an attractive central potential $V = \frac{-k}{r^n}$, where k is a positive constant, n is a positive integer and r is the distance from a fixed centre of force O. Suppose that the conserved angular momentum $J \neq 0$. Show that a necessary condition for the particle to fall into the centre of force O is $n \geq 2$. What is the value of n for the standard model of our Universe and what implications can be drawn for the dynamics of our planets and their satellites.