

UNIVERSITY OF SURREY<sup>©</sup>

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT3008 — 15 Credits

**LAGRANGIAN AND HAMILTONIAN DYNAMICS**

Level HE3 Examination

Time allowed: Two hours

Summer Resit 2010/11

Answer **THREE** questions only

If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Each question carries 25 marks.

Approved calculators are allowed.

*Additional material:*

1 handout

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**Question 1**

- (a) A particle of mass  $m$  slides under gravity on a smooth wire of shape

$$z = \cosh x,$$

where the  $x$ -axis is horizontal and the  $z$ -axis is vertically upwards.

- (i) Show that the Lagrangian  $L$  of the motion for the above particle is given by

$$L = \frac{m}{2}\dot{x}^2 \cosh^2 x - mg \cosh x,$$

where  $g$  is the constant gravitational acceleration and the dot denotes differentiation with respect to time. [5]

- (ii) Find the corresponding Hamiltonian. [3]

- (b) A simple pendulum consists of a particle of mass  $m$  attached to the end of a light stiff rod of length  $l$ . The pendulum is in a constant gravitational field, with gravity acting downwards. The pivot  $P$  executes a motion on the downward oriented  $y$ -axis with the temporal law

$$y_p = \lambda(t),$$

where the variable  $t$  represents the time.

- (i) Show that the Lagrangian for the motion of the simple pendulum is

$$\bar{L} = \frac{m}{2}(l^2\dot{\theta}^2 - 2l\dot{\theta}\dot{\lambda} \sin \theta) + mgl \cos \theta + \frac{m}{2}\dot{\lambda}^2 + mg\lambda,$$

where  $\theta$  is measured from the downward  $y$ -axis,  $g$  is the constant acceleration due to gravity, and the dots denote differentiation with respect to time. [7]

- (ii) Show that an equivalent Lagrangian for the above system is given by

$$L = \frac{ml^2}{2}\dot{\theta}^2 + ml(g - \ddot{\lambda}) \cos \theta.$$

[6]

- (iii) Explain the dynamical significance of the equivalent Lagrangian. [4]

**Question 2**

- (a) Consider the system having Lagrangian

$$L(q, \dot{q}, t) = \frac{1}{2}G(q, t)\dot{q}^2 + F(q, t)\dot{q} - V(q, t).$$

Find the corresponding Hamiltonian.

[6]

- (b) Suppose a particle of mass  $m$  is sliding on a smooth wire in a vertical plane. The wire is defined parametrically by

$$\begin{aligned} x &= l(\theta + \sin \theta) \\ y &= l(1 - \cos \theta), \end{aligned}$$

where  $-\pi < \theta < \pi$  and  $l$  is a positive constant parameter. Gravity acts in the negative vertical  $y$ -direction.

- (i) Compute the Lagrangian  $L$  for the above system.

[8]

- (ii) Show that the Lagrangian in terms of  $\phi$  and  $\dot{\phi}$ , where  $\phi = \sin(\frac{\theta}{2})$ , is given by

$$L = 8ml^2\dot{\phi}^2 - 2mgl\phi^2.$$

[5]

- (iii) Find the equation of motion for  $\phi$  from the transformed Lagrangian.

[3]

- (iv) Hence deduce that  $\phi$  varies harmonically and compute its period.

[3]

**Question 3**

- (a) Consider the time-independent canonical transformation from the  $(q, p)$  representation to the  $(Q, P)$  representation, with corresponding conservative Hamiltonians functions  $H(q, p)$  and  $K(Q, P)$  describing the same flow in each representation. Show that

$$\frac{\partial H}{\partial P} = \frac{\partial K}{\partial P} \quad \text{and} \quad \frac{\partial H}{\partial Q} = \frac{\partial K}{\partial Q},$$

so that we have  $K(Q, P) = H(q(Q, P), p(Q, P))$ , up to an integration constant which is irrelevant for the equations of motion. [9]

- (b) Determine which of the following transformations are canonical

i)  $Q = \frac{1}{2}q^2, \quad P = \frac{p}{q};$

[1]

ii)  $Q = p \tan q, \quad P = (p - 3) \cos^2 q;$

[2]

iii)  $Q = \sqrt{q}e^t \cos p, \quad P = \sqrt{q}e^{-t} \sin p,$  where  $t$  is the time variable.

[2]

- c) Consider the transformation from a fixed reference frame  $(q, p)$  to a moving reference frame  $(Q, P)$  given by  $Q = q - D(t)$ ,  $P = p$ , where  $D(t)$  is the distance between the origins of the coordinate systems at time  $t$ .

- i) Show that  $F_2(P, q, t) = P(q - D(t))$  is a generating function for the above transformation. [4]

- ii) Given the Hamiltonian  $H = \frac{p^2}{2m} + V(q)$  in the  $(p, q)$  representation, find its transformed form in the  $(P, Q)$  representation. [4]

- iii) Find the equations of motion in the  $(P, Q)$  representation. [3]

**Question 4**

Consider the Lagrangian  $L(q, \dot{q}) = \frac{q^2 \dot{q}^2}{2} - \frac{q^4}{8}$ .

- (i) Show that the corresponding Hamiltonian function is given by  $H(q, p) = \frac{p^2}{2q^2} + \frac{q^4}{8}$ , and hence find Hamilton's equations of motion. [4]
- (ii) By using a generating function of the first kind  $F_1(q, Q) = \frac{q^2 Q}{2}$ , transform the above Hamiltonian from the  $(q, p)$  representation into the Hamiltonian  $K(Q, P)$  in the  $(Q, P)$  representation. [7]
- (iii) By using the new Hamiltonian  $K(Q, P)$ , solve the equations of motion with initial conditions  $q(0) = 1, p(0) = 0$ . [6]
- (iv) Solve the equations of motion found in (i) by using the method of Hamilton-Jacobi. [8]

**END OF PAPER**

### Question -1- Solution

(a)  $Z(x) = \cosh x$

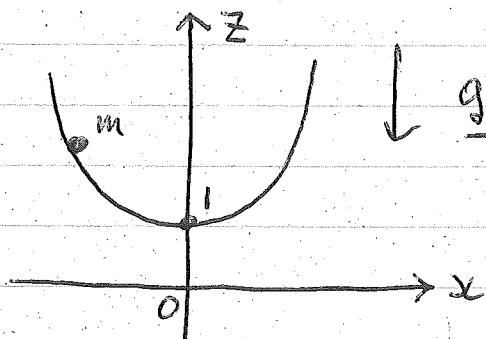
Kinetic Energy

$$(i) T = \frac{m}{2} (\dot{x}^2 + \dot{z}^2) = \frac{m}{2} \left( \dot{x}^2 + \left( \frac{dz}{dx} \dot{x} \right)^2 \right)$$

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$$\therefore T = \frac{m}{2} (1 + \sinh^2 x) \dot{x}^2$$

$$= \frac{m}{2} \dot{x}^2 \cosh^2 x$$



Potential energy

$$V(Z) = mgZ(x) = mg \cosh x$$

Thus the Lagrangian function is given by

$$L = T - V = \frac{m}{2} \dot{x}^2 \cosh^2 x - mg \cosh x$$

The corresponding Hamiltonian is given by

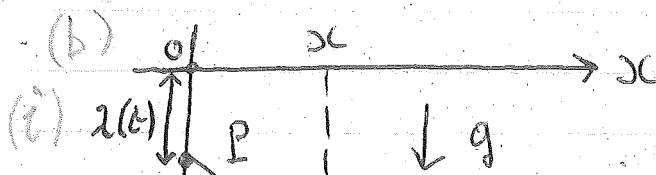
$$(ii) H(x, P_x) = \dot{x} P_x - L(x, \dot{x}, P_x)$$

3 where  $P_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \cosh^2 x$

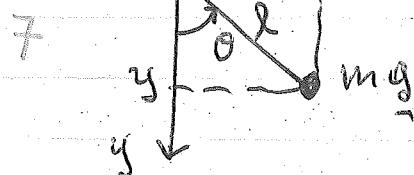
$$\therefore \dot{x} = \frac{P_x}{m \cosh^2 x}$$

$$\text{Hence } H = \frac{P_x^2}{m \cosh^2 x} - \frac{m}{2} \frac{P_x^2 \cosh^2 x}{m^2 \cosh^4 x} + mg \cosh x ;$$

$$\text{Therefore } H = \frac{P_x^2}{2m \cosh^2 x} + mg \cosh x.$$



$$\begin{cases} x = l \sin \theta \\ y = l \cos \theta + 2t \end{cases}$$



$$\begin{cases} \dot{x} = l \dot{\theta} \cos \theta \\ \dot{y} = -l \dot{\theta} \sin \theta + 2 \end{cases}$$

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Question -1 - Solution

So the Kinetic energy of the pendulum is

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) = \frac{m}{2} (l^2 \dot{\theta}^2 + \dot{z}^2 - 2 l \dot{\theta} \dot{z} \sin \theta)$$

The potential energy of the pendulum is

$$U(\theta, z) = -mgz = -mg(l \cos \theta + z(t))$$

∴ the Lagrangian of this system is

$$\bar{L} = T - U = \frac{m}{2} (l^2 \dot{\theta}^2 - 2 l \dot{\theta} \dot{z} \sin \theta + \dot{z}^2)$$

$$+ mg l \cos \theta + mg z(t).$$

Rearranging we obtain

$$\bar{L} = \frac{m}{2} l^2 \dot{\theta}^2 + mg l \cos \theta - m l \dot{\theta} \dot{z} \sin \theta + \frac{m}{2} \dot{z}^2 + mgz.$$

(ii) In order to obtain an equivalent Lagrangian

we first note that the term

$$- m l \dot{\theta} \dot{z} \sin \theta = \frac{d}{dt} (m l \dot{z} \cos \theta) - m l \ddot{z} \cos \theta.$$

∴ the Lagrangian function transforms into

$$\bar{L} = \frac{m}{2} l^2 \dot{\theta}^2 + m l (g - \ddot{z}) \cos \theta + \frac{1}{m} (m l \dot{z} \cos \theta) + \frac{m}{2} \dot{z}^2 + mgz.$$

Now observe that the terms  $\frac{m}{2} \dot{z}^2 + mgz$  are a function of time only and therefore can

Question - 3 - Solution

be ignored; furthermore the term  $\frac{d}{dt} (ml^2 \dot{\theta}^2)$  represents the total derivative of a function of the coordinate  $\theta = \theta(t)$  and the time  $t$  and therefore can also be ignored. Thus an equivalent Lagrangian is given by

$$L = \frac{m}{2} l \ddot{\theta}^2 + ml(g - \ddot{z}) \cos \theta.$$

(iii) One can see that if  $z(t) \equiv 0$  we obviously have the simple pendulum with constant gravity acceleration  $g$ . If  $z(t)$  is not zero then the form of the Lagrangian above clearly shows that the movement of the pivot P has the same effect as a time varying gravitational field having the form  $(g - \ddot{z})$ .

25 [a) This question is essentially unanswered;  
b) book work done during lectures.]

## Question - 2 - Solution

(a) Let  $L = \frac{1}{2} G(q_1, t) \dot{q}^2 + F(q_1, t) \dot{q} - V(q_1, t)$

We know that the generalized momentum

$P$  is given by

$$P = \frac{\partial L}{\partial \dot{q}} = G \dot{q} + F$$

$$\therefore \dot{q} = \frac{P - F}{G}$$

From  $H(q, P, t) = P \dot{q} - L$  we have

$$H = \left( \frac{P - F}{G} \right) P - \frac{G}{2} \left( \frac{P - F}{G} \right)^2 - \frac{F}{G} (P - F) + V$$

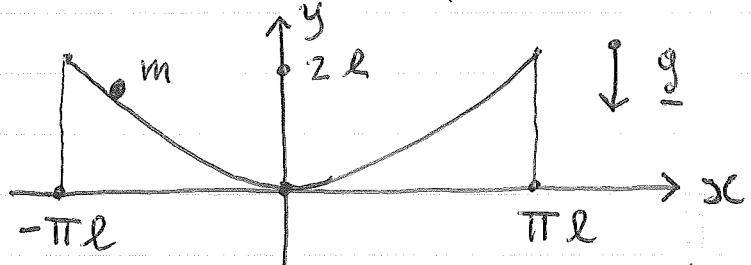
$$\therefore H = \left( \frac{P - F}{G} \right) \left[ P - F - \frac{P}{2} + \frac{F}{2} \right] + V =$$

$$= \left( \frac{P - F}{G} \right) \left[ \frac{P - F}{2} \right] + V$$

$$\therefore H = \frac{(P - F)^2}{2G} + V.$$

Question -2 - Solution

(b)  $\begin{cases} x = l(\theta + \sin\theta) \\ y = l(1 - \cos\theta) \end{cases}$



(i) The Kinetic energy of the sphere is given by

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2)$$

$$\begin{cases} \dot{x} = l\dot{\theta} + l\dot{\theta}\cos\theta \\ \dot{y} = -l\dot{\theta}\sin\theta \end{cases}$$

$$\text{Hence we obtain } T = \frac{m}{2} (l^2\dot{\theta}^2 + l^2\dot{\theta}^2\cos^2\theta + 2l^2\dot{\theta}^2\cos\theta)$$

The potential energy is given by

$$U(y) = mg y = mgl(1 - \cos\theta)$$

$$\therefore L = T - U = m\dot{\theta}^2 + m\dot{\theta}^2\cos^2\theta - mgl(1 - \cos\theta).$$

(ii) Let us put  $\varphi = \sin\left(\frac{\theta}{2}\right)$ ,  $\therefore \dot{\varphi} = \frac{\dot{\theta}}{2}\cos\left(\frac{\theta}{2}\right)$

$$5 \quad \dot{\varphi}^2 = \frac{\dot{\theta}^2}{4}\cos^2\left(\frac{\theta}{2}\right); \quad \dot{\theta} = \frac{2\dot{\varphi}}{\cos\left(\frac{\theta}{2}\right)}.$$

From  $(1 + \cos\theta) = 2\cos^2\left(\frac{\theta}{2}\right)$  we obtain

$$2\dot{\varphi}^2 = \frac{\dot{\theta}^2}{4}(1 + \cos\theta) \quad \therefore 8\dot{\varphi}^2 = \dot{\theta}^2(1 + \cos\theta)$$

$$1 + \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) = 2\dot{\varphi}^2$$

Question -2 - Solution

i. The Lagrangian transforms into

$$L = 8m\ell^2\ddot{\varphi} - 2mgl\dot{\varphi}^2.$$

(iii) Equation of motion for  $\varphi$  is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0.$$

$$\text{Thus } 16m\ell^2\ddot{\varphi} + 4mgl\dot{\varphi} = 0$$

$$\therefore \ddot{\varphi} + \frac{g}{4\ell} \varphi = 0, \text{ Harmonic oscillator.}$$

(iv) The period of the solution of the  
3 harmonic oscillator is

$$T = \frac{2\pi}{\omega} \quad \text{where} \quad \omega^2 = \frac{g}{4\ell}$$

$$\therefore T = 4\pi\sqrt{\frac{\ell}{g}}.$$

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[ Question unpern, but similar systems  
have been analyzed during lectures and  
also given as part of the homework.]

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Question - 3. Solution

(a)  
g

We have to show that

$$K(Q, P) = H(q(Q, P), p(Q, P)).$$

So we must express  $H$  as a function of  $(Q, P)$ .

This is done as follows:

$$\begin{aligned} \frac{\partial K}{\partial P} &= \dot{Q} = \frac{\partial Q}{\partial q} \dot{q} + \frac{\partial Q}{\partial P} \dot{P} \\ &= \frac{\partial Q}{\partial q} \frac{\partial H}{\partial P} - \frac{\partial Q}{\partial P} \frac{\partial H}{\partial q}. \end{aligned}$$

∴ by using the chain rule we obtain:

$$\begin{aligned} \frac{\partial K}{\partial P} &= \frac{\partial Q}{\partial q} \left( \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial P} + \frac{\partial H}{\partial P} \frac{\partial P}{\partial P} \right) + \\ &\quad - \frac{\partial Q}{\partial P} \left( \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P} \frac{\partial P}{\partial q} \right) = \\ &= \frac{\partial H}{\partial P} \left( \frac{\partial Q}{\partial q} \frac{\partial P}{\partial P} - \frac{\partial Q}{\partial P} \frac{\partial P}{\partial q} \right). \end{aligned}$$

Now we observe that

$$\frac{\partial Q}{\partial q} \frac{\partial P}{\partial P} - \frac{\partial Q}{\partial P} \frac{\partial P}{\partial q} = 1 \quad (*)$$

and therefore we obtain  $\frac{\partial K}{\partial P} = \frac{\partial H}{\partial P}$ .

(\*) is equal to 1 because the transformation

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Question - 2 - Solution

is canonical. Similarly, by considering

$$\begin{aligned}\frac{\partial K}{\partial Q} &= -\dot{P} = -\frac{\partial P}{\partial q}\dot{q} - \frac{\partial P}{\partial p}\dot{p} = \\ &= -\frac{\partial P}{\partial q}\frac{\partial H}{\partial P} + \frac{\partial P}{\partial P}\frac{\partial H}{\partial q}\end{aligned}$$

We can show that  $\frac{\partial K}{\partial Q} = \frac{\partial H}{\partial Q}$ . (\*\*)

$\therefore$  From  $\frac{\partial K}{\partial P} = \frac{\partial H}{\partial P}$  and (\*\*) we have that

$K = H + \text{constant}$ , as required.

(b) (i)  $Q = \frac{q^2}{2}, P = \frac{p}{q}$  (a)

$$\frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} = \frac{q}{2} - 0 = 1$$

$\therefore$  (a) is a canonical transformation.

(ii)  $Q = p \tan q, P = (p-3) \cos^2 q$  (b)

Q  $\frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} = p + \tan q (2 \cos q \sin q)(p-3)$  \*

$\therefore$  (b) is not canonical because in general  
(\*) is not equal to one.

(iii)  $Q = e^{t\sqrt{2}} \cos p, P = e^{-t\sqrt{2}} \sin p$ , t time. (c)

Q  $\frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} - \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} = \frac{\cos^2 p \sqrt{2}}{2\sqrt{2}} + \frac{\sqrt{2} \sin p}{2\sqrt{2}} =$   
 $= \frac{1}{2} (\cos^2 p + \sin^2 p) \neq 1;$

$\therefore$  (iii) is not a canonical transformation.

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### Question-3 - Solution

(C) Given the transformation

(i)  $Q = q - D(t)$  between the  $(P, q)$  and the  $(P, Q)$  representations, let us first find a generating function for this transformation.

Because  $Q$  and  $q$  are not independent the generating function cannot be of  $F_1(Q, q)$  type.

Thus we choose as independent variables

$(P, q)$  and therefore we choose a generating

function of  $F_2 = F_2(P, q, \epsilon)$  type. For

$Q = q - D(\epsilon)$  an  $F_2$  generating function is

$$F_2(P, q, \epsilon) = P Q = P(q - D(\epsilon)).$$

$$\therefore p = \frac{\partial F_2}{\partial q} = P; \therefore F_2 = P q + g(P) = P(q + D) + g(P)$$

$$\therefore \begin{cases} Q = q - D(\epsilon) & \frac{\partial F_2}{\partial P} = Q + D + \frac{\partial g}{\partial P} = Q \\ P = P \end{cases}$$

$$\therefore g = -PD \text{ and } F_2 = PQ.$$

(ii) Given the Hamiltonian  $H(P, q) = \frac{P^2}{2m} + V(q)$

in the  $(P, q)$  representation, the transformed

Hamiltonian in the  $(P, Q)$  representation is

given by  $K(P, Q, \epsilon) = H(P(P, Q, \epsilon), q(P, Q, \epsilon), t) -$

$$+ \frac{\partial F_2}{\partial \epsilon};$$

$$\therefore K(P, Q, \epsilon) = \frac{P^2}{2m} + V(Q + D(\epsilon)) - PD$$

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### Question -3 - Solution

(iii) The equations of motion in the  $(P, Q)$  representation are therefore given by

$$3 \quad \begin{cases} \dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m} - \ddot{D}(t) \\ \dot{P} = -\frac{\partial K}{\partial Q} = \frac{-\partial V(Q + D(t))}{\partial Q} \end{cases}$$

$$\therefore m \ddot{Q} = -\frac{\partial V(Q + D)}{\partial Q} - m \ddot{D}(t).$$

The  $m \ddot{D}(t)$  term is naturally the acceleration of the reference frame. For uniform translation this term is naturally zero.

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[**(a)** Book work ; **(b)** unseen ; **(c)** unseen]

## Question 4 - Solution

(a) Consider the following transformation:

$$\begin{cases} Q = -\ln(P-2q) \\ P = -2q^2 + 2q \end{cases} \quad (*)$$

where  $P-2q > 0$ .

(i) We wish to show that it is canonical, namely

$$2 \cdot \frac{\frac{\partial Q}{\partial P} \frac{\partial P}{\partial q}}{\frac{\partial P}{\partial q}} - \frac{\frac{\partial Q}{\partial P} \frac{\partial P}{\partial q}}{\frac{\partial P}{\partial q}} = 1. \quad \text{We have:}$$

$$\frac{2q}{P-2q} + \frac{P-4q}{P-2q} = 1$$

(ii) In order to find the generating function  
of type one, namely  $F_2 = F_1(q, Q)$  we use  
the equations

$$P = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}.$$

Using (ii) we obtain

$$\begin{cases} P = 2q + e^{-Q} \\ P = -2q^2 + 2q + 2qe^{-Q} \end{cases}$$

$$\therefore \begin{cases} q = P \cdot e^Q \\ P = 2Pe^Q + e^{-Q} \end{cases} \quad (***)$$

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Question - 4 - Solution

Hence we have

$$\frac{\partial F_i}{\partial g} = P = 2g + e^{-Q}$$

$$\therefore F_i = g^2 + 2e^{-Q} + g(Q)$$

where  $g(Q)$  is an arbitrary differentiable function.

$$\begin{aligned} \text{From } \frac{\partial F_i}{\partial Q} &= -P = -2e^{-Q} + g'(Q) = 2g - gP = \\ &= 2g - 2g - 2e^{-Q} = -2e^{-Q} \end{aligned}$$

$$\therefore g'(Q) = 0 \text{ and } \therefore g(Q) = \text{constant.}$$

Because the generating function is determined up to an additive constant, we can then take

$$g(Q) = 0. \quad \therefore F_i(g, Q) = g^2 + 2e^{-Q}$$

12  
Question - 4 - Solution

Hence we have

$$\frac{\partial F}{\partial q} = p = 2q + e^{-Q}$$

$$\therefore F = q^2 + q e^{-Q} + g(Q)$$

where  $g(Q)$  is an arbitrary differentiable function.

$$\begin{aligned} \text{From } \frac{\partial F}{\partial Q} &= -P = -q e^{-Q} + g'(Q) = 2q - qP = \\ &= 2q - 2q - q e^{-Q} = -q e^{-Q} \end{aligned}$$

$$\therefore g'(Q) = 0 \text{ and } \therefore g(Q) = \text{constant}.$$

Because the generating Function is determined up to an additive constant, we can then take

$$g(Q) = 0. \therefore F(q, Q) = q^2 + q e^{-Q}.$$

The bit below is not required for the exam question.

We consider now the Hamiltonian

$$H(q, p) = (2q^2 - pq)^2; \therefore H = P^2 = K(Q, P).$$

$$\dot{P} = 0 \therefore P(t) = P(0) = \text{constant}$$

$$\dot{Q} = + \frac{\partial K}{\partial P} = 2P \therefore Q(t) = 2P(0)t + Q(0)$$

Inserting these equations into (\*\*) we find:

$$\begin{cases} q = P(0)e^Q = P(0)e^{2P(0)t + Q(0)} \\ P = 2P(0)e^{2P(0)t + Q(0)} - (2P(0)t + Q(0)) \end{cases}$$

The constants  $P(0)$  and  $Q(0)$  are determined so as to satisfy the initial conditions for  $q(t=0)$  and  $P(t=0)$ .