MAT1017: SOLUTIONS FOR ASSIGNMENT 1

Question 1: If a and b are real positive numbers with $a \neq b$, prove by contraposition that $\sqrt{ab} \neq \frac{a+b}{2}$.

Solution: Proof (by contraposition): Assume the negation of the conclusion, namely that $(ab)^{1/2} = (a+b)/2$. Then $ab = (a+b)^2/4 = (a^2 + 2ab + b^2)/4 = (a^2 - 2ab + b^2)/4 + ab = (a-b)^2/4 + ab$. Thus $(a-b)^2 = 0$ and so $a = b$, which is the negation of the premise.

Question 2: Prove that $x^2 - y^2 = p$ with p a prime larger then 2 has always solutions within the set of positive integers.

Solution: $x^2 - y^2 = (x+y)(x-y) = p$. As x and y are natural numbers and p is a natural number, it follows that $x + y$ is a natural number and also $x - y$ is a natural number (from $(x+y)(x-y) = p > 0$). Therefore $x + y = p$ and $x - y = 1$. Hence we obtain $y = (p-1)/2$ which is a natural number and so is x. Notice that the propositions $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$ have no solutions in the set of natural numbers!

Question 3: If a and b are positive integers, prove by contradiction that $ax^2 + bx + (b-a) = 0$ has no positive integer root.

Solution: Proof (by contradiction): Assume that m is a positive integer root of $ax^2 + bx + (b-a) = 0$, with a and b natural numbers. Then $am^2 + bm + (b-a) = 0$. If $b \geq a$, then $am^2 + bm + (b-a) = 0$ which is impossible. If $b < a$, then $m(ma+b) = a - b > 0$, but then $a - b = m(ma+b) \geq ma + b \geq a + b$. Therefore $2b \leq 0$ which is impossible as $b > 0$.

Question 4: Prove by contradiction that the sum of the squares of three consecutive integers cannot leave remainder -1 on division by 12.

Solution: Assume by contradiciton that the statement P of the question is false. Let then $n-1, n, n+1$ be integers for which it fails. Thus there is an integer m such that $12m - 1 = (n-1)^2 + n^2 + (n+1)^2$, that is $12m = 3(n^2+1)$. Thus $n^2 + 1 = 4m$, so that n^2 leaves remainder -1 on division by 4 which is impossible. In fact if $n = 2k$ is even then n^2 leaves remainder 0 on division by 4, where as if $n = 2p+1$ is odd, then n^2 leaves remainder 1 on division by 4.

Question 5: Prove by contradiction that if p is a prime dividing a product ab of positive integers, then either $p|a$ or $p|b$ or both.

Solution: The proof is by contradiction. So assume that n is a prime number with $n|ab$ (read n divides ab), but that n divides neither a nor b. Since n is prime the greatest common divisor between n and a, defined as $h(n,a)$, can be n or 1; but because n and a are relatively prime it follows that $h(n,a) = 1$. Similarly $h(n,b) = 1$. So we can find integers s, t, u, v such that $1 = sn + ta$ and $1 = un + vb$. Then

$$1 = (sn+ta)(un+vb) = sun^2 + (uta + vbs)n + tvab.$$

Since $n|ab$, n divides the right-hand side, whence also $n|1$. Contradiction.