MS111: ASSESSED TEST ONE - SOLUTIONS

Question 1: Question 1: Show by induction that for every natural number n, the sum

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: For $n = 1$ it is true. Assume it is true for n, namely that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

We wish to show that it is true for $n + 1$, that is we wish to show that

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}.$$

Now we have

$$\sum_{k=1}^{n+1} k^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2[\frac{n^2}{4} + n + 1] = \frac{(n+1)^2(n+2)^2}{4}.$$

Question 2: Given a and b integers with $b > 0$. Then we have $a = qb+r = q'b+r'$ with $0 \le r < b$ and $0 \le r' < b$.
Thus we have $r - r' = (q' - q)b$.
If $q' \neq q$ then $|q - q'| \ge 1$.
So $|r - r'| \ge b$, which is impossible since r and r' lie between 0 and $b - 1$ inclusive.
Hence $q = q'$ and $r = r'$.

Question 3: Assume that the greatest common divisor of the two integers a and b is $g(a, b) = 1$.

(i) Show that if $a|c$ and $b|c$ than $ab|c$;

(ii) If $a|bc$ then $a|c$.

Solution:

(i) From the fundamental property of the highest common factor of two coprime integers we have that $ax + by = 1$, for some integers x, y.
Also by hypothesis we have $c = au$ and $c = bv$ for some integers u, v.
Then $c = cax + cby = bvax + auby = ab(vx + uy)$, and so $ab|c$.

(ii) As in (i), $c = cax + cby$.
Since $a|bc$ and $a|a$, then we know (from a previous Corollary which states that if c divides a and b, then c divides $au + bv$ for all integers u and v) that $a|(cax + cby) = c$.

Question 4: Show that for each prime $p \ge 5$ the natural number p^2+2 is divisible by 3 and therefore it is a composite number. *Hint: Work modulus* 3.

Solution: We work modulus 3; thus we write $p = 3q + r$, with $r = 1, 2$.
The $r = 0$ case has to be excluded otherwise p would not be prime.
Then $p^2 + 2 = 9q^2 + 6qr + r^2 + 2$. Hence we obtain:

- $r = 1$, $p^2 + 2 = 9q^2 + 6qr + 3$ which is divisible by 3;
- $r = 2$, $p^2 + 2 = 9q^2 + 6qr + 6$ which is divisible by 3.

Therefore $p^2 + 2$ is divisible by 3 for each prime $p > 3$ (and as a matter of fact for each natural number $n > 3$).

Question 5: Show by induction that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

Solution: For $n = 1$ it is true. Assume it is true for n, namely assume that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

We wish to show that it is true for $n + 1$, that is we wish to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}.$$

In fact from

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}},$$

and therefore we want to show that

$$\frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} \geq \sqrt{n+1}.$$

Hence $\sqrt{n}\sqrt{n+1} + 1 \geq n + 1$ which is equivalent to $n(n+1) \geq n^2$. Thus by induction the desired result follows.