Questions and Solutions for ASSIGNMENT 1, MAT1026: PROOF

Question 1: Prove, by contradiction, that the polynomial $f(x) = x^4 + 2x^2+2x+1$ cannot be written as the product of two quadratic polynomials with integer coefficients.

Proof: Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are quadratic polynomials with integer coefficients. Comparing coefficients of x^4, the leading coefficients of $g(x)$ and $h(x)$ are both equal to 1 or -1. In the latter case, replace $g(x)$ and $h(x)$ with their negatives. Then we can write $g(x) = x^2 + ax + b$, and $h(x) = x^2 + cx + d$, with a, b, c, d integers. Multiplying and comparing coefficients we obtain: $a + c = 0$, $ac + b + d = 2$, $ad + bc = 2$ and $bd = 1$. Use the first two equations to eliminate $c = -a$, $d = a^2 - b + 2$ from the third: $a(a^2 - 2b + 2) = 2$. Thus a is even and 2 is the product of two even numbers, which is impossible. There is an easier route but it works only in the case where tha constant term is 1. I have given here the general proof which works for $f(x) = x^4 + 2x^2 + 2x + \alpha$ for any $\alpha \neq 0$.

Question 2: Prove that at a party of at least two people, there are at least two who have the same number of friends at the party.

Proof: Let the people be $1, 2, ..., n$ with $f_1, f_2, ..., f_n$ friends present, respectively. Assume for a contradiction that the numbers $f_1, f_2, ..., f_n$ are all different. Since each of the numbers is between 0 and $n-1$ inclusive, they must be equal to $0, 1, ..., n-1$ in some order. So we can find i and j such that $f_i = 0$ and $f_j = n-1$. So j is everyone's friend and in particular i and j are friends, and also i is no-one's friend, and so in particular i and j are not friends. Contradiction.

Question 3: If a and b are positive integers, then $ax^2 + bx + (b - a) = 0$ has no positive integer root.

Proof: Assume that m is a positive integer root of $ax^2 + bx + (b - a) = 0$, with a and b are natural numbers. Then $am^2 + bm + (b - a) = 0$. If $b \geq a$, then $am^2 + bm + (b - a) = 0$ which is impossible. If $b < a$, then $m(ma + b) = a - b > 0$, but then $a - b = m(ma + b) \geq ma + b \geq a + b$. Therefore $2b \leq 0$ which is impossible as $b > 0$.

Question 4: Prove by contradiction that the sum of the squares of three consecutive integers cannot leave remainder -1 on division by 12.

Proof: Assume by contradiction that the statement P of the question is false. Let then $n - 1, n, n + 1$ be integers for which it fails. Thus there is an integer m such that $12m - 1 = (n-1)^2 + n^2 + (n+1)^2$, that is $12m = 3(n^2+1)$. Thus $n^2 + 1 = 4m$, so that n^2 leaves remainder -1 on division by 4 which is impossible. In fact if $n = 2k$ is even then n^2 leaves remainder 0 on division by 4, where as if $n = 2p+1$ is odd, then n^2 leaves remainder 1 on division by 4.

Question 5: Show that the statements P and $\sim\sim P$ are the same. Further suppose that P is a statement from which you can deduce $\sim P$. Which (if any) of the following conclusions can you draw:

(i) P is true;

(ii) P is false;

(iii) $\sim P$ is true;

(iv) $\sim P$ is false.

The negation of "P is false"(that is $\sim\sim P$) is "P is not false" which is the same as P is true.
Now assume P and deduce $\sim P$. We now have P and $\sim P$ which is a contradiction. We conclude that P is false, that is (ii). We can also conclude (iii) as it is the same as (ii). (i) and (iv) are also the same, and cannot be concluded.