SOLUTIONS UNASSESSED ASSIGNMENT 1

Question 1: Prove that the polynomial $f(x) = x^4 + 2x^2 + 2x + 1$ cannot be written as the product of two quadratic polynomials with integer coefficients.

Solution: Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are quadratic polynomials with integer coefficients. Comparing coefficients of x^4, the leading coefficients of $g(x)$ and $h(x)$ are both equal to 1 or -1. In the latter case, replace $g(x)$ and $h(x)$ with their negatives. Then we can write $g(x) = x^2 + ax + b$, and $h(x) = x^2 + cx + d$, with a, b, c, d integers. Multiplying and comparing coefficients we obtain: $a+c = 0$, $ac+b+d = 2$, $ad + bc = 2$ and $bd = 1$. Use the first two equations to eliminate $c = -a$, $d = a^2 - b + 2$ from the third: $a(a^2 - 2b + 2) = 2$. Thus a is even and 2 is the product of two even numbers, which is impossible. There is an easier route but it works only in the case where tha constant term is 1. I have given here the general proof which works for $f(x) = x^4 + 2x^2 + 2x + \alpha$ for any $\alpha \neq 0$.

Question 2: If a and b are positive integers, then $ax^2 + bx + (b - a) = 0$ has no positive integer root.

Solution: Proof (by contradiction): Assume that m is a positive integer root of $ax^2 + bx + (b - a) = 0$, with a and b are natural numbers. Then $am^2 + bm + (b - a) = 0$. If $b \geq a$, then $am^2 + bm + (b - a) = 0$ which is impossible. If $b < a$, then $m(ma + b) = a - b > 0$, but then $a - b = m(ma + b) \geq ma + b \geq a + b$. Therefore $2b \leq 0$ which is impossible as $b > 0$.

Question 3: Prove by contradiction that the sum of the squares of three consecutive integers cannot leave remainder -1 on division by 12.

Solution: Assume by contradiction that the statement P of the question is false. Let then $n - 1, n, n + 1$ be integers for which it fails. Thus there is an integer m such that $12m - 1 = (n - 1)^2 + n^2 + (n + 1)^2$, that is $12m = 3(n^2 + 1)$. Thus $n^2 + 1 = 4m$, so that n^2 leaves remainder -1 on division by 4 which is impossible. In fact if $n = 2k$ is even then n^2 leaves remainder 0 on division by 4, where as if $n = 2p+1$ is odd, then n^2 leaves remainder 1 on division by 4.

Question 4: Prove by contradiction that if p is a prime dividing a product ab of positive integers, then either $p|a$ or $p|b$.

Solution: The proof is by contradiction. So assume that p is a prime number with $p|ab$ (read p divides ab), but that p divides neither a nor b. Since p is prime the greatest common divisor between p and a, defined as $h(p, a)$, can be p or 1; but because p and a are relatively prime it follows that $h(p, a) = 1$. Similarly $h(p, b) = 1$. So we can find integers s, t, u, v such that $1 = sp + ta$ and $1 = up + vb$. Then

$$1 = (sp + ta)(up + vb) = sup^2 + (uta + vbs)p + tvab.$$

Since $p|ab$, p divides the right-hand side, whence also $p|1$. Contradiction.

Question 5: Show by induction that for every natural number n, the sum of the first n odd numbers is n^2.

Solution: For $n = 1$ it is true. Assume now it is true for the first $(n-1)$ odd numbers, namely assume that the sum of the first $(n-1)$ odd numbers is $(n-1)^2$. Then the sum of the first n odd numbers is obtained from this by adding the n^{th} odd number which is $2n-1$. Thus $(n-1)^2+(2n-1) = n^2$.

Question 6: Show by contraposition that if $a \geq 2$ and $a^m + 1$ is a prime number, with m any natural number, then a must be even.

Solution: Assume $a \geq 2$ is odd, then $a^m + 1$ is even for every natural number m and therefore it cannot be a prime.