

SHARP CONSTANTS FOR THE L^∞ -NORM ON THE TORUS AND APPLICATIONS TO DISSIPATIVE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Sharp estimates are obtained for the constants appearing in the Sobolev embedding theorem for the L^∞ norm on the d -dimensional torus for $d = 1, 2, 3$. The sharp constants are expressed in terms of the Riemann zeta-function, the Dirichlet beta-series and various lattice sums. We then provide some applications including the two dimensional Navier-Stokes equations.

1. INTRODUCTION AND NOTATION

The classical interpolation inequalities of functional analysis are of primary importance in the analysis of solutions of partial differential equations of mathematical physics (see for example [1, 33, 36, 24, 12, 13, 7, 8, 27, 17, 31, 9] and references there-in). Among the many fundamental interpolation inequalities, the Sobolev Embedding Theorem (SET) plays a central role as an indispensable tool in the analysis of solutions of nonlinear partial differential equations. In this paper, we extend the analysis reported in [5] to the case of the multidimensional torus. In [5] sharp estimates are obtained for the constants appearing in the Sobolev embedding theorem on the two dimensional torus. Here, we wish to derive similar estimates for the case of spatial dimensions $d = 1, 2, 3$ in a unified way.

Let us first give some standard preliminary functional setting and notations [34, 1, 26, 30, 37]. Denote by $\Omega = [0, L]^d$ the d -dimensional torus; for any scalar and mean-zero function $\phi(x)$ with $x \in \Omega$ let $\|\phi\|_p^p = \int_\Omega |\phi(x)|^p dx$ be the norm associated to the Banach space of Ω -periodic functions; we also define the L^∞ norm as

$$\|\phi(x)\|_\infty = \sup_{x \in \Omega} |\phi(x)|. \quad (1.1)$$

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For $p = 2$, we denote by $L^2(\Omega)$ the Hilbert space of Ω -periodic functions; given $n = n_1 + n_2 + \cdots + n_d$ with all the n_i non-negative integers, let

$$D^n := D^{n_1, n_2, \dots, n_d} = \frac{\partial^{n_1+n_2+\dots+n_d}}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_d^{n_d}}, \quad (1.2)$$

and let

$$\dot{H}^n := \left\{ \phi : \int_{\Omega} \phi dx = 0, \int_{\Omega} (D^{n_1, n_2, \dots, n_d} \phi)^2 dx < +\infty \text{ for } n_1 + n_2 + \cdots + n_d = n \right\} \quad (1.3)$$

together with

$$\|\phi\|_{\dot{H}^n}^2 := \sum_{n=n_1+\dots+n_d} \frac{n!}{n_1! \cdots n_d!} \|D^{n_1, n_2, \dots, n_d} \phi\|_{L^2}^2, \quad (1.4)$$

be the Sobolev space of mean zero Ω -periodic functions with up to n -derivatives in $L^2(\Omega)$. It then follows from Parseval's identity that

$$\sum_{n=n_1+\dots+n_d} \frac{n!}{n_1! \cdots n_d!} \|D^{n_1, n_2, \dots, n_d} \phi\|_{L^2}^2 = L^d \left(\frac{2\pi}{L}\right)^{2n} \sum_{n=n_1+\dots+n_d} \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^{2n} |\phi_{\vec{k}}|^2. \quad (1.5)$$

In (1.5) the Fourier series expansion has been used for the mean zero function

$$\phi = \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i \vec{k} \cdot \vec{x}/L}, \quad (1.6)$$

and $\left(\frac{2\pi}{L}\right)^2 \vec{k} \cdot \vec{k} = \left(\frac{2\pi}{L}\right)^2 (k_1^2 + k_2^2 + \cdots + k_d^2)$. By the same token the corresponding Sobolev space of mean zero periodic functions can be defined as \dot{H}^s for every *real* number s ; this is the same as

$$\begin{aligned} \dot{H}^s = \left\{ \phi : \phi = \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i \vec{k} \cdot \vec{x}/L}, \right. \\ \left. \overline{\phi_{\vec{k}}} = \phi_{-\vec{k}}, \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^{2s} |\phi_{\vec{k}}|^2 < +\infty \right\}. \end{aligned} \quad (1.7)$$

Hence, by extending (1.4) to non-integer positive values, we have $\dot{H}^s = \{\phi : \|\phi\|_{\dot{H}^s}^2 < +\infty\}$. These Sobolev spaces, defined on the d -dimensional torus, are used below as we need to deal with the negative Laplacian $A = -\Delta$ (as a self-adjoint unbounded operator) and its fractional powers. More precisely, we have that the eigenvalues of the negative Laplacian $A = -\Delta$ are given by

the numbers $(\frac{2\pi}{L})^2 |\vec{k}|^2$, so the domain of its powers A^s is the set of functions such that

$$L^d \left(\frac{2\pi}{L}\right)^{4s} \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^{4s} |\phi_{\vec{k}}|^2 = \|A^s \phi(x)\|_2^2 < +\infty. \quad (1.8)$$

In particular, for $s = \frac{1}{2}$ (on the torus) we have

$$\|A^{\frac{1}{2}} \phi(x)\|_2^2 = \|\nabla \phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^2 |\phi_{\vec{k}}|^2, \quad (1.9)$$

while for $s = 1$ we have (on the torus)

$$\|A\phi(x)\|_2^2 = \|(-\Delta)\phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^4 \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^4 |\phi_{\vec{k}}|^2. \quad (1.10)$$

In the rest of the paper, with a minor abuse of notation, for any $s > 0$, we make the *formal* identification

$$\|A^{\frac{s}{2}} \phi(x)\|_2^2 = \|(-\Delta)^{\frac{s}{2}} \phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k} \in \mathbb{Z}^d \setminus \{\vec{0}\}} |\vec{k}|^{2s} |\phi_{\vec{k}}|^2, \quad (1.11)$$

provided it is understood that these operators are being used as differential operators “acting” on functions in \dot{H}^s , according to (1.7) and (1.8).

The layout of the paper is as follows: in Section 2, we prove results on the sharp estimates of the constants on the torus in the SET (see Theorems 1, 2 and 3). Then in Section 3, we apply the results obtained in section 2 in order to find estimates for the L^∞ norm of solutions of the dissipative partial differential equation known as the Kolmogorov-Petrovskii-Piscuinov-Fisher equation (KPPE) (see Theorems 4 and 5). In Section 3, we also consider the two dimensional Navier-Stokes equations and we prove the asymptotic bound in Theorem 6.

2. SHARP CONSTANTS IN THE SOBOLEV EMBEDDING THEOREM FOR THE L^∞ -NORM ON THE TORUS IN ONE, TWO AND THREE SPACE DIMENSIONS

In this section, we wish to estimate explicitly the constants on the torus in the SET, for the L^∞ norm of any mean-zero function $\phi \in \dot{H}^s$, which depends upon one, two or three space variables. First note that on the d -dimensional torus Ω , if $s > d/2$, for any mean zero function $\phi \in \dot{H}^s$ it is true that [35]

$$\|\phi\|_\infty \leq c_s \|\phi\|_{\dot{H}^s},$$

where c_s is a positive constant depending upon s only. Our aim in this section is to obtain a sharp estimate of the constant c_s , appearing in the

above inequality. For simplicity, we consider the d -dimensional torus of length 2π , namely, $\Omega = [0, 2\pi]^d$.

One-Dimensional Case $d = 1$. We begin with space dimension $d = 1$. From here on, we use the more flexible notation $\sum' := \sum_{k \in \mathbb{Z}^d \setminus \{0\}}$, that is, the prime in the symbol \sum' meaning that the sum is to be taken all over the lattice \mathbb{Z}^d excluding only the case where the indices are all simultaneously zero. So we start by proving the following:

Theorem 1. *On the one-dimensional torus $\Omega = [0, 2\pi]$, for every positive real number $s = \frac{1+\epsilon}{2}$ with $\epsilon > 0$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^s$ satisfies the estimate*

$$\|\phi(x)\|_\infty^2 \leq \frac{\zeta(1+\epsilon)}{\pi} \|(-\Delta)^{\frac{1+\epsilon}{4}} \phi(x)\|_2^2, \quad (2.1)$$

where the coefficient

$$\zeta(1+\epsilon) = \sum_{n \geq 1} \frac{1}{n^{1+\epsilon}}, \quad (2.2)$$

is sharp and coincide with the Riemann zeta-function.

Proof. We first expand our function in Fourier series $\phi(x) = \sum' \phi_k e^{ikx}$; to give

$$\begin{aligned} \|\phi(x)\|_\infty &\leq \sum' |\phi_k| = \sum' \frac{|k|^{(1+\epsilon)/2}}{|k|^{(1+\epsilon)/2}} |\phi_k| \\ &\leq \left(\sum' \frac{1}{|k|^{1+\epsilon}} \right)^{1/2} \left(\sum' |k|^{1+\epsilon} |\phi_k|^2 \right)^{1/2} \\ &= \sqrt{2} (\zeta(1+\epsilon))^{\frac{1}{2}} (2\pi)^{-1/2} \|(-\Delta)^{\frac{1+\epsilon}{4}} \phi\|_2. \end{aligned} \quad (2.3)$$

which, when squared up, gives (2.1). In order to see that $c(\epsilon) = \zeta(1+\epsilon)$ is sharp, we use the extremal functions [32]

$$\phi = \sum' |k|^{-(1+\epsilon)} e^{ikx}. \quad (2.4)$$

Now, first note that from the definition of ϕ it follows that,

$$\|\phi(x)\|_\infty \leq \sum' |k|^{-(1+\epsilon)}. \quad (2.5)$$

Secondly,

$$\|\phi(x)\|_\infty \geq |\phi(0)| = \phi(0) = \sum' |k|^{-(1+\epsilon)}, \quad (2.6)$$

so we obtain,

$$\|\phi(x)\|_\infty = \phi(0) = \sum' |k|^{-(1+\epsilon)}. \quad (2.7)$$

It follows that all the above inequalities become equalities and hence $c(\epsilon) = \frac{\zeta(1+\epsilon)}{\pi}$ cannot be improved, namely, it is sharp or optimal. The proof is now complete. \square

Remark. Note that in the one-dimensional case the embedding $H^s \rightarrow L^\infty$ for integer values of $s \geq 1$, was found by Stechkin and it is reported in the appendix by V.Y. Levin and S.B. Stechkin in the Russian edition of [18].

Formula (2.1) gives the explicit value of the constant in front of *all* the Sobolev spaces \dot{H}^s with $s = \frac{1+\epsilon}{2}$ for *every* $\epsilon > 0$. For instance, by choosing the value $\epsilon = 1$, then

$$\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}; \quad (2.8)$$

hence in this case we have,

$$\|\phi(x)\|_\infty^2 \leq \frac{\pi}{6} \|(-\Delta)^{\frac{1}{2}} \phi(x)\|_2^2 \equiv \frac{\pi}{6} \|\nabla \phi(x)\|_2^2. \quad (2.9)$$

If we take the value $\epsilon = 2$, we have,

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = B = 1.20205690032\dots, \quad (2.10)$$

and thus for the $\|\phi(x)\|_\infty^2$, we have,

$$\|\phi(x)\|_\infty^2 \leq \frac{B}{\pi} \|(-\Delta)^{\frac{3}{4}} \phi(x)\|_2^2. \quad (2.11)$$

We wish now to find the asymptotic of the Riemann zeta-function for small values of the positive parameter ϵ . In this case one obtains,

$$\zeta(1 + \epsilon) = \frac{1}{\epsilon} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \gamma_n \epsilon^n, \quad (2.12)$$

where $\gamma \simeq 0.58$ is the Euler-Mascheroni constant and γ_n are the Stieltjes constants. Therefore, we can state the following Corollary which provides the explicit computation of the constant in the Sobolev embedding theorem on the one-dimensional torus of length 2π :

Corollary 1. *On the one-dimensional torus $\Omega = [0, 2\pi]$, in the limit $\epsilon \rightarrow 0^+$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{\frac{1+\epsilon}{2}}$ obeys the estimate*

$$\|\phi(x)\|_\infty^2 \leq c_\epsilon^2 \|(-\Delta)^{\frac{1+\epsilon}{4}} \phi\|_2^2, \quad (2.13)$$

where $c_\epsilon^2 = \frac{1}{\pi} (\frac{1}{\epsilon} + \gamma) + O(\epsilon)$ for $\epsilon \rightarrow 0^+$.

Two-Dimensional - Case $d = 2$. This case has been investigated in detail in [5] and for the convenience of the reader, we report here a brief summary. First we state

Theorem 2. *On the two-dimensional torus $\Omega = [0, 2\pi]^2$, for every positive real number $s = 1 + \epsilon$ with $\epsilon > 0$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{1+\epsilon}$ satisfies the estimate*

$$\|\phi(x)\|_\infty \leq [4\zeta(1 + \epsilon)\beta(1 + \epsilon)]^{\frac{1}{2}} (2\pi)^{-1} \|(-\Delta)^{\frac{1+\epsilon}{2}} \phi(x)\|_2, \quad (2.14)$$

where $C(\epsilon) = 4\zeta(1 + \epsilon)\beta(1 + \epsilon)$ is sharp, and where $\zeta(1 + \epsilon)$ and $\beta(1 + \epsilon)$

$$\zeta(1 + \epsilon) = \sum_{n \geq 1} \frac{1}{n^{1+\epsilon}}, \quad \beta(1 + \epsilon) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^{1+\epsilon}}, \quad (2.15)$$

are the Riemann zeta-function and Dirichlet series, respectively.

For the proof see [5], where one can also find some particular cases giving the values of the sharp constants. Furthermore, the asymptotics regime $\epsilon \rightarrow 0^+$ in the two-dimensional case is provided by:

Corollary 2. *On the two-dimensional torus $\Omega = [0, 2\pi]^2$, in the limit $\epsilon \rightarrow 0^+$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{1+\epsilon}$ satisfies the estimate*

$$\|\phi(x)\|_\infty^2 \leq c_\epsilon^2 \|(-\Delta)^{\frac{1+\epsilon}{2}} \phi\|_2^2, \quad (2.16)$$

where $c_\epsilon^2 = \frac{1}{4\pi} \left(\frac{1}{\epsilon} + \hat{\gamma} \right) + O(\epsilon)$ for $\epsilon \rightarrow 0^+$, and $\hat{\gamma} := \gamma + \frac{4\beta'(1)}{\pi}$, with $\gamma \simeq 0.58$ and $\beta'(1) \simeq 0.19$.

Here as $\epsilon \rightarrow 0^+$

$$4\zeta(1 + \epsilon)\beta(1 + \epsilon) \simeq \pi(\gamma + \epsilon^{-1}) + 4\beta'(1) + O(\epsilon), \quad (2.17)$$

provided ϵ is small enough. For more details, see [5].

Three-Dimensional Case $d = 3$. In three spatial dimensions, we prove the following:

Theorem 3. *On the three-dimensional torus $\Omega = [0, 2\pi]^3$, for every positive real number $s = \frac{3}{2} + \epsilon$ with $\epsilon > 0$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{\frac{3}{2}+\epsilon}$ satisfies the estimate*

$$\|\phi(x)\|_\infty^2 \leq \frac{a(2s)}{(2\pi)^3} \|(-\Delta)^{\frac{s}{2}} \phi(x)\|_2^2, \quad (2.18)$$

where [39, 40, 41, 10, 11]

$$a(2s) = \sum' (\vec{k} \cdot \vec{k})^{-(\frac{3}{2}+\epsilon)} = \frac{3b(2s) + 3c(2s) + d(2s)}{2^{3-2s} - 1}$$

with

$$\begin{aligned} b(2s) &= \sum' (-1)^{k_1} (k_1^2 + k_2^2 + k_3^2)^{-s}, \\ c(2s) &= \sum' (-1)^{k_1+k_2} (k_1^2 + k_2^2 + k_3^2)^{-s}, \\ d(2s) &= \sum' (-1)^{k_1+k_2+k_3} (k_1^2 + k_2^2 + k_3^2)^{-s}. \end{aligned}$$

Remark. Note that for $s > \frac{3}{2}$, $3b(2s) + 3c(2s) + d(2s) < 0$ and so

$$a(2s) = \frac{3b(2s) + 3c(2s) + d(2s)}{2^{3-2s} - 1} > 0$$

as it should.

Proof. The proof is similar to the one and two dimensional case and we give here the main steps for the reader's convenience: expand $\phi(x)$ in Fourier series

$$\phi(x) = \sum' \phi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}; \quad (2.19)$$

to give

$$\begin{aligned} \|\phi(x)\|_\infty^2 &\leq \left(\sum' |\phi_{\vec{k}}| \right)^2 \\ &\leq \left(\sum' \frac{1}{(\vec{k} \cdot \vec{k})^{\frac{3}{2}+\epsilon}} \right) \left(\sum' (\vec{k} \cdot \vec{k})^{\frac{3}{2}+\epsilon} |\phi_{\vec{k}}|^2 \right) \\ &= \frac{1}{(2\pi)^3} \left(\sum' (\vec{k} \cdot \vec{k})^{-(\frac{3}{2}+\epsilon)} \right) \|(-\Delta)^{\frac{3}{2}+\epsilon} \phi(x)\|_2^2 = \frac{a(2s)}{(2\pi)^3} \|(-\Delta)^{\frac{s}{2}} \phi(x)\|_2^2. \end{aligned} \quad (2.20)$$

with $s = \frac{3}{2} + \epsilon$. The sharpness of the coefficient $a(2s)$ is shown in the same way as in the one-dimensional case.

3. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

In this section, we apply the results obtained above to find estimates for the L^∞ norm of solutions of Partial Differential Equations (PDEs). Before beginning, let us make clear the notations we will use in this section: because, we are going to deal with powers of functions in the Sobolev space

$$\dot{H}^n := \left\{ u : \int_\Omega u dx = 0, \int_\Omega (D^{n_1, n_2, \dots, n_d} u)^2 dx < +\infty \text{ for } n_1 + n_2 + \dots + n_d = n \right\}, \quad (3.1)$$

in order to avoid confusion, we will define in a similar way to (1.4)

$$J_n := \|u\|_{J_n}^2 = \sum_{n=n_1+\dots+n_d} \frac{n!}{n_1! \cdot \dots \cdot n_d!} \|D^{n_1, n_2, \dots, n_d} u\|_{L^2}^2, \quad (3.2)$$

in addition to the fact that this notation appears in previous works in the literature (see for example, [3, 16]).

3.1. Estimates for the Kolmogorov-Petrovskii-Piscuinov-Fisher equation. The first PDE we wish to study is the well known Kolmogorov-Petrovskii-Piscuinov-Fisher equation (KPPE), namely

$$u_t = \alpha \Delta u + \lambda u - u^3 \quad (3.3)$$

with α and λ positive constants, in the domain $\Omega = [0, 2\pi]^d$ with periodic boundary conditions. Equation (3.3) is known to have a unique solution for every initial datum $u_0 \in L_2(\Omega)$; the solution $u \in C([0, T]; H)$, where $H = L_2(\Omega)$, and $T > 0$; in addition the corresponding semigroup $S_t u_0 = u(t)$ has a global attractor $\mathcal{A} \subset\subset H$ (for details see [2, 37, 26]). We wish to find estimates for the J_n as accurate as possible and then use them to obtain the corresponding estimates for the L^∞ norm of the solutions by using the sharp estimate found in Section 2.

First note that one can show that the time-dependent functionals J_n satisfy, a so-called *ladder differential inequality* ([3, 16, 4]), namely for any $en > \frac{d}{2}$, where d is the spatial dimension, we have that

$$\frac{1}{2} \dot{J}_n \leq -\alpha J_{n+1} + \lambda J_n + c_n \|u\|_\infty^2 J_n, \quad (3.4)$$

where the constants c_n do not depend upon the solution function $u(x, t)$. It is immediately apparent that computing all the constants c_n appearing in the ladder is a formidable task and so, because we wish to be as explicit and sharp as possible, we are somehow forced to restrict ourselves to the first few J_n , where we can obtain explicitly all the constants involved in the ladder inequality. In particular, we know that for the one-dimensional case $d = 1$, we can restrict ourselves to the analysis of J_0 and J_1 , which together are sufficient for having an upper bound on the $\|u\|_\infty$ norm of the solution of the PDE. On the other hand, for the $d = 2, 3$ case we will have to analyze J_2 also. Before we start our analysis, we wish to make the following important remark:

From now on with an overbar we mean the so-called limit superior taken over all the initial conditions as time goes to plus infinity; more formally we mean that we are using the classical Gronwall inequality and then we take the limit over all the initial conditions as time goes to infinity; in other words, an overbar over a time-dependent quantity X , which also depends upon the initial condition, namely $\overline{X}(u_0, t)$ means the least majorant of the set comprised of all the limits of $X(u_0, t)$ as t tends to infinity for all the possible set of initial data. Occasionally, the set of initial data may be restricted to the

global attractor of the PDE under investigation, but this will be clear from the context if not explicitly stated.

Also, because our analysis above is done for mean zero functions and, in addition, because we may need occasionally to use the Poincaré inequality, we restrict ourselves to the sub-space S_0 of solutions of the KPPF equation containing only periodic odd solutions. One can see that the KPPF flow does have the property of leaving invariant the sub-space of odd solutions (or even ones as well).

So let us now start with the analysis of J_0 :

Lemma 1. *Given any (smooth) solution $u(x, t) \in S_0$ of the KPPF equation the time-asymptotic behaviour of $J_0(t)$, namely \bar{J}_0 , is given by,*

$$\bar{J}_0 := \lim_{t \rightarrow \infty} J_0(t) \leq (2\pi)^d (\lambda - \alpha). \quad (3.5)$$

Proof. Differentiating $J_0(t)$ with respect to time and using equation (3.3) we find

$$\frac{1}{2} \dot{J}_0 = -\alpha J_1 + \lambda J_0 - \int_{\Omega} (u)^4 dx. \quad (3.6)$$

For non-trivial behaviour, one can see that we must have $\lambda > \alpha$, and of course we will assume that this condition is satisfied. Thus, by applying the Poincaré inequality to the term J_0 , we obtain $J_0 \leq J_1$ with the constant being exactly one on the torus of length 2π [35]; also by applying a Cauchy-Schwarz inequality to the term $\int_{\Omega} (u)^2 dx$ we get

$$\left(\int_{\Omega} (u)^2 dx \right)^2 \leq (2\pi)^d \int_{\Omega} (u)^4 dx;$$

inserting these estimates into (3.6) we obtain,

$$\dot{J}_0 \leq 2(\lambda - \alpha)J_0 - \frac{2J_0^2}{(2\pi)^d}. \quad (3.7)$$

We need to study the solutions of the above differential inequality. By a comparison principle, they are bounded above by the solutions of the one-dimensional ODE

$$\dot{J}_0 = 2(\lambda - \alpha)J_0 - \frac{2J_0^2}{(2\pi)^d}.$$

One can see that the fixed points are $J_0 = 0, (2\pi)^d(\lambda - \alpha)$ with 0 being unstable and $(2\pi)^d(\lambda - \alpha)$ being stable. Thus, the long-time asymptotic behaviour of J_0 (denoted with \bar{J}_0) is given by,

$$\bar{J}_0 := \lim_{t \rightarrow \infty} J_0(t) \leq (2\pi)^d (\lambda - \alpha), \quad (3.8)$$

which is independent of the initial condition $u(x, t = 0)$. \square

We now estimate J_1 with a similar strategy; we have:

Lemma 2. *The time-asymptotic behaviour of $J_1(t)$, namely \bar{J}_1 , is given by*

$$\bar{J}_1 := \lim_{t \rightarrow \infty} J_1(t) \leq \frac{(2\pi)^d \lambda (\lambda - \alpha)}{\alpha}. \quad (3.9)$$

Proof.

$$\frac{1}{2} \dot{J}_1 = -\alpha J_2 + \lambda J_1 - \sum_{n=n_1+\dots+n_d=1} \int_{\Omega} 3u^2 (Du)^2 dx; \quad (3.10)$$

the last term is negative (or zero) and so it can be neglected. So we obtain the estimate

$$\frac{1}{2} \dot{J}_1 \leq -\alpha \frac{J_1^2}{J_0} + \lambda J_1, \quad (3.11)$$

where we have used again the Cauchy-Schwarz inequality on the J_1 term, $J_1 \leq J_2^{\frac{1}{2}} J_0^{\frac{1}{2}}$. A similar analysis to that used in obtaining the estimate (3.8) gives for J_1 the result

$$\bar{J}_1 := \lim_{t \rightarrow \infty} J_1(t) \leq \frac{\lambda}{\alpha} \bar{J}_0 \leq \frac{(2\pi)^d \lambda (\lambda - \alpha)}{\alpha}. \quad (3.12)$$

\square

We are then ready to obtain the estimate for the $\|u(x)\|_{\infty}$ of the solution in the one-dimensional case:

Theorem 4. *The time-asymptotic upper bound for the $\|u(x)\|_{\infty}$ of the solution of (3.3) in the $d = 1$ case is given by*

$$\overline{\|u(x)\|_{\infty}^2} \leq \frac{\pi^2}{3} \frac{\lambda (\lambda - \alpha)}{\alpha}. \quad (3.13)$$

Proof. First we take the sharp result (2.1), namely,

$$\|u(x)\|_{\infty}^2 \leq \frac{\zeta(1+\epsilon)}{\pi} \|(-\Delta)^{\frac{1+\epsilon}{4}} u(x)\|_2^2. \quad (3.14)$$

By taking the value $\epsilon = 1$, we therefore obtain,

$$\|u(x)\|_{\infty}^2 \leq \frac{\pi}{6} \|(-\Delta)^{\frac{1}{2}} u(x)\|_2^2 = \frac{\pi}{6} \|\nabla u(x)\|_2^2 = \frac{\pi}{6} J_1; \quad (3.15)$$

using (3.12) in the estimate (3.15) we obtain

$$\overline{\|u(x)\|_{\infty}^2} \leq \frac{\pi^2}{3} \frac{\lambda (\lambda - \alpha)}{\alpha}. \quad \square$$

We now provide the estimate for the two-dimensional case. Here, of course we need to integrate over the periodic domain $\Omega = [0, 2\pi]^2$. We want to

obtain as sharp as possible estimates for J_2 . We begin with the study of $J_2(t)$:

Lemma 3. *The time-asymptotic estimate of \bar{J}_2 is given by,*

$$\bar{J}_2 \leq \frac{4\pi^2\lambda(\lambda - \alpha)}{\alpha^3} (\alpha\lambda + 312\pi\lambda(\lambda - \alpha)). \quad (3.16)$$

Proof. The differential inequality for $J_2(t)$ is given by,

$$\frac{1}{2}\dot{J}_2 \leq -\alpha J_3 + \lambda J_2 - \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_{\Omega} (D^2u)(D^2u^3) dx dy. \quad (3.17)$$

We first analyze the nonlinear term:

$$\begin{aligned} & - \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_{\Omega} (D^2u)(D^2u^3) dx dy \quad (3.18) \\ & = -6 \int_{\Omega} u(u_x)^2 u_{xx} - 3 \int_{\Omega} u^2(u_{xx})^2 dx dy - 6 \int_{\Omega} u(u_y)^2 u_{yy} dx dy \\ & - 3 \int_{\Omega} u^2(u_{yy})^2 dx dy - 6 \int_{\Omega} u^2(u_{xy})^2 dx dy - 12 \int_{\Omega} uu_x u_y u_{xy} dx dy; \end{aligned}$$

integrating by parts the first, the third and the last term on the right hand side and then rearranging we obtain,

$$\begin{aligned} & - \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_{\Omega} (D^2u)(D^2u^3) dx dy = 2 \int_{\Omega} (u_x)^4 dx dy \quad (3.19) \\ & - 3 \int_{\Omega} u^2(u_{xx})^2 dx dy + 2 \int_{\Omega} (u_y)^4 dx dy - 3 \int_{\Omega} u^2(u_{yy})^2 dx dy \\ & - 6 \int_{\Omega} u^2(u_{xy})^2 dx dy + 6 \int_{\Omega} (u_x)^2(u_y)^2 dx dy + 6 \int_{\Omega} uu_{xx}(u_y)^2 dx dy; \end{aligned}$$

by splitting the last two terms by applying first a Cauchy-Schwarz inequality and then a Young inequality we get,

$$\begin{aligned} & - \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_{\Omega} (D^2u)(D^2u^3) dx dy = 2 \int_{\Omega} (u_x)^4 dx dy \quad (3.20) \\ & - 3 \int_{\Omega} u^2(u_{xx})^2 dx dy + 2 \int_{\Omega} (u_y)^4 dx dy - 3 \int_{\Omega} u^2(u_{yy})^2 dx dy \\ & - 6 \int_{\Omega} u^2(u_{xy})^2 dx dy + 3 \int_{\Omega} (u_x)^4 dx dy + 3 \int_{\Omega} (u_y)^4 \\ & + 3 \int_{\Omega} u^2(u_{xx})^2 dx dy + 3 \int_{\Omega} (u_y)^4 dx dy; + 3 \int_{\Omega} (u_y)^4. \end{aligned}$$

Simplifying we finally obtain that the nonlinear term can be estimated as follows:

$$- \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_{\Omega} (D^2 u)(D^2 u^3) dx dy \leq 5 \int_{\Omega} (u_x)^4 dx dy + 8 \int_{\Omega} (u_y)^4 dx dy. \quad (3.21)$$

Thus, we have to estimate the terms $5 \int_{\Omega} (u_x)^4 dx dy$ and $8 \int_{\Omega} (u_y)^4 dx dy$. In the two-dimensional case, we can use an improved version of the Ladyzhenskaya inequality [21], namely for any mean zero function $\phi(x, y)$ on the $2d$ torus we have the inequality

$$\int_{\Omega} (\phi(x, y))^4 dx dy \leq \frac{6}{\pi} \int_{\Omega} (\phi(x, y))^2 dx dy \int_{\Omega} |\nabla \phi|^2 dx dy;$$

hence we can estimate the term $5 \int_{\Omega} (u_x)^4 dx dy$ as

$$5 \int_{\Omega} (u_x)^4 dx dy \leq \frac{30}{\pi} \left(\int_{\Omega} (u_x)^2 dx dy \right) \left(\int_{\Omega} (u_{xx}^2 + u_{xy}^2) dx dy \right)$$

and similarly,

$$8 \int_{\Omega} (u_y)^4 dx dy \leq \frac{48}{\pi} \left(\int_{\Omega} (u_y)^2 dx dy \right) \left(\int_{\Omega} (u_{yy}^2 + u_{xy}^2) dx dy \right).$$

By noting that

$$\int_{\Omega} (u_x)^2 dx dy \leq J_1 \quad \text{and} \quad \int_{\Omega} (u_{xx}^2 + u_{xy}^2) dx dy \leq J_2$$

we can write for (3.17) the inequality

$$\frac{1}{2} \dot{J}_2 \leq -\alpha J_3 + \lambda J_2 + \frac{78}{\pi} J_1 J_2. \quad (3.22)$$

By using $J_2 \leq J_3^{\frac{1}{2}} J_1^{\frac{1}{2}}$ and

$$\left(\frac{1}{\alpha} (\lambda + \frac{78}{\pi} J_1)^2 J_1 \right)^{\frac{1}{2}} (\alpha J_3)^{\frac{1}{2}} \leq \frac{\alpha}{2} J_3 + \frac{1}{2\alpha} (\lambda + \frac{78}{\pi} J_1)^2$$

(3.22) becomes

$$\dot{J}_2 \leq -\alpha \frac{J_2^2}{J_1} + \frac{1}{\alpha} (\lambda + \frac{78}{\pi} J_1)^2 J_1. \quad (3.23)$$

By means of an analysis similar to the one done for \bar{J}_0 and \bar{J}_1 , we have that the time asymptotic behaviour for \bar{J}_2 is given by

$$\bar{J}_2 \leq \frac{1}{\alpha} (\lambda + \frac{78}{\pi} \bar{J}_1) \bar{J}_1. \quad (3.24)$$

By substituting the estimate (3.9) (which holds in any spatial dimension) for \bar{J}_1 we obtain

$$\bar{J}_2 \leq \frac{4\pi^2 \lambda (\lambda - \alpha)}{\alpha^3} (\alpha \lambda + 312\pi \lambda (\lambda - \alpha)). \quad (3.25)$$

By using the estimate (3.25), one can then state the

Theorem 5. *The time-asymptotic upper bound for the $\|u(x)\|_\infty$ of the solution of (3.3) in the $d = 2$ case is given by*

$$\overline{\|u(x)\|_\infty}^2 \leq \frac{24K\lambda}{\pi^2\alpha^3}(\lambda - \alpha)(\alpha\lambda + 312\pi\lambda(\lambda - \alpha)). \quad (3.26)$$

Proof. We use formula (2.14) obtained in Theorem 2 with $\epsilon = 1$, namely,

$$\overline{\|u(x)\|_\infty}^2 \leq \frac{1}{\pi^2}\zeta(2)\beta(2) \|\Delta u(x)\|_2^2 \leq \frac{1}{\pi^2}\zeta(2)\beta(2) \bar{J}_2. \quad (3.27)$$

By using the values for $\zeta(2)\beta(2) = 6\pi^{-2}K$ with $K = 0.915965594\dots$ and inserting the estimate (3.25) for \bar{J}_2 , we finally obtain

$$\overline{\|u(x)\|_\infty}^2 \leq \frac{24K\lambda}{\pi^2\alpha^3}(\lambda - \alpha)(\alpha\lambda + 312\pi\lambda(\lambda - \alpha)). \quad (3.28)$$

3.2. Estimates for the two-dimensional Navier-Stokes equations.

Consider the incompressible Navier-Stokes equations on the two-dimensional periodic domain $\Omega = [0, 2\pi]^2$,

$$u_t + (u \cdot \nabla)u = \nu\Delta u - \nabla\mathcal{P} + f, \quad \operatorname{div} u = 0, \quad \operatorname{div} f = 0, \quad u(0) = u_0. \quad (3.29)$$

Here, as usual, we denote with $u = (u_1, u_2)$ the velocity vector, with \mathcal{P} the pressure, with ν the constant kinematic viscosity and with f the external forces applied to the fluid. Following rigorous results [2, 37, 26, 35, 28] for the Navier-Stokes flow on the two-dimensional torus, it is rigorously proved that for any periodic and divergence free initial condition $u_0 \in J_1$ and any force $f \in L_2(\Omega)$ there is a unique solution $u \in C([0, T]; H)$ which is also divergence free, and in addition, it depends continuously on the initial condition u_0 . We take the spatial average of both the velocity field and the force field to be zero, namely we suppose that (all the integrals are evaluated on the domain $\Omega = [0, 2\pi]^2$)

$$\int u(x, t)dx = 0, \quad \text{and} \quad \int f(x)dx = 0.$$

Define the functional quantities (which are Sobolev norms by virtue of the mean-zero assumptions on u, f and the Poincaré inequality)

$$J_n = \sum_{i=1}^2 \sum_{n=n_1+n_2} \frac{n!}{n_1!n_2!} \int |D^n u_i|^2 dx \equiv \|D^n u\|_2^2, \quad (3.30)$$

$$F_n = \sum_{i=1}^2 \sum_{n=n_1+n_2} \frac{n!}{n_1!n_2!} \int |D^n f_i|^2 dx \equiv \|D^n f\|_2^2. \quad (3.31)$$

Then for $n \geq 1$ the J_n satisfy a ladder theorem [3, 16, 23]

$$\frac{1}{2} \dot{J}_n \leq -\nu J_{n+1} + c_n \|Du\|_\infty J_n + J_n^{1/2} F_n^{1/2}, \quad (3.32)$$

where the c_n are constants not dependent upon the solution function u .

By using the inequality

$$J_n^{p+q} \leq J_{n+p}^q J_{n-q}^p \quad 0 < q \leq n, \quad p \geq 1, \quad (3.33)$$

we can cast (3.32) as

$$\frac{1}{2} \dot{J}_n \leq -\nu \frac{J_n^{1+1/s}}{J_{n-s}^{1/s}} + c_n J_n \|Du\|_\infty + J_n^{1/2} F_n^{1/2}, \quad (3.34)$$

where $1 \leq s < n$. As we mentioned above, it is clear that computing all the constants c_n appearing in the ladder is extremely hard and so we will restrict ourselves to the first few J_n ; more precisely, we consider the 2 dimensional Navier-Stokes equations, and hence, we need to analyze in details the quantities J_0, J_1, J_2 . As it is well known for J_0 we have

$$\frac{1}{2} \dot{J}_0 \leq -\nu J_1 + J_0^{1/2} F_0^{1/2}; \quad (3.35)$$

by using a Poincaré inequality on the J_1 term we obtain

$$\frac{1}{2} \dot{J}_0 \leq -\nu J_0 + J_0^{1/2} F_0^{1/2}; \quad (3.36)$$

thus, by an analysis similar to the one employed for the KPPF equation above we obtain

$$\bar{J}_0 := \lim_{t \rightarrow \infty} J_0(t) \leq \frac{F_0}{\nu^2}. \quad (3.37)$$

For J_1 a similar strategy yields the same estimate obtained for J_0 , namely

$$\frac{1}{2} \dot{J}_1 \leq -\nu J_2 + J_2^{1/2} F_0^{1/2}, \quad (3.38)$$

where we have performed first an integration by parts and then a Cauchy-Schwarz inequality to obtain the last term; by splitting the last term by using a Young inequality, simplifying and then using a Poincaré inequality on the $-\frac{\nu}{2} J_2$ term and rearranging we have

$$\frac{1}{2} \dot{J}_1 \leq -\frac{\nu}{2} J_1 + \frac{1}{2\nu} F_0 \quad (3.39)$$

and thus,

$$\bar{J}_1 := \lim_{t \rightarrow \infty} J_1(t) \leq \frac{F_0}{\nu^2}. \quad (3.40)$$

We now turn our attention to the analysis of J_2 : because we need to analyze with great care both the nonlinear term and the term originating from the forcing, we write the ladder (3.32) with $n = 2$ from the beginning, namely, by denoting the velocity field by $u = (u_1, u_2)$, we have

$$\begin{aligned} \frac{1}{2} J_2 &\leq -\nu J_3 + \sum_{i=1}^2 \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})) \\ &+ \sum_{i=1}^2 \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 f_i) dx. \end{aligned} \quad (3.41)$$

We wish now to estimate the nonlinear term and the forcing term as accurately as we possibly can; we start with

Lemma 4. *For the incompressible two-dimensional Navier-Stokes equations on the two-dimensional torus $\Omega = [0, 2\pi]^2$, the nonlinear term obeys the estimate*

$$\sum_{i=1}^2 \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})) \leq 3|Du|_\infty J_2, \quad (3.42)$$

where

$$|Du|_\infty = \sup_{x \in \Omega} \left\{ \left| \frac{\partial u_i}{\partial x_j} \right|, i, j = 1, 2 \right\} \text{ with } x_1 = x, x_2 = y$$

is the maximum component of the various spatial derivatives of the velocity field

$$u = (u_1, u_2).$$

Proof. See Appendix A.

Thus, in the light of Lemma 4 we can state the

Lemma 5. *The time-asymptotic estimate of $J_2(t)$ is given by*

$$\bar{J}_2 \leq \left(\frac{2F_0}{\nu^3} \left(\frac{3^4 F_0^3}{2\pi^2 \nu^9} + \frac{(F_0 F_3)^{\frac{1}{2}}}{\nu} \right) \right)^{\frac{1}{2}}. \quad (3.43)$$

Proof. We start from the inequality (3.32), which then becomes

$$\frac{1}{2} J_2 \leq -\nu J_3 + 3|Du|_\infty J_2 + J_1^{1/2} F_3^{1/2}, \quad (3.44)$$

where the last term has been obtained by first integrating by parts and then using a Cauchy-Schwarz inequality. By using [22]

$$|Du|_\infty \leq \sqrt{\frac{1}{\pi} J_3^{\frac{1}{4}} J_1^{\frac{1}{4}}} \quad \text{and} \quad J_2 \leq J_3^{\frac{1}{2}} J_1^{\frac{1}{2}},$$

the inequality (3.44) becomes,

$$\frac{1}{2} \dot{J}_2 \leq -\nu J_3 + 3\sqrt{\frac{1}{\pi}} J_3^{\frac{3}{4}} J_1^{\frac{3}{4}} + J_1^{1/2} F_3^{1/2}. \quad (3.45)$$

Splitting the term $3\sqrt{\frac{1}{\pi}} J_3^{\frac{3}{4}} J_1^{\frac{3}{4}}$ by using a Young inequality and simplifying and rearranging we obtain

$$\dot{J}_2 \leq -\frac{\nu}{2} J_3 + \frac{3^4}{2\pi^2\nu^3} J_1^3 + 2J_1^{1/2} F_3^{1/2}. \quad (3.46)$$

By using $-J_3 \leq \frac{-J_2^2}{J_1}$, we have that the long-time asymptotic behaviour of J_2 is given by

$$\bar{J}_2 \leq \left(\frac{2F_0}{\nu^3} \left(\frac{3^4 F_0^3}{2\pi^2\nu^9} + \frac{(F_0 F_3)^{\frac{1}{2}}}{\nu} \right) \right)^{\frac{1}{2}}. \quad (3.47)$$

Because we are going to use the Brezis-Gallouet inequality with explicit constants, we need to have an accurate pointwise in time estimate of the $J_1(t)$ norm, and then take the limit as time goes to infinity. In fact, in the Brezis-Gallouet inequality the $J_1(t)$ norm appears in the denominator and so, it is a detailed time-pointwise analysis that is needed. Recall that the Brezis-Gallouet inequality states that for any J^2 mean-zero function $\phi(x, y)$ on the two-dimensional torus we have (see formula (48) in [5])

$$\|\phi\|_{\infty}^2 \leq \frac{\|\nabla\phi\|_2^2}{4\pi} [\eta + \ln \delta], \quad (3.48)$$

where $1 \ll \delta := \frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}$, $\epsilon = \frac{1}{\ln \delta} \ll 1$, and $\eta := 1 + \gamma + \frac{4\beta'(1)}{\pi} + O(\epsilon) \simeq 1.83 + O(\epsilon)$ with $\gamma \simeq 0.58$ being the Euler-Mascheroni constant and $\beta'(1) \simeq 0.19$, or alternatively (see *Theorem 3.8* in [6])

$$\|\phi\|_{\infty}^2 \leq \frac{\|\nabla\phi\|_2^2}{4\pi} \left[\ln \delta + \ln(1 + \ln \delta) + \hat{L} \right], \quad (3.49)$$

where $\delta \geq 1$ and $\hat{L} \simeq 2.15$. In the vector case $u = (u_1, u_2)$, as for example in the Navier-Stokes equations, the explicit constants are essentially the same (see Appendix B), and so for (3.48), we correspondingly obtain the estimate

$$\|u\|_{\infty}^2 \leq \frac{J_1}{4\pi} \left[\hat{\eta} + \frac{1}{2} \ln \left(\frac{J_2}{J_1} \right) \right] \text{ with } \frac{J_2}{J_1} \gg 1, \quad (3.50)$$

where $\hat{\eta} = \eta - \frac{1}{4} \ln(4c_1c_2)$, and a similar one for (3.49); c_1, c_2 are two positive constants such that $c_1 + c_2 = 1$.

So in order to use (3.50), we need an estimate of the lower bound of $J_1(t)$, as this term appears in the denominator of the Brezis-Gallouet inequality.

Its behaviour will naturally depend on the structure of the forcing function, and its estimate is a very tricky business; the only “current” estimate of a lower bound is that contained in [15] for the energy of solutions inside the attractor of the two-dimensional NSE on the torus; taking the length of the torus to be 2π , the estimate in [15] reads

$$J_1 \geq J_0 > \Gamma \Lambda, \quad (3.51)$$

where

$$\Gamma = (6c_3)^{-6} \left(\frac{\pi\nu}{4c_0}\right)^2 G^{38} (\ln(G+1))^{18}, \quad (3.52)$$

$$\Lambda = 3^{-8} \left(\frac{\pi}{4c_0}\right)^{-24} G^{-48} (\ln(G+1))^{-24}, \quad G = \frac{F_0^{\frac{1}{2}}}{\nu^2}, \quad c_3 = \frac{7}{8}(21)^{\frac{1}{3}};$$

in the above formula G is the so-called Grashof number and c_0 is an absolute constant of order one.

Hence, we have effectively proved the following result:

Theorem 6. *Assume that the kinematic viscosity is small enough, namely, there exists a ν_0 such that $0 < \nu_0 \ll 1$. Then for $\nu < \nu_0$, the long-time behaviour of the $\|u\|_\infty^2$ norm of the solution of the two-dimensional Navier-Stokes equations on the torus of length 2π is obtained by taking (3.47) and (3.51) and by inserting them into (3.50). One obtains*

$$\overline{\|u\|_\infty^2} \leq \frac{1}{4\pi} \frac{F_0}{\nu^2} [\hat{\eta} + \ln(\bar{J}_2(\Gamma\Lambda)^{-1})]. \quad (3.53)$$

It is of course interesting to compute the estimate (3.53) in the limit of very small ν obtaining

$$\overline{\|u\|_\infty^2} \leq \frac{1}{4\pi} \frac{F_0}{\nu^2} [-28 \ln \nu + \tilde{\eta}], \quad \nu \rightarrow 0^+, \quad (3.54)$$

where $\tilde{\eta} = \hat{\eta} + \ln\left(\frac{3^{10}}{\pi} \left(\frac{\pi}{4}\right)^{22} \frac{(6c_3)^6}{c_0^{22}} F_0^7 (\ln(\frac{F_0^{\frac{1}{2}}}{\nu^2} + 1))^6\right)$.

Appendix A. Here we wish to prove Lemma 4 which we re-write for the convenience of the reader:

Lemma 4. *For the incompressible 2-dimensional Navier-Stokes equations on the two-dimensional torus $\Omega = [0, 2\pi]^2$, the nonlinear term obeys the estimate*

$$\sum_{i=1}^2 \sum_{n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})) \leq 3|Du|_\infty J_2, \quad (3.55)$$

where

$$|Du|_\infty = \sup_{x \in \Omega} \left\{ \left| \frac{\partial u_i}{\partial x_j} \right|, i, j = 1, 2 \right\} \text{ with } x_1 = x, x_2 = y$$

is the maximum component of the various spatial derivatives of the velocity field $u = (u_1, u_2)$.

Proof. Take the terms

$$\sum_{i=1}^2 \sum_{n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})) \quad (3.56)$$

and insert the calculations of the derivatives $D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})$ obtaining

$$\begin{aligned} & \sum_{i=1}^2 \sum_{n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) \left[(D^2 u_1) \frac{\partial u_i}{\partial x_1} + 2(Du_1) \left(D \frac{\partial u_i}{\partial x_1} \right) + u_1 (D^2 \frac{\partial u_i}{\partial x_1}) \right. \\ & \left. + (D^2 u_2) \frac{\partial u_i}{\partial x_2} + 2(Du_2) \left(D \frac{\partial u_i}{\partial x_2} \right) + u_2 (D^2 \frac{\partial u_i}{\partial x_2}) \right], \end{aligned} \quad (3.57)$$

where D^2 is equal in turn to $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y}$. After an integration by parts, the third and the last terms in the square brackets disappear due to the divergence theorem; for the remaining terms we make the following transformations: whenever any of the spatial derivatives of any component of the velocity field appears, we take it out of the integral sign in the L^∞ norm; hence any one of the terms $\int (D^2 u_i) (D^2 u_j) \frac{\partial u_i}{\partial x_j}$ becomes

$$\int (D^2 u_i) (D^2 u_j) \frac{\partial u_i}{\partial x_j} \leq \left| \frac{\partial u_i}{\partial x_j} \right|_\infty \int |(D^2 u_i) (D^2 u_j)|. \quad (3.58)$$

Furthermore, observe that if $i = j$ in (3.58), then

$$\int |(D^2 u_i) (D^2 u_i)| = J_{2, u_i}^{x_i, x_i},$$

where by $J_{2, u_i}^{x_i, x_i}$, we denote one of the components of J_2 “along” the u_i component of the velocity field. On the other hand, if $i \neq j$, then to terms of the form

$$\left| \frac{\partial u_i}{\partial x_j} \right|_\infty \int |(D^2 u_i) (D^2 u_j)|, \quad i \neq j, \quad (3.59)$$

we first apply a Cauchy-Schwarz inequality to the integral, and then a Young inequality to the term as a whole; this gives as a result terms of the form

$$\frac{1}{2} \left| \frac{\partial u_i}{\partial x_j} \right|_\infty J_{2, u_i}^{x_i, x_j}, \quad i, j = 1, 2.$$

Hence, by doing the above transformations to all the terms forming

$$\sum_{i=1}^2 \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int (D^2 u_i) (D^2 (u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y})), \quad (3.60)$$

then collecting all the terms having the same $|\frac{\partial u_i}{\partial x_j}|_\infty$, and using that $|\frac{\partial u_i}{\partial x_j}|_\infty \leq |Du|_\infty$, we obtain the result stated in the Lemma. \square

Appendix B. Consider a real vector valued function in two spatial dimensions $u = (u_1(x_1, x_2), u_2(x_1, x_2))$ defined on the torus $T^2 = [0, 2\pi]^2$. Its modulus squared is given by $|\vec{u}|^2 = u_1^2 + u_2^2 = |u_1^2 + u_2^2|$. Thus, its sup norm over $(x, y) \in T^2$ is given by

$$|\vec{u}|_\infty^2 = |u_1^2 + u_2^2|_\infty \leq |u_1|_\infty^2 + |u_2|_\infty^2 \leq \frac{\|\nabla u_1\|_2^2}{4\pi} [\eta + \ln \delta_1] + \frac{\|\nabla u_2\|_2^2}{4\pi} [\eta + \ln \delta_2], \quad (3.61)$$

where $\delta_i = \frac{\|\Delta u_i\|_2^2}{\|\nabla u_i\|_2^2}$ $i = 1, 2$ (see formula (3.48)). Now we observe that $\delta_i \leq \delta_1 \delta_2$, $i = 1, 2$, and so

$$|\vec{u}|_\infty^2 \leq \frac{\|\nabla u_1\|_2^2}{4\pi} [\eta + \ln \delta_1] + \frac{\|\nabla u_2\|_2^2}{4\pi} [\eta + \ln \delta_2] \leq \frac{J_1}{4\pi} [\eta + \ln(\delta_1 \delta_2)]. \quad (3.62)$$

Now, we use the important result that, provided the vorticity field is sufficiently regular and not zero, both components of the divergence free velocity field are not zero and (generally) of the same order; this follows from solving the Poisson equation on the two-dimensional torus [29]. More precisely, let $x = (x_1, x_2)$ and $u = u(x) = (u_1(x_1, x_2), u_2(x_1, x_2))$ be the divergence free velocity field in the two-dimensional Navier-Stokes equations on the torus $T = [0, 2\pi]^2$. Then assuming that the vorticity field is known, the solution of the equations

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0 \quad (3.63)$$

in the unknown quantity u , is given by

$$u(x) = \frac{i}{2\pi} \sum' (k_2, -k_1) \frac{\omega(k)}{k^2} e^{ik \cdot x}, \quad (3.64)$$

where $k = (k_1, k_2)$, $k \cdot k = k^2$ and $\omega(k)$ is the Fourier transform of $\omega(x)$. Thus, provided that in the forced two-dimensional incompressible Navier-Stokes equations, the *curl* of the forcing $f(x) = (f_1(x), f_2(x))$ is not zero, namely $\nabla \times f \neq \mathbf{0}$, it is reasonable to suppose that the two components of the velocity field, namely $(u_1(x_1, x_2), u_2(x_1, x_2))$ are bounded away from

zero in L^2 , namely $\|u_1\|_2^2 > 0$, $\|u_2\|_2^2 > 0$. Also on the torus one has

$$\|\nabla u\|_2^2 = \|\omega\|_2^2 \geq (2\pi)^{-2} \left(\int |\omega| \right)^2 > 0,$$

provided that $|\omega(x, t)|$ is not identically zero, and so for generic non-zero vorticity fields one has that both components of the spatial gradient of the velocity field are non-zero. Hence, if the identically zero solution of the forced Navier-Stokes equations $u(x)$ is not in the global attractor \mathcal{A} , then for all $t > 0$, one has that the L^2 norm (squared) of each component of the velocity field $\|u_1(t)\|_2^2 > 0$, $\|u_2(t)\|_2^2 > 0$ [15]. Then noting that $J_1 = \|\nabla \vec{u}\|_2^2 = \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$ it follows that (by using the Poincaré inequality)

$$J_1 > \|\nabla u_1\|_2^2 \geq \|u_1\|_2^2 > 0, \quad J_1 > \|\nabla u_2\|_2^2 \geq \|u_2\|_2^2 > 0. \quad (3.65)$$

We now wish to estimate as accurately as we possibly can the term $\ln(\delta_1 \delta_2)$. First, from the above considerations, one can assume that $\|\nabla u_1\|_2^2 = c_1 J_1$ with the constant $0 < c_1 < 1$, and similarly, $\|\nabla u_2\|_2^2 = c_2 J_1$ with the constant $0 < c_2 < 1$, and $c_1 + c_2 = 1$. Thus, from

$$\ln(\delta_1 \delta_2) = \ln \left(\frac{\|\Delta u_1\|_2^2 \|\Delta u_2\|_2^2}{\|\nabla u_1\|_2^2 \|\nabla u_2\|_2^2} \right)$$

we obtain

$$\frac{1}{2} \ln \left(\frac{\|\Delta u_1\|_2 \|\Delta u_2\|_2}{\|\nabla u_1\|_2 \|\nabla u_2\|_2} \right) \leq \frac{1}{2} \ln \left(\frac{\|\Delta u_1\|_2^2 + \|\Delta u_2\|_2^2}{2\sqrt{c_1 c_2} J_1} \right) \leq \frac{1}{2} \ln \left(\frac{J_2}{2\sqrt{c_1 c_2} J_1} \right). \quad (3.66)$$

Rearranging we finally obtain

$$\|u\|_\infty^2 \leq \frac{J_1}{4\pi} \left[\hat{\eta} + \frac{1}{2} \ln \left(\frac{J_2}{J_1} \right) \right], \quad (3.67)$$

where $\hat{\eta} = \eta - \frac{1}{4} \ln(4c_1 c_2)$ which is (3.50) as required.

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