Numerical Solutions for PDEs
spring 2012

Your are required to submit your report to the teaching support office and to e-mail your matlab code to me at s.b.delahaies@surrey.ac.uk by the **30th of April**. Work in group is accepted, precise your co-worker(s).

Part A:

We consider the following partial differential equation

$$u_t - 5u_x = 0, \quad x \in (0,3), \quad t > 0,$$

together with the boundary condition

$$u_x(0,t) = 0, \quad u_x(3,t) = 0,$$

and initial condition

$$u(x,0) = \begin{cases} 1, & \text{if } x \geq 2.5, \\ 0, & \text{if } x < 2.5. \end{cases}$$

The Leapfrog scheme is the central time central space given by

$$\frac{\delta_t U_{j,n}}{2\Delta t} - 5\frac{\delta_x U_{j,n}}{2\Delta x} = 0,$$

for $U_{j,n} \approx u(x_j, t_n)$.

1. Work out the local truncation error and show that the scheme is consistent with the PDE.

2. Carry out a von Neumann stability analysis and find under what restrictions the approximate solution converges toward a solution of the PDE.

3. Consider a spatial discretization of the domain into N intervals of same size. Show that

$$U_{j,2} = f(x_j) + 5\frac{\Delta t}{\Delta x}\left(f(x_{j+1}) - f(x_{j-1})\right) + 5\frac{\Delta t^2}{\Delta x}\left(g(x_{j+1}) - g(x_{j-1})\right),$$

   for $j = 1, \ldots, N-1$ and

$$U_{0,2} = U_{1,2}, \quad U_{N,2} = U_{N-1,2}.$$

4. Write a matlab script to find an approximate solution to the initial-bounday value problem described above using the Leapfrog scheme:

   (i) Briefly describe the structure of your code, state how you implemented the first time step. Add your code at the end of your report in appendix.

   (ii) Print plots showing d'Alembert's solution to this problem together with 1) a stable solution, 2) a unstable solution, for each case precise the values of Δt, Δx and the number of timesteps.

TURN PAGE FOR PART B

Part B:
We consider Poisson equation given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in [0,1]^2, \tag{1}$$

together with boundary conditions at $u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0$. We choose $N \in \mathbb{N}$ and we discretize the domain as follows $x_j = j/N$, $y_k = k/N$, $j,k = 0,\dots,N$. We approximate the differential operator using the following five point scheme

$$u_{xx} + u_{yy} \approx \alpha_1 U_{j-1,k-1} + \alpha_2 U_{j+1,k-1} + \alpha_3 U_{j+1,k+1} + \alpha_4 U_{j-1,k+1} + \alpha_0 U_{j,k}. \tag{2}$$

1. Find $\alpha_0, \dots, \alpha_4$. Show that the approximate solution satisfies the equation

   $$\mathbf{MU} = \mathbf{F},$$

   where $\mathbf{U}$ and $\mathbf{F}$ are defined by

   $$\mathbf{U} = \begin{pmatrix} U_{1,1} \\ \vdots \\ U_{1,N-1} \\ \vdots \\ U_{N-1,1} \\ \vdots \\ U_{N-1,N-1} \end{pmatrix}, \quad \mathbf{F} = 2h^2 \begin{pmatrix} f(x_1,y_1) \\ \vdots \\ f(x_1,y_{N-1}) \\ \vdots \\ f(x_{N-1},y_1) \\ \vdots \\ f(x_{N-1},y_{N-1}) \end{pmatrix}$$

   where $h = 1/N$, and $\mathbf{M}$ is the blok matrix given by

   $$\mathbf{M} = \begin{pmatrix} A & B & 0 & \dots & 0 \\ B & A & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \dots & 0 & B & A \end{pmatrix}$$

   where A and B are the $(N-1)\times(N-1)$ matrices given by

   $$A = \begin{pmatrix} -4 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

2. Write a matlab code to find an approximate solution to the boundary value problem defined above using the matrix formulation given above with

   $$f(x,y) = \mathrm{e}^{-100(x-0.5)^2 - 100(y-0.5)^2}.$$

   (i) Describe briefly the structure of your code.

   (ii) Take N=4, find the value of the approximate solution by hand, compare this with the result of your code.

   (iii) Show one plot of the function f, show one plot of your approximate solution $\mathbf{U}$.