Numerical Solutions for PDEs: Exercise sheet 3

Exercise 1:
Write down the θ-method for the heat diffusion equation $u_t - u_{xx} = 0$. Assuming zero boundary conditions, show that the scheme can written in the matrix form

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{B}\mathbf{u}_n,$$

where $\mathbf{u}_s = (U_{j+1,s}, U_{j,s}, U_{j-1,s})^T$. Use this formulation to find approximation solution of the heat diffusion equation with initial condition $u(x,0) = \sin \pi x$ and zero boundary conditions for $N = 4$ and $r = 0.4$ on $x \in [0,1]$, $t \geq 0$.

Show that with non-zero boundary conditions the scheme can be written in the form

$$\mathbf{A}\mathbf{u}_{n+1} = \mathbf{B}\mathbf{u}_n + \mathbf{b},$$

Use this formulation to find approximation solution of the heat diffusion equation with initial condition $u(x,0) = \sin \pi x$ and boundary conditions $u(0,t) = a$ and $u(1,t) = b$ for $N = 4$ and $r = 0.4$ on $x \in [0,1]$, $t \geq 0$.

Exercise 2:
Write down the Crank-Nicolson scheme for the reaction-diffusion equation

$$u_t = u_{xx} - 10u, \quad x \in (0,1), \quad t > 0,$$

with

$$u(0,t) = 1, \quad u(1,t) = 0, \quad u(x,0) = 1 - x.$$

The term $-10u$ is approximated by averaging over t_n and t_{n+1}. Using $\Delta x = 1/N = 1/2$ and $\Delta t = \Delta x^2$, calculate the approximate solution after the first two timesteps.

Exercise 3:
Consider the parabolic boundary value problem

$$\begin{cases} u_t = u_{xx}, \quad x \in (0,1), t > 0 & , \quad u_x(0,t) = a(t), \\ u(x,0) = f(x) & , \quad u(1,t) = b(t). \end{cases}$$

Let u be a solution of this problem, then the maximum principle states that the maximum of u is obtained on the boundary of the domain

$$\partial\Omega = \{(x,0)\} \cup \{(0,t)\} \cup \{(x,t)\} \cup \{(1,t)\}.$$

Show that under the restriction $\theta \leq 1/2(1-\theta)$ the θ-method satisfies this principle.

Exercise 4:
Consider the following problem

$$\begin{cases} u_t = u_{xx}, \quad x \in (0,1) & , \quad u_x(0,t) = 1, \\ u(x,0) = \frac{1}{2}(x-1)^2 & , \quad u(1,t) = 0. \end{cases}$$

Write down a FTCS scheme for the PDE. Using 1. a one-sided scheme, 2. ghost points, with $\Delta x = 1/2$, and $r = 0.25$, calculate the approximate solution after the first two timesteps.