# Numerical Solutions for PDEs: Exercise sheet 4

**Exercise 1:**
Show that the FTFS scheme for $u_t + au_{xx} = 0$ is consistent. Show that it is stable if $a < 0$ and $\Delta t \leq \Delta x/a$.

**Exercise 2:**
The Leapfrog scheme for $u_t + au_{xx} = 0$ is the CTCS scheme given by

$$\frac{U_{j,n+1} - U_{j,n-1}}{2\Delta t} + a\frac{U_{j+1,n} - U_{j-1,n}}{2\Delta x} = 0,$$

Show that it is consistent. Show that it is stable for $-1 \leq p \leq 1$.

**Exercise 3:**
The Lax-Wendroff scheme for for $u_t + au_x = 0$ is given by

$$U_{j,n+1} = U_{j,n} - a\frac{D_x U_{j,n}}{2\Delta x}\Delta t + \frac{a^2}{2}\frac{\delta_x^2 U_{j,n}}{2\Delta x}\Delta t^{\,2}$$

Show that the LTE is given by

$$\frac{1}{6}\left(\Delta t^{\,2} u_{ttt} + a\Delta x^{\,2} u_{xxx}\right) + O(\Delta x^{\,3}, \Delta t^{\,3}).$$

Show that the amplification factor is given by

$$\xi = 1 - 2p^2 \sin^2\frac{\omega}{2} - 2ip\sin\frac{\omega}{2}\cos\frac{\omega}{2},$$

where $p = a\Delta t/\Delta x$, then show that it is stable $-1 \leq p \leq 1$.

**Exercise 4:**
The Crank-Nicolson scheme for $u_t + au_x = 0$ is given by

$$\frac{U_{j,n+1} - U_{j,n}}{\Delta t} + \frac{a}{2}\frac{D_x U_{j,n}}{2\Delta x} + \frac{a}{2}\frac{D_x U_{j,n+1}}{2\Delta x} = 0.$$

Show that the LTE is given by

$$\mathcal{L}_\Delta u = au_{xxx}\left(\frac{1}{6} + \frac{p^2}{12}\right)\Delta x^{\,2} + O(\Delta x^{\,3}, \Delta t^{\,3}),$$

where $p = a\Delta t/\Delta x$. Find the amplification factor and find the conditions for stability.

**Exercise 5:**
We consider the second order wave equation $u_{tt} = a^2 u_{xx}$. We use a CSCT scheme to approximate this equation, that is

$$\frac{\delta_t^2 U_{j,n}}{\Delta t^{\,2}} = a^2\frac{\delta_x^2 U_{j,n}}{\Delta x^{\,2}}.$$

Show that the LTE is given by

$$\mathcal{L}_\Delta u = \frac{a^2}{12}(p^2 - 1)u_{4x}\Delta x^{\,2} + O(\Delta x^{\,4}, \Delta t^{\,4}),$$

where $p = a\Delta t/\Delta x$ and that the scheme is stable if $-1 \leq p \leq 1$.