Numerical Solutions for PDEs: Lab 5

The barotropic vorticity equation describes the evolution of a homogeneous, divergence free, incompressible (2d) fluid on the surface of the sphere. In the absence of any non-conservative forces, Kelvin's circulation theorem tells us that any line integral of the flow around any material loop is conserved in time. Using Stokes' theorem this can be related to the conservation of the absolute vorticity $\omega = \zeta + f$, that is:

$$\frac{D}{Dt}(\zeta + f) = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)(\zeta + f) = 0,$$

where $\mathbf{u} = (u, v)$ is the fluid velocity, with zonal and meridional components, $f = f_0 + \beta y$ is the so-called β-plane approximation to the Coriolis force, and ζ is the relative vorticity defined by $\zeta = \nabla \times \mathbf{u} = v_x - u_y$. Since the flow is divergence free, we can introduce a streamfunction ϕ such that $\mathbf{u} = \nabla \times \phi$, and we have

$$\zeta = \nabla^2 \phi.$$

Therefore the set of equations

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v &= 0 && (1)\\
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \zeta, && (2)\\
u &= -\frac{\partial \phi}{\partial y}, && (3)\\
v &= \frac{\partial \phi}{\partial x}. && (4)
\end{aligned}$$

defined a closed set of equations for u, v and ζ. We want to use the finite difference methods studied in the course to find numerical solutions to this equation on the rectangular domain $(x, y) \in [0, 1]^2$. Consider the following algorithm:

1. Consider $\beta = 1$, start with the initial relative vorticity given by

   $$\zeta_0 = \mathrm{e}^{-100(x-0.5)^2 - 100(y-0.4)^2} - 0.5\mathrm{e}^{-100(x-0.5)^2 - 100(y-0.6)^2}$$

2. Considering periodic boundary condtions solve the Poisson equation (2).
3. Solve equations (3) and (4) using first order central difference operators and periodic boundary conditions.
4. set $n = n + 1$ and use the folloving scheme to approximate equation (1) :

   $$\frac{F_t \zeta}{\Delta t} + u\frac{\delta_x \zeta}{2\Delta x} + v\frac{\delta_y \zeta}{2\Delta y} + v = 0,$$

   where we assume periodic boundary conditions. Then repeat 2. 3. 4. .