Newton Raphson Method

Notice: this material must not be used as a substitute for attending the lectures

0.1 Newton Raphson Method

The Newton Raphson method is for solving equations of the form $f(x) = 0$. We make an initial guess for the root we are trying to find, and we call this initial guess x_0. The sequence $x_0, x_1, x_2, x_3, \ldots$ generated in the manner described below should converge to the exact root.
To implement it analytically we need a formula for each approximation in terms of the previous one, i.e. we need x_{n+1} in terms of x_n.
The equation of the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The tangent line intersects the x-axis when $y = 0$ and $x = x_1$, so

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

Solving this for x_1 gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and, more generally,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

You should **memorize** the above formula. Its application to solving equations of the form $f(x) = 0$, as we now demonstrate, is called the **Newton Raphson method**. It is guaranteed to converge if the initial guess x_0 is close enough, but it is hard to make a clear statement about what we mean by 'close enough' because this is highly problem specific. A sketch of the graph of $f(x)$ can help us decide on an appropriate initial guess x_0 for a particular problem.

0.2 Example

Let us solve $x^3 - x - 1 = 0$ for x.
In this case $f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$. So the recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 1)}{3x_n^2 - 1}$$

Need to decide on an appropriate initial guess x_0 for this problem. A rough graph can help. Note that $f(1) = -1 < 0$ and $f(2) = 5 > 0$. Therefore, a root of $f(x) = 0$ must exist between 1 and 2. Let us take $x_0 = 1$ as our initial guess. Then

$$x_1 = x_0 - \frac{(x_0^3 - x_0 - 1)}{3x_0^2 - 1}$$

and with $x_0 = 1$ we get $x_1 = 1.5$.
Now

$$x_2 = x_1 - \frac{(x_1^3 - x_1 - 1)}{3x_1^2 - 1}$$

and with $x_1 = 1.5$ we get $x_2 = 1.34783$. For the next stage,

$$x_3 = x_2 - \frac{(x_2^3 - x_2 - 1)}{3x_2^2 - 1}$$

and with the value just found for x_2, we find $x_3 = 1.32520$.
Carrying on, we find that $x_4 = 1.32472$, $x_5 = 1.32472$, etc. We can stop when the digits stop changing to the required degree of accuracy. We conclude that the root is 1.32472 to 5 decimal places.

0.3 Example

Let us solve $\cos x = 2x$ to 5 decimal places.
This is equivalent to solving $f(x) = 0$ where $f(x) = \cos x - 2x$. [**NB: make sure your calculator is in radian mode**]. The recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(\cos x_n - 2x_n)}{(-\sin x_n - 2)}$$

With an initial guess of $x_0 = 0.5$, we obtain:

$$\begin{aligned} x_0 &= 0.5 \\ x_1 &= 0.45063 \\ x_2 &= 0.45018 \\ x_3 &= 0.45018 \\ &\vdots \end{aligned}$$

with no further changes in the digits, to five decimal places. Therefore, to this degree of accuracy, the root is $x = 0.45018$.

0.4 Possible problems with the method

The Newton-Raphson method works most of the time if your initial guess is good enough. Occasionally it fails but sometimes you can make it work by changing the initial guess. Let's try to solve $x = \tan x$ for x. In other words, we solve $f(x) = 0$ where $f(x) = x - \tan x$. The recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(x_n - \tan x_n)}{1 - \sec^2 x_n}$$

Let's try an initial guess of $x_0 = 4$. With this initial guess we find that $x_1 = 6.12016$, $x_2 = 238.40428$, $x_3 = 1957.26490$, etc. Clearly these numbers are not converging.
We need a new initial guess. Let's try $x_0 = 4.6$. Then we find $x_1 = 4.54573$, $x_2 = 4.50615$, $x_3 = 4.49417$, $x_4 = 4.49341$, $x_5 = 4.49341$, etc. A couple of further iterations will confirm that the digits are no longer changing to 5 decimal places. As a result, we conclude that a root of $x = \tan x$ is $x = 4.49341$ to 5 decimal places.