Grad operator, triple and line integrals

Notice: this material must not be used as a substitute for attending the lectures

0.1 The grad operator ∇

Let $f(x_1, x_2, \ldots, x_n)$ be a function of the n variables $x_1, x_2, \ldots, x_n$. The grad of f, written grad f or ∇f is defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Often we are working in 3 dimensions, so the grad of f would be

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \quad \text{or} \quad \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Also recall the notation we use in 3d: $\mathbf{r} = (x, y, z)$ or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of a point, r (also denoted $|\mathbf{r}|$) is the length of the vector $\mathbf{r}$ and is given by $r = \sqrt{x^2 + y^2 + z^2}$ in 3d.

0.2 Example

If $f(x, y) = x^2 + y^2$ then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 2y).$$

0.3 Example

If $f(x, y, z) = 2xy - 3z^2$ then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2y, 2x, -6z).$$

0.4 Example

Show that $\nabla r = \frac{1}{r}\mathbf{r}$, where $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Solution. From the definition of the grad, we have

$$\nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right)$$

Now

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = \tfrac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$. Hence

$$\nabla r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = \frac{1}{r}(x, y, z) = \frac{1}{r}\mathbf{r}.$$

0.5 Properties of the grad operator

- $\nabla(f+g) = \nabla f + \nabla g$
- $\nabla(\alpha f) = \alpha \nabla f$ if α is constant
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

We will not prove all of the above properties here, only the third rule (the product rule for the grad operator). We have

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}\right) \\
&= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}, f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right) \\
&= f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) + g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= f\nabla g + g\nabla f.
\end{aligned}$$

0.6 Gradient as normal to a surface

The general form of the equation of a surface is $f(x,y,z) = 0$. (Often a surface will come to us in the form $z = g(x,y)$, but we can always write this as $g(x,y) - z = 0$.)

Fact: ∇f is perpendicular (normal) to the surface $f(x,y,z) = 0$ at every point.

Proof. Let P be a typical point on the surface $f(x,y,z) = 0$. Let C be *any* curve lying in the surface and passing through P. Assume this curve has been parametrised so that it is given by $\mathbf{r}(t) = (x(t), y(t), z(t))$ for some parameter t. Since C lies in the surface,

$$f(x(t), y(t), z(t)) = 0$$

Differentiating with respect to t using the chain rule for partial derivatives gives

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

i.e.

$$\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}_{=\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}_{=d\mathbf{r}/dt} = 0.$$

Hence ∇f is perpendicular to $d\mathbf{r}/dt$.

However, $d\mathbf{r}/dt$ is tangent to C, because

$$\frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Hence ∇f is perpendicular to the curve C at the point P. All this is true for *any* curve C passing through P. Hence ∇f is perpendicular to the surface at P and the proof is complete.

0.7 Example

Find a unit normal to the surface $z = x^2 + y^2$.

Solution. First write the surface in the form $x^2 + y^2 - z = 0$, i.e. $f(x, y, z) = 0$ where in this case $f(x, y, z) = x^2 + y^2 - z$. By the above theory, the vector

$$\nabla f = (2x, 2y, -1)$$

is normal to the surface.

This example asks for a **unit** normal so we must divide the above vector by its length. A unit normal $\mathbf{n}$ is then given by

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$$

0.8 Tangent plane to a surface

In two dimensions we can find the equation of the tangent line to a curve at any point using A-Level methods.

In three dimensions the analogous concept is the **tangent plane**. At any point on a (smooth) surface there will be a plane which is tangent to the surface at that point. We want to know how to find the equation of the tangent plane, and the next example illustrates how to do so.

0.9 Example (tangent plane)

Find the equation of the tangent plane to the surface $xy + yz + zx = 11$ at the point $(1, 2, 3)$.

Solution. Recall that the equation of a plane is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

where $\mathbf{a}$ is the position vector of some point in the plane, and $\mathbf{n}$ is a normal to the plane. We can find a suitable normal $\mathbf{n}$ by using the ideas just recently discussed, i.e. we write the surface in the form $f(x, y, z) = 0$ where, in this case, $f(x, y, z) = xy + yz + zx - 11$. Then

$$\nabla f = (y + z, x + z, y + x)$$

Thus, at $(1, 2, 3)$,

$$\nabla f = (5, 4, 3)$$

so a normal is $\mathbf{n} = (5, 4, 3)$. So the tangent plane has equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$, i.e.

$$\begin{aligned} ((x, y, z) - (1, 2, 3)) \cdot (5, 4, 3) &= 0 \\ (x - 1, y - 2, z - 3) \cdot (5, 4, 3) &= 0 \end{aligned}$$

so that $5(x - 1) + 4(y - 2) + 3(z - 3) = 0$. Thus, the equation of the tangent plane is

$$5x + 4y + 3z = 22.$$

0.10 Example

Find the equation of the tangent plane to the surface $xyz = a^3$ at the point (a, a, a).

Solution. The surface has equation $f(x, y, z) = 0$ where $f(x, y, z) = xyz - a^3$. We want a normal to the surface at the point (a, a, a). Such a normal is

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= (yz, xz, xy) \\ &= (a^2, a^2, a^2) \quad \text{at the given point.} \end{aligned}$$

The equation of the tangent plane is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ with $\mathbf{a} = (a, a, a)$ and $\mathbf{n} = (a^2, a^2, a^2)$ which simplifies to

$$x + y + z = 3a.$$

1 Method of Lagrange multipliers

The method of Lagrange multipliers is for the finding the maximum or minimum of a function $f(\mathbf{x})$ subject to a side constraint of the form $g(\mathbf{x}) = 0$.

To find such critical points, we form the equation

$$\nabla f = \lambda \nabla g$$

in which λ is called a *Lagrange multiplier*. The above equation is a vector equation, so we can compare components. The resulting equations are then solved together with $g(\mathbf{x}) = 0$ to find $\mathbf{x}$ and λ. In the general setting $\mathbf{x}$ is an n-dimensional vector, so we have $n + 1$ equations for $n + 1$ unknowns. So, in principle, the problem can be solved. How to do so in practice is highly problem-specific.

1.1 Example

Maximise $f(x, y, z) = xyz$ subject to $x^3 + y^3 + z^3 = 1$.

Solution. We can write the constraint in the form $g(x, y, z) = 0$ by taking $g(x, y, z) = x^3 + y^3 + z^3 - 1$. Now

$$\nabla f = (yz, xz, xy) \quad \text{and} \quad \nabla g = (3x^2, 3y^2, 3z^2)$$

so the equation $\nabla f = \lambda \nabla g$ becomes

$$(yz, xz, xy) = 3\lambda(x^2, y^2, z^2).$$

Comparing components gives

$$\begin{aligned} yz &= 3\lambda x^2 \\ xz &= 3\lambda y^2 \\ xy &= 3\lambda z^2 \end{aligned}$$

If we multiply the first of these equations by x, the second by y and the third by z we find that $3\lambda x^3 = 3\lambda y^3 = 3\lambda z^3$ and so, for this problem, $x = y = z$. Therefore, the equation $x^3 + y^3 + z^3 = 1$ becomes $3x^3 = 1$. So $x = y = z = (1/3)^{1/3}$. Therefore the maximum value of the function subject to the given constraint is

$$f(x, y, z) = xyz = \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{3}\right)^{\frac{1}{3}} = \frac{1}{3}.$$

1.2 Example

Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and furthest from the point $(1, 2, 2)$.

Solution. The distance of any point (x, y, z) to $(1, 2, 2)$ is given by

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Minimising or maximising this is equivalent to minimising or maximising its square. So, we want to minimise/maximise the function

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

subject to $x^2 + y^2 + z^2 = 36$. The latter is our constraint and so we take $g(x, y, z) = x^2 + y^2 + z^2 - 36$. Now

$$\begin{aligned} \nabla f &= (2(x-1), 2(y-2), 2(z-2)) \\ \nabla g &= (2x, 2y, 2z) \end{aligned}$$

so the equation $\nabla f = \lambda \nabla g$ becomes, on equating components,

$$2(x-1) = 2\lambda x, \quad 2(y-2) = 2\lambda y, \quad 2(z-2) = 2\lambda z$$

giving

$$x = \frac{1}{1-\lambda}, \quad y = \frac{2}{1-\lambda}, \quad z = \frac{2}{1-\lambda}.$$

Putting these expressions into $x^2 + y^2 + z^2 = 36$ gives

$$\frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 36$$

so that $\frac{9}{36} = (1-\lambda)^2$. Hence $1 - \lambda = \pm\frac{1}{2}$ giving $\lambda = \frac{3}{2}$ or $\frac{1}{2}$.

$\lambda = \frac{3}{2}$ gives $(x, y, z) = (-2, -4, -4)$, this is the furthest point.

$\lambda = \frac{1}{2}$ gives $(x, y, z) = (2, 4, 4)$ which is the nearest point. (NB: a diagram shows us which point is nearest/furthest).

1.3 Example

Fermat, in 1661, stated that light always travels along the path that minimises the time taken for the journey. The velocity of light depends on the medium it is travelling through. So the optimum path is not necessarily a straight line.

Medium 1 is located above medium 2, the two being separated by a horizontal straight line. In medium 1 light travels with speed v, while in medium 2 it travels with speed V. A light source is located in medium 1, and an observer in medium 2 a horizontal distance L to the right of the source. A light ray travels from source to observer along the path which minimises time taken, which consists of two straight line paths that meet at the interface. The path in medium 1 makes an angle α with the vertical, that in medium 2 makes an angle β with the vertical. We want to find the relationship between α and β.

We shall do this using the method of Lagrange multipliers. We shall calculate an expression for the time taken for the ray of light to get from source to observer, and then minimise this expression with α and β as the variables. Now

$$\text{time from source to interface} = \frac{\text{dist}}{\text{speed}} = \frac{a\sec\alpha}{v}$$

and

$$\text{time from interface to observer} = \frac{\text{dist}}{\text{speed}} = \frac{b\sec\beta}{V}$$

Therefore the total time taken by the ray of light is $f(\alpha,\beta)$ where

$$f(\alpha,\beta) = \frac{a\sec\alpha}{v} + \frac{b\sec\beta}{V}.$$

Also, a diagram shows that

$$L = a\tan\alpha + b\tan\beta.$$

So our constraint is $g(\alpha,\beta) = 0$ where $g(\alpha,\beta) = a\tan\alpha + b\tan\beta - L$. Applying the method of Lagrange multipliers, the equation $\nabla f = \lambda\nabla g$ becomes

$$\left(\frac{a}{v}\sec\alpha\tan\alpha,\ \frac{b}{V}\sec\beta\tan\beta\right) = \lambda(a\sec^2\alpha,\ b\sec^2\beta)$$

comparing components,

$$\frac{\tan\alpha}{v} = \lambda\sec\alpha, \qquad \frac{\tan\beta}{V} = \lambda\sec\beta$$

i.e.

$$\frac{\sin\alpha}{v} = \lambda, \quad \text{and} \quad \frac{\sin\beta}{V} = \lambda$$

So

$$\frac{\sin\alpha}{v} = \frac{\sin\beta}{V} \quad \text{or} \quad \frac{\sin\alpha}{\sin\beta} = \frac{v}{V}$$

This is called *Snell's Law.* It can be solved simultaneously with $L = a\tan\alpha + b\tan\beta$ to find α and β.

2 Cylindrical and spherical polar coordinates

Recall that in two dimensions, polar coordinates give us a way of describing the position of a point in the (x, y) plane. We describe the position by its distance r to the origin and the angle θ that the point makes with the positive x-axis, measured anticlockwise. From elementary trigonometry we have the following relations between a point's cartesian coordinates (x, y) and its polar coordinates (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

This section is about polar coordinates in three dimensional space. In three dimensions there are two different polar coordinate systems. These are: *cylindrical polars* and *spherical polars*. It is important not to confuse the two as they are quite separate systems and the meanings of the variables in the two systems are different.

2.1 Cylindrical polars (r, θ, z)

In this system the position of a point is described by the three coordinates (r, θ, z), in which r and θ have the same meaning as in plane polars and describe the location of the point's projection onto the (x, y) plane, while z is the point's cartesian z coordinate. In cylindrical polars the relationship between a points cartesian coordinates (x, y, z) and its cylindrical polar coordinates (r, θ, z) is given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Cylindrical polars are commonly used in practical problems where there is cylindrical symmetry, for example, fluid flow problems in pipes of circular cross-section.

2.2 Spherical polars (r, θ, ϕ)

In this system the position of a point is described by: its distance r to the origin, its angle θ measured from the positive z axis, and the angle ϕ that its projection onto the (x, y) plane makes with the positive x-axis.

Simple trigonometry shows that the relationship between a point's cartesian coordinates (x, y, z) and its spherical polar coordinates (r, θ, ϕ) is given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2.1}$$

Some books write θ and ϕ the other way round, which can be a source of confusion. One has to adopt a convention and stick to it.

Note that θ and ϕ are essentially latitude and longitude. Applications of the spherical polar coordinate system include aircraft navigation and the cataloguing of celestial objects like stars and galaxies. In the latter context θ and ϕ are called *declination* and *right ascension*.

Note the following useful fact which follows from the above three equations:

$$x^2 + y^2 + z^2 = r^2 \quad \text{in spherical polars.}$$

Note also this result does **not** hold in cylindrical polars.

Various surfaces have equations both in cartesian coordinates and in cylindrical or spherical polars. The following examples illustrate this.

2.3 Example

Let us write the cylinder $(x-2)^2 + y^2 = 4$ in terms of cylindrical polar coordinates.

First we expand out to get $x^2 - 4x + 4 + y^2 = 4$. In cylindrical polars $x = r\cos\theta$ and $y = r\sin\theta$. Inserting these expressions gives

$$r^2\cos^2\theta - 4r\cos\theta + 4 + r^2\sin^2\theta = 4$$

Simplifying this gives $r = 4\cos\theta$.

2.4 Example

The cartesian equation of a sphere of radius R centred at the origin is

$$x^2 + y^2 + z^2 = R^2.$$

Recall that in spherical polars $x^2 + y^2 + z^2 = r^2$. Therefore in spherical polars the equation of the sphere is simply $r = R$.

Let's look next at the sphere of radius R centred at the point $(0, 0, a)$ rather than the origin. Its cartesian equation would be

$$x^2 + y^2 + (z-a)^2 = R^2$$

Let's convert this to spherical polars. Expanding out,

$$x^2 + y^2 + z^2 - 2az + a^2 = R^2.$$

Using (2.1), this becomes $r^2 - 2ar\cos\theta + a^2 = R^2$ which is the sphere's equation in spherical polars.

3 Triple integrals

Triple integrals are integrals of the form

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

in which V is a subset of three dimensional space. An important particular case is:

$$\text{volume of a 3-d object} = \iiint_{\text{object}} dx\,dy\,dz$$

Such an integral is worked out similarly to the method for working out a double integral (i.e. start from the inside and work out). However it is often advantageous to make a substitution. As for double integrals, which substitution to use will depend as much on the region of integration V as on the integrand $f(x, y, z)$. Changing variables in a triple integral is like changing variables in a double integral and involves working out the **Jacobian** (it will be a 3×3 determinant this time) and working out the limits of integration.

If we change the variables x, y and z to new variables u, v and w defined by

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w),$$

then the **Jacobian** J is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We find the modulus $|J|$ of the Jacobian J. It is then an important fact that

$$dx\, dy\, dz = |J|\, du\, dv\, dw$$

Thus after making the substitution the integral becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_{\text{newlimits}} f(x(u, v, w), y(u, v, w), z(u, v, w))\, |J|\, du\, dv\, dw$$

In situations where the region of integration V is a cylinder or a sphere (or part thereof) it is common to transform into the appropriate polar coordinate system. Let us, therefore, calculate the Jacobian J for cylindrical polars and for spherical polars.

3.1 Jacobian in cylindrical polars (r, θ, z)

Recall that, in cylindrical polars, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Therefore the Jacobian J is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

which gives:

$$J = r \quad \text{in cylindrical polars}$$

3.2 Jacobian in spherical polars (r, θ, ϕ)

In spherical polars,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Therefore in this coordinate system the Jacobian J is given by

$$
\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\
&= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(-r\sin\theta\cos\theta\cos\phi) \\
&\quad - r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) \\
&= r^2\sin\theta\left[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi\right] \\
&= r^2\sin\theta(\cos^2\phi + \sin^2\phi) \\
&= r^2\sin\theta
\end{aligned}
$$

We have shown that

$$J = r^2\sin\theta \quad \text{in spherical polars}$$

3.3 Example

Find the volume of a sphere of radius a.

Solution. Let's take the sphere to be centred at the origin. In terms of spherical polar coordinates (r, θ, ϕ), the sphere is then described by

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi.$$

The volume of the sphere is given by

$$
\begin{aligned}
\text{volume} &= \iiint_{\text{sphere}} dx\,dy\,dz. \qquad \text{Now switch to spherical polars} \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \underbrace{r^2\sin\theta}_{\text{Jacobian}}\,dr\,d\theta\,d\phi \\
&= \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi \\
&= \frac{2a^3}{3}\int_0^{2\pi}d\phi \\
&= \frac{4}{3}\pi a^3
\end{aligned}
$$

3.4 Example

Evaluate

$$\iiint_V (y^2 + z^2)\,dV \qquad \text{[NB: } dV \text{ means } dx\,dy\,dz\text{]}$$

where V is the cylinder of radius a whose axis lies along the z-axis, occupying the portion $0 \le z \le h$ of the z-axis.

Solution. Switch to cylindrical polars $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The given cylinder V is then described by

$$0 \le r \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le h$$

and we have

$$\begin{aligned}
\iiint_V (y^2 + z^2)\, dV &= \int_0^h \int_0^{2\pi} \int_0^a (r^2 \sin^2\theta + z^2)\, r\, dr\, d\theta\, dz \\
&= \int_0^h \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\, dr\, d\theta\, dz + \int_0^h \int_0^{2\pi} \int_0^a r z^2\, dr\, d\theta\, dz \\
&= \frac{a^4}{4} \int_0^h \int_0^{2\pi} (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta)\, d\theta\, dz + \frac{a^2}{2} \int_0^h \int_0^{2\pi} z^2\, d\theta\, dz \\
&= \frac{\pi h a^4}{4} + \frac{\pi a^2 h^3}{3}
\end{aligned}$$

3.5 Example (volume of ellipsoid)

Let's find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The volume is equal to

$$\iiint_V dx\, dy\, dz$$

where in this case V consists of all points (x, y, z) which make up the ellipsoid.

The problem is simplified by the transformation

$$x = au, \qquad y = bv, \qquad z = cw$$

under which the equation of the ellipse transforms into $u^2 + v^2 + w^2 = 1$. So the ellipsoid V manifests itself in the (u, v, w) variables as a sphere of radius 1.

The Jacobian of the transformation is

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \\
&= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\
&= abc
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{volume} &= \iiint_V dx\, dy\, dz \\
&= \iiint_{\{u^2+v^2+w^2 \le 1\}} \overbrace{abc}^{\text{Jacobian}}\ du\, dv\, dw = abc \iiint_{\{u^2+v^2+w^2 \le 1\}} du\, dv\, dw \\
&= abc\ (\text{volume of sphere of radius 1}) = \frac{4}{3}\pi abc.
\end{aligned}$$

3.6 Example

Find the volume of the solid common to the two spheres given in terms of spherical polar coordinates by $r = 2$ and $r = 2\sqrt{2}\cos\theta$.

Solution. The sphere $r = 2$ is centred at the origin and has radius 2.

The other sphere $r = 2\sqrt{2}\cos\theta$ is centred at the point $(x, y, z) = (0, 0, \sqrt{2})$ and has radius $\sqrt{2}$ (to see this compare with the second part of Example 2.4).

We are considering a region of space defined by the intersection of two spheres, one of radius 2 centred at the origin, and the other a smaller sphere centred at a point on the z axis. A diagram, viewed with the (x, y) plane edge-on, shows two circles. Let A denote the centre of the offset one, and B one of the two points where the two circles intersect. Then $AB = \sqrt{2}$, $OA = \sqrt{2}$, and $OB = 2$, so that triangle OAB is right angled. Hence the angle AOB is $\pi/4$. A line drawn from O to B shows that the volume we are trying to calculate is divided into two parts, one of which is the $0 \le \theta \le \pi/4$ part which resembles an ice cream cone. In the other 'lower' part, θ is from $\pi/4$ to $\pi/2$ but the upper limit for r in this part will depend on θ.

Let us deal with 'ice cream' part first. This part is described in spherical polar variables by $0 \le r \le 2$, $0 \le \theta \le \pi/4$, $0 \le \phi \le 2\pi$. The volume of this part is $\iiint dx\, dy\, dz$. In spherical polars $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$ so the volume of this part is

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r^2 \sin\theta\, dr\, d\theta\, d\phi &= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\theta\, d\theta\, d\phi \\
&= \frac{8}{3}\int_0^{2\pi} [-\cos\theta]_0^{\pi/4}\, d\phi \\
&= \frac{8}{3}\int_0^{2\pi} \left(1 - \frac{1}{\sqrt{2}}\right) d\phi \\
&= \frac{16\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)
\end{aligned}$$

The other part of the volume is described by

$$\pi/4 \le \theta \le \pi/2, \quad 0 \le \phi \le 2\pi, \quad 0 \le r \le 2\sqrt{2}\cos\theta.$$

The volume of this part is

$$\begin{aligned}
\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\sqrt{2}\cos\theta} r^2 \sin\theta\, dr\, d\theta\, d\phi &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3}\sin\theta\right]_{r=0}^{r=2\sqrt{2}\cos\theta} d\theta\, d\phi \\
&= \frac{16\sqrt{2}}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \cos^3\theta \sin\theta\, d\theta\, d\phi \\
&= \frac{16\sqrt{2}}{3}\int_0^{2\pi} \left[-\frac{\cos^4\theta}{4}\right]_{\pi/4}^{\pi/2} d\phi \\
&= \frac{16\sqrt{2}}{3}\int_0^{2\pi} \frac{1}{16}\, d\phi \\
&= \frac{2\pi\sqrt{2}}{3}
\end{aligned}$$

The total volume of the region common to the spheres is therefore

$$\frac{16\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)+\frac{2\pi\sqrt{2}}{3}$$

which equals $(\pi/3)(16-6\sqrt{2})$.

4 Line integrals

Line integrals are integrals of the form

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $d\mathbf{r}=(dx,dy,dz)$ and C is a curve (usually in 3d).

A common interpretation of the line integral is that it equals the work done by a force on a particle as it moves along the curve C.

The curve C must have a definite start and end point. It may be a closed curve, but in the latter case we still have to be clear about the sense in which the curve is traversed (e.g. if the curve is in 2d, whether it is traversed clockwise or anticlockwise).

4.1 Example

Evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and C is the curve parametrised by $\mathbf{r}(t)=t\mathbf{i}+3t^2\mathbf{j}+2t^3\mathbf{k}$, for $0\le t\le 1$.

Solution. The curve C is given parametrically by $x=t$, $y=3t^2$, $z=2t^3$. We have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \underbrace{(x,y,z)}_{\mathbf{F}}\cdot\underbrace{(dx,dy,dz)}_{d\mathbf{r}} \\
&= \int_C (x\,dx+y\,dy+z\,dz)=\int_C\left(x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)dt \\
&= \int_0^1 (t(1)+3t^2(6t)+2t^3(6t^2))\,dt=\int_0^1(t+18t^3+12t^5)\,dt \\
&= \left[\frac{t^2}{2}+\frac{18t^4}{4}+\frac{12t^6}{6}\right]_0^1 \\
&= 7.
\end{aligned}$$

4.2 Example

Evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F} = (3z, y^2, 6z)$ and C is the curve $(\cos\theta, \sin\theta, \theta/3)$ for $0 \le \theta \le 4\pi$. This curve is a helix.

Solution.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (3z, y^2, 6z)\cdot(dx, dy, dz)\\
&= \int_C \left(3z\frac{dx}{d\theta} + y^2\frac{dy}{d\theta} + 6z\frac{dz}{d\theta}\right) d\theta\\
&= \int_0^{4\pi} \left(\theta(-\sin\theta) + \sin^2\theta\cos\theta + \frac{2\theta}{3}\right) d\theta\\
&= -\int_0^{4\pi} \theta\sin\theta\, d\theta + \left[\frac{\sin^3\theta}{3}\right]_0^{4\pi} + \left[\frac{\theta^2}{3}\right]_0^{4\pi}\\
&= 4\pi + \frac{16\pi^2}{3}.
\end{aligned}$$

4.3 Green's theorem

Let C be a closed curve in the (x, y) plane. If the curve is traversed anticlockwise then

$$\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where D is the region in the (x, y) plane enclosed by C.

4.4 Example of use of Green's theorem

Evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ and C is the closed curve consisting of $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by $y = x$ from $(1, 1)$ to $(0, 0)$.

Solution. Since the curve C in this problem lies entirely in the (x, y) plane we may evaluate this line integral by using Green's theorem. We have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (xy, y^2)\cdot(dx, dy)\\
&= \int_C xy\,dx + y^2\,dy\\
&= \iint_D (0 - x)\,dx\,dy \qquad \text{by Green's theorem}\\
&= -\iint_D x\,dx\,dy \qquad \text{evaluate this using double integral methods}\\
&= -\int_0^1\int_{x^2}^{x} x\,dy\,dx = -\int_0^1 [xy]_{y=x^2}^{y=x}\,dx\\
&= -\int_0^1 (x^2 - x^3)\,dx = -\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1\\
&= -\frac{1}{12}.
\end{aligned}$$

4.5 A formula for area

In Green's theorem, if we take $N = x$ and $M = -y$ then we get

$$\iint_D (1 - (-1))\, dx\, dy = \int_C (-y\, dx + x\, dy)$$

i.e.

$$2 \underbrace{\iint_D dx\, dy}_{=\text{area of } D} = \int_C (-y\, dx + x\, dy)$$

Hence we have the following formula:

$$\text{area of a region} = \tfrac{1}{2} \int_C (-y\, dx + x\, dy) \tag{4.2}$$

where C is the boundary curve of the region, traversed anticlockwise.

4.6 Area of ellipse using above formula

Let us find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

We parametrise the ellipse:

$$x = a \cos t, \quad y = b \sin t, \quad 0 \le t \le 2\pi.$$

Then we have

$$\begin{aligned}
\text{area of ellipse} = \tfrac{1}{2} \int_C (-y\, dx + x\, dy) & \\
&= \tfrac{1}{2} \int_0^{2\pi} \left(-b \sin t \frac{dx}{dt} + a \cos t \frac{dy}{dt} \right) dt \\
&= \tfrac{1}{2} \int_0^{2\pi} [-b \sin t(-a \sin t) + (a \cos t)(b \cos t)]\; dt \\
&= \tfrac{1}{2} ab \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \pi ab.
\end{aligned}$$

4.7 Area of a general polygon

The area of a polygon with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in anticlockwise order is given by

$$A = \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$$

Proof. By the general area formula (4.2), the area of a general polygon will be given by

$$\begin{aligned}
A &= \tfrac{1}{2} \int_C (-y\, dx + x\, dy) \\
&= \tfrac{1}{2} \sum_j \int_{C_j} (-y\, dx + x\, dy)
\end{aligned}$$

where C is the overall boundary of the polygon, and C_j is the straight line segment joining (x_j, y_j) to (x_{j+1}, y_{j+1}) for $j = 1, 2, \ldots, n-1$, and C_n joins (x_n, y_n) to (x_1, y_1).

We can parametrise the segment C_j by

$$(x, y) = (1-t)(x_j, y_j) + t(x_{j+1}, y_{j+1}) \qquad \text{for } 0 \le t \le 1$$

(this is basically using the vector equation of a straight line), so that

$$\begin{aligned} x &= (1-t)x_j + t x_{j+1}, \\ y &= (1-t)y_j + t y_{j+1}. \end{aligned}$$

Thus

$$\begin{aligned}
A &= \tfrac{1}{2} \sum_j \int_{C_j} \left(-y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt \\
&= \tfrac{1}{2} \sum_j \int_0^1 \{ -[(1-t)y_j + t y_{j+1}](x_{j+1} - x_j) + [(1-t)x_j + t x_{j+1}](y_{j+1} - y_j) \} \, dt \\
&= \tfrac{1}{2} \sum_j \left\{ (x_{j+1} - x_j) \left[-\frac{t^2}{2} y_{j+1} + \frac{(1-t)^2}{2} y_j \right]_0^1 + (y_{j+1} - y_j) \left[-\frac{(1-t)^2}{2} x_j + \frac{t^2}{2} x_{j+1} \right]_0^1 \right\} \\
&= \tfrac{1}{2} \sum_j \left\{ (x_{j+1} - x_j)(-\tfrac{1}{2} y_{j+1} - \tfrac{1}{2} y_j) + (y_{j+1} - y_j)(\tfrac{1}{2} x_{j+1} + \tfrac{1}{2} x_j) \right\} \\
&= \tfrac{1}{2} \sum_j (x_j y_{j+1} - x_{j+1} y_j)
\end{aligned}$$

and the proof is complete.