UNIVERSITY OF SURREY ©

Faculty of Engineering and Physical Sciences

Department of Mathematics

Undergraduate Programmes in Mathematical Studies

Module MAT1030 — 15 Credits

Calculus

Level 1 Examination

Time allowed: Two hours

Semester 1, 2011/12

Answer **ALL** questions. All working must be shown. Please note that some questions carry more marks than others.

Where appropriate the mark carried by an individual part of a question is indicated in square brackets [ ].

Approved calculators are allowed.

Additional material:
None

Question 1

(a) Write down the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials. From these definitions show that
(i) $e^x = \cosh x + \sinh x$, (ii) $\cosh^2 x - \sinh^2 x = 1$, (iii) $\sinh 2x = 2\sinh x \cosh x$. [6]

(b) Given that $\sinh x = \frac{12}{5}$, find $\cosh x$, $\sinh 2x$ and e^x. [4]

Question 2

Assuming that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ where $i = \sqrt{-1}$, show that

$$\sin^4 x = \tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x$$ [7]

Question 3

Assuming that the Maclaurin expansion for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

deduce expansions, with the coefficients simplified as far as possible, for (i) $\cos 2x$, (ii) $\cos 4x$ and (iii) $\sin^4 x$, in each case up to and including the x^6 term. For part (iii) you may use the result of Question 2. [9]

Question 4

Evaluate the following limits using L'Hopital's rule, with a substitution where necessary:

(a) $\displaystyle\lim_{x\to 7} \frac{x^2 - 49}{x - 7}$ (b) $\displaystyle\lim_{x\to 0} \frac{\cos 5x - \cos 3x}{x^2}$ (c) $\displaystyle\lim_{x\to\infty} \left(x - x^2 \ln\left(\frac{1+x}{x}\right)\right)$ [10]

Question 5

Differentiate each of the following, simplifying your answers

(a) $\dfrac{x^3}{1+x^2}$ (b) $\cos 3x^2$ (c) $x \sin^{-1} x$ [7]

Question 6

Find the sum to infinity of the geometric series $\displaystyle\sum_{n=1}^{\infty} x^{n+1}$.

By differentiating your answer show that, for $-1 < x < 1$,

$$\sum_{n=1}^{\infty} (n+1)x^n = \frac{2x - x^2}{(1-x)^2}$$

and hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)}{2^n}$. [7]

Question 7

Evaluate the integrals

(a) $\displaystyle\int x \cos x \, dx$ (b) $\displaystyle\int \frac{x}{\sqrt{4 - x^2}} \, dx$ (c) $\displaystyle\int \frac{3x + 7}{x^2 + 3x - 4} \, dx$ [11]

Question 8

By writing the integrand as $\cos^{n-1} x \cos x$ and then integrating by parts show that, if n is an integer with $n \geq 2$, then

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx. \qquad [5]$$

Find $\displaystyle\int_0^{\pi/2} \cos^9 x\, dx$. [4]

Question 9

Solve the following differential equations, expressing the solution in the form $y = f(x)$:

(i) $\dfrac{dy}{dx} = -2y^3 x$ [3]

(ii) $\dfrac{dy}{dx} = \dfrac{y^2+1}{y}$ subject to $y(0) = 2$ [4]

(iii) $(x+1)\dfrac{dy}{dx} + 4y = 1$ [4]

Question 10

Solve the differential equations:

(i) $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 26y = 0$ [4]

(ii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 2e^{-4x}$ subject to $y(0) = 1$ and $y'(0) = 2$ [8]

Question 11

Show that the transformation $v = \dfrac{1}{y^2}$ transforms the differential equation

$$\frac{dy}{dx} + y = 3e^x y^3$$

into

$$\frac{dv}{dx} - 2v = -6e^x. \qquad [4]$$

Hence solve the original differential equation, expressing the answer in the form $y = f(x)$. [3]

END OF PAPER

INTERNAL EXAMINER: S.A. Gourley